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One Kind of Optimal International Security Investment Portfolio and Consumption Choice Problem[1)]

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Abstract One kind of optimal security investment portfolio and consumption choice problem in international securities markets when the securities pay dividends is studied by using classical dynamic programming method. The economical analysis to the optimal choice of the investor is given through the investment theory when the investor invests two kinds of securities in different countries. The explicit optimal solution is presented using very simple and direct method for two kinds of typical utility cases, the idea comes from the technique which is used for solving the celebrated LQ problem in optimal control theory. At last, some simulation results are given to illustrate the influence of the parameters on optimal choice.

Key words Consumption/investment optimization, dynamic programming principle, stochastic analysis and control

1 Introduction

We consider the optimal international security investment portfolio and consumption choice problem in this paper. The international portfolio diversification depends on several factors. These factors include the different expected rates of return, the paying dividends of the securities, the volatilities, the regulations and transaction costs in different countries and also depends on the effect of exchange risk. Some explanations can be seen in Choi[1] and Black[2].

In Merton[3] and Duffie[4,5], the stochastic models for security market are given. They used Brownian motion to represent the uncertain resource in the market. Karatzas[6] used one unified approach, based on stochastic analysis, to study the consumption and investment problem systematically. Bellalah and Wu[7] studied one kind of corporate investment optimal problem using optimal control theory.

In our paper we use the stochastic models of security market to deal with one kind of optimal international security investment problem. In Section 2, we study the optimal problem for two kinds of stocks in two different countries. The investor can choose his proportion of the wealth in these two kinds of stocks and he wants to maximize his expected utility of both consumption and terminal wealth. His investment and consumption strategy can also be influenced by the exchange rate. We give the solving frame using classical stochastic dynamical programming principle and give the economical analysis to the optimal choice of the investor using the investment theory.

For two kinds of typical cases, the important constant relative risk aversion case and another case, we use a simple and direct method to give the explicit optimal choice for the investor in Section 3. The idea comes from the technique which is used for solving the celebrated LQ problem in optimal control theory (see Yong and Zhou[8]). In the last section, we give some numerical examples to illustrate the influence of parameters on the optimal

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choice which is obtained in Section 3 for two typical cases.

2 The optimal international investment problem

We consider a "two-country " security market model where an investor can invest his money in two different security markets , one is in the domestic , another is in a foreign country. The foreign prices are influenced by uncertain exchange rate. The numeration is the domestic currency and the investor retains his domestic consumption habitat. This reflects an implicit assumption according to which all projects are evaluated in domestic currency unit.

We suppose the investor wants to invest one kind of stock in his home country and another kind of stock in a foreign country, the price of domestic stock and the stock in the foreign country satisfy

$$\mathrm{d}P(t) = P(t)b_1(t)\mathrm{d}t + P(t)\sigma_1(t)\mathrm{d}B_1(t) \tag{1}$$

$$\mathrm{d}\bar{P}(t) = \bar{P}(t)\bar{b}_1(t)\mathrm{d}t + \bar{P}(t)\bar{\sigma}_1(t)\mathrm{d}B_2(t) \tag{2}$$

respectively. The currency exchange rate for the two countries satisfies

$$\mathrm{d}e(t) = e(t)b_e(t)\mathrm{d}t + e(t)\sigma_e(t)\mathrm{d}B_3(t) \tag{3}$$

p_0, $\bar{p}_0$ and e_o are initial values, respectively. Here $b_1(t)$, $\bar{b}_1(t)$ and $b_e(t)$ represent the instantaneous expected rates in security and currency exchange markets, respectively. $\sigma_1(t)$, $\bar{\sigma}_1(t)$ and $\sigma_e(t)$ are instantaneous volatilities, respectively. They are all assumed to be deterministic and bounded. The terms $B_1(t)$, $B_2(t)$ and $B_3(t)$ are three one-dimensional mutually dependent Brownian motions which represent the external sources of uncertainty in the different markets with correlation coefficients $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$. We need change the price of the stock in foreign country into domestic currency and let $R(t)=e(t)\bar{P}(t)$, $b_R(t)=b_e(t)+\bar{b}_1(t)+\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3}$. Then

$$\mathrm{d}R(t) = R(t)b_R(t)\mathrm{d}t + R(t)\bar{\sigma}_1(t)\mathrm{d}B_2(t) + R(t)\sigma_e(t)\mathrm{d}B_3(t) \tag{4}$$

Let $W(t)$ represent the wealth of the investor with the initial wealth W_0, $x(t)$ represent the proportion in foreign stock investment, $1-x(t)$ be the proportion in home stock market, and $c(t)$ be the consumption rate of the investor in home country. We suppose the stocks pay dividends continuously at the rates $d_1(t)$ and $\bar{d}_1(t)$ in domestic and foreign markets, respectively. Then

$$\begin{aligned}\mathrm{d}W(t) =& W(t)[x(t)(b_R(t)+\bar{d}_1(t))+(1-x(t))(b_1(t)+d_1(t))]\mathrm{d}t + \\ & W(t)(1-x(t))\sigma_1(t)\mathrm{d}B_1(t) + W(t)x(t)\bar{\sigma}_1(t)\mathrm{d}B_2(t) + \\ & W(t)x(t)\sigma_e(t)\mathrm{d}B_3(t) - c(t)\mathrm{d}t\end{aligned} \tag{5}$$

There are several points of view to deal with the problem of stocks paying dividends in the existing literature. The one we do in this paper can be seen in Duffie[4], and the wealth process of the investor is modelled such that the wealth rises for acquiring the dividends, yet the downfall of the stock price after paying dividends is implicitly addressed in the model.

Suppose the investor wants to maximize the expected utility of both consumption and terminal wealth at some future time $T>0$ by choosing his portfolio and consumption rate

$$J(W_0) = \max_{(x(\cdot),c(\cdot))} \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}U(c(t))\mathrm{d}t + \mathrm{e}^{-rT}h(W(T))\right] \tag{6}$$

where $(x(\cdot), c(\cdot))$ is the proportion in foreign stock investment and the consumption rate, $r>0$ is the discount rate. U and h are called utility functions if U: $(0,\infty)\to\mathbb{R}$ is C^1 function and satisfies i) $U(\cdot)$ is strictly increasing and strictly concave; ii) The derivative $U'(c)=\frac{\partial U(c)}{\partial c}$ satisfies $\lim_{c\to\infty}U'(c)=0$ and $U'(0_+)=\lim_{c\to 0}U'(c)=\infty$, where U can take U or h. These ordinary properties of the utility functions can be seen in Section 7 of [6].

We first give the definition of the admissible strategy of the investor including consumption rate and proportion of the stocks.

Definition 1. An admissible strategy of the investor is the consumption rate and proportion $(x(t), c(t))$, $t \in [0, T]$, where $x(\cdot)$ is bounded process, $c(\cdot)$ is nonnegative progressively measurable process and satisfies $\mathbb{E}\int_0^T c^2(t)\mathrm{d}t < \infty$.

So the problem is to look for admissible strategy $(x^*(t), c^*(t))$ which is called an optimal strategy if it attains the maximum of $J(W_0)$. In the following part of this section we use the classical dynamical programming to show how to solve this problem. We let

$$\bar{J}(W(t), t; T) = \max_{(x(\cdot), c(\cdot))} \mathbb{E}\left[\int_t^T e^{-rs}U(c(s))\mathrm{d}s + e^{-rT}h(W(T))\right]$$

where $(x(\cdot), c(\cdot))$ is the admissible strategy. Then $\bar{J}(W_0, 0; T) = J(W_0)$ and from the classical dynamic programming principle, we can get the following Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial \bar{J}}{\partial t} + \max\limits_{(x,c)}\Big\{ \frac{\partial \bar{J}}{\partial W}W[x(t)(b_R(t) + \bar{d}_1(t)) + (1 - x(t))(b_1(t) + d_1(t))] + \\ \quad \frac{1}{2}\frac{\partial^2 \bar{J}}{\partial W^2}W^2(1 - x(t))^2\sigma_1^2(t) + \frac{1}{2}\frac{\partial^2 \bar{J}}{\partial W^2}W^2x^2(t)\bar{\sigma}_1^2(t) + \\ \quad \frac{1}{2}\frac{\partial^2 \bar{J}}{\partial W^2}W^2x^2(t)\sigma_e^2(t) + \frac{\partial^2 \bar{J}}{\partial W^2}W^2x(t)(1 - x(t))\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} + \\ \quad \frac{\partial^2 \bar{J}}{\partial W^2}W^2x(t)(1 - x(t))\sigma_1(t)\sigma_e(t)\rho_{1,3} + \frac{\partial^2 \bar{J}}{\partial W^2}W^2\, x^2(t)\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3} - \\ \quad \frac{\partial \bar{J}}{\partial W}c(t) + e^{-rt}U(c)\Big\} = 0, \\ \bar{J}(W, T) = e^{-rT}h(W), \qquad t \in [0, T] \end{cases} \tag{7}$$

We can try to find the optimal consumption rate by $e^{-rt}U'(c) = \frac{\partial \bar{J}}{\partial W}$. Similar to that in Karatzas[6], we let $I(\cdot)$ be the inverse of the strictly decreasing function U', so the optimal consumption rate $c^*(t)$ under the reasonable assumptions for $U(\cdot)$ and $h(\cdot)$ can be obtained by

$$c^*(t) = I\left(\frac{\partial \bar{J}}{\partial W}e^{rt}\right) \tag{8}$$

From (7), the optimal value of $x(t)$ corresponding to the optimum proportion of foreign stock investment in the investor's total capital budget can be obtained by

$$x^*(t) = \frac{\Gamma^1}{\Gamma^2} \tag{9}$$

where

$$\Gamma^1 = \frac{\partial^2 \bar{J}}{\partial W^2}W\sigma_1^2(t) + \frac{\partial \bar{J}}{\partial W}(b_1(t) + d_1(t)) - \frac{\partial \bar{J}}{\partial W}(b_R(t) + \bar{d}_1(t)) - \frac{\partial^2 \bar{J}}{\partial W^2}W\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - \frac{\partial^2 \bar{J}}{\partial W^2}W\sigma_1(t)\sigma_e(t)\rho_{1,3}$$

$$\Gamma^2 = \frac{\partial^2 \bar{J}}{\partial W^2}W[\sigma_1^2(t) + \bar{\sigma}_1^2(t) + \sigma_e^2(t) - 2\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - 2\sigma_1(t)\sigma_e(t)\rho_{1,3} + 2\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3}]$$

We have Proposition 1.

Proposition 1. Under reasonable assumptions for utility functions U, h and the coefficients of the models in different markets, the optimal proportion $x^*(t)$ for the investor can be obtained by (9), the optimal consumption rate $c^*(t)$ can be obtained by (8), and the optimal value function $J(W_0) = \bar{J}(W, 0)$, where $\bar{J}(W, t)$ is the solution of partial differential equation(PDE) (7) when we take place (8) and (9) to (7).

Now we give the economical analysis for $x^*(t)$ in (9). In fact, we can write our optimal proportion of foreign stock investment in the investor's total capital budget in the following form.

$$x^*(t) = \frac{1}{S_w^2(t)}\left[\frac{\alpha_R(t) - \alpha_p(t)}{A} + (S_p^2(t) - S_{pR}(t))\right] = H_1 + H_2$$

$$H_1 = \frac{1}{AS_w^2(t)}(\alpha_R(t) - \alpha_p(t)),\ H_2 = \frac{1}{S_w^2(t)}(S_p^2(t) - S_{pR}(t))$$

$$A = -\frac{\overline{J}_{ww}W}{\overline{J}_w},\ \alpha_p(t) = b_1(t) + d_1(t),\ \alpha_R(t) = b_R(t) + \overline{d}_1(t)$$

and

$$S_w^2(t) = S_R^2(t) + S_p^2(t) - 2S_{pR}(t),\ S_{pR}(t) = \sigma_1(t)\overline{\sigma}_1(t)\rho_{1,2} + \sigma_1(t)\sigma_e(t)\rho_{1,3}$$

$$S_p^2(t) = \sigma_1^2(t),\ S_R^2(t) = \overline{\sigma}_1^2(t) + \sigma_e^2(t) + 2\overline{\sigma}_1(t)\sigma_e(t)\rho_{2,3}$$

The term A corresponds to the Pratt-Arrow measure of relative risk aversion. It reflects the investor's attitude in risk taking for domestic and foreign stock investments (see Adler M and Dumas B[9]). This equation is of the general form as the one in Dornbush[10] in stock markets. The term S_w^2 refers to the variability of the portfolio.

The first term H_1 in $x^*(t)$ is refered to as the "speculative demand" or the aggressive demand and depends on the measure of relative risk aversion. The second term H_2 in $x^*(t)$ is called the hedging demand. This term describes the demand for foreign stock investments on a risk-hedged basis. It is independent of the investor's attitude toward risk. The value of H_1 can be expressed as

$$H_1 = \frac{1}{A\,S_w^2(t)}[\,b_e(t) + \overline{\sigma}_1(t)\sigma_e(t)\rho_{2,3} + \overline{b}_1(t) + \overline{d}_1(t) - b_1(t) - d_1(t)\,] \tag{10}$$

This expression of the speculative demand embraces some theories on international investment. Some theories of investment incorporate the demand side effect. In this case, foreign stock investment is affected by whether the expectation and dividends rates of stock abroad are higher than in the domestic or home market. This idea is reflected in the term $\overline{b}_1(t)+\overline{d}_1(t)-b_1(t)-d_1(t)$. The term $b_e(t)+\overline{\sigma}_1(t)\sigma_e(t)\rho_{2,3}$ indicates how the flow of investment is affected by the exchange rate, accounting for the degree of competition in the economy of the foreign country. It also reflects that the way the investment flow from domestic stock market to the foreign country depends on the stochastic changes.

Now, we turn to the second term H_2 in the above equation of $x^*(t)$ which reflects the passive hedging demand:

$$H_2 = \frac{1}{S_w^2(t)}[\,(S_p^2(t) - S_{pR}(t))]\ \text{or}$$

$$H_2 = \frac{1}{S_w^2(t)}[\sigma_1^2(t) - \sigma_1(t)\overline{\sigma}_1(t)\rho_{1,2}(t) - \sigma_1(t)\sigma_e(t)\rho_{1,3}] \tag{11}$$

The first term in the right hand side $\sigma_1^2(t)$ represents the home country volatility of the stock. The second term $\sigma_1(t)\overline{\sigma}_1(t)\rho_{1,2}$ reflects the home and foreign stock correlation volatility. The third term $\sigma_1(t)\sigma_e(t)\rho_{1,3}$ shows the home stock and exchange rate correlation volatility. This demand can be traced to uncertainty in price. For $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$ are the correlation coefficients, it is easy to prove that $S_w^2(t)>0$, so we have

"The great variability in domestic price can stimulate foreign investment."

In theory, we can solve the above partial differential equations to get the optimal (x^*, c^*) and the optimal value function $J(W_0)$ for some kinds of utility functions. But in fact, it is difficult to get the explicit solution through the partial differential equations. In the next section, we will study two typical utility functions and get the explicit solution.

3 Explicit solutions to two typical cases

In this section we consider the utility functions for two typical cases and give the explicit optimal proportion, the consumption rate and the value function for these two cases.

First case: CRRA utility function

We first study one important typical case called constant relative risk aversion (CRRA) case. The idea to get optimal decision and consumption rate comes from the technique to solve celebrated LQ (linear quadratic) problem in optimal control theory (see [8]). In (6), we let

$$U(c(t)) = \frac{c^{1-a}(t)}{1-a}, \quad h(W(T)) = \frac{W^{1-a}(T)}{1-a} \tag{12}$$

where a is constant, $0<a<1$. The admissible strategy $(x^*(t), c^*(t))$ is called an optimal strategy which attains the maximum of $J(W_0)$.

The solving method is constructive. We let $Q(t)$ be a nonnegative deterministic smooth function satisfying $Q(T)=1$, whose dynamics will be given later. Applying Itô′s formula to $\frac{e^{-rt}}{1-a}W^{1-a}(t)Q(t)$ from 0 to T and taking expectation on both sides, we can write

$$J(W_0) = \frac{W_0^{\ 1-a}}{1-a}Q(0) + I + II + III$$

where

$$I = \max_{(x,c)} \mathbb{E}\int_0^T \frac{e^{-rt}}{1-a}\left[c^{1-a}(t) - (1-a)W^{-a}(t)Q(t)c(t) - aW^{1-a}(t)Q^{1-\frac{1}{a}}(t)\right]\mathrm{d}t,\ a \in (0,1)$$

$$II = \max_{(x,c)} \mathbb{E}\int_0^T e^{-rt}W^{1-a}(t)Q(t)L(x(t))\mathrm{d}t$$

$$\begin{aligned} L(x(t)) = {} & (b_R(t)+\bar{d}_1(t))x(t) - (b_1(t)+d_1(t))x(t) + a\sigma_1^2(t)x(t) - a\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2}x(t) - \\ & a\sigma_1(t)\sigma_e(t)\rho_{1,3}x(t) - \frac{1}{2}a\sigma_1^2(t)x^2(t) - \frac{1}{2}a\bar{\sigma}_1^2(t)x^2(t) + a\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2}\,x^2(t) - \\ & \frac{1}{2}a\sigma_e^2(t)x^2(t) + a\sigma_1(t)\sigma_e(t)\rho_{1,3}x^2(t) - a\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3}x^2(t) - \Delta_t = \\ & \Big[-\frac{1}{2}a\sigma_1^2(t) - \frac{1}{2}a\bar{\sigma}_1^2(t) - \frac{1}{2}a\sigma_e^2(t) + \\ & a\sigma_1(t)\sigma_e(t)\rho_{1,3} + a\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - a\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3}\Big]x^2(t) + \\ & \left[b_R(t)+\bar{d}_1(t) - b_1(t) - d_1(t) + a\sigma_1^2(t) - a\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - a\sigma_1(t)\sigma_e(t)\rho_{1,3}\right]x(t) - \Delta_t \end{aligned}$$

$$\Delta_t = \frac{(\Delta_t^u)^2}{2\Delta_t^d}$$

$$\Delta_t^u = b_R(t) + \bar{d}_1(t) - b_1(t) - d_1(t) + a\sigma_1^2(t) - a\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - a\sigma_1(t)\sigma_e(t)\rho_{1,3}$$

$$\Delta_t^d = a\left[\sigma_1^2(t) + \bar{\sigma}_1^2(t) + \sigma_e^2(t) - 2\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - 2\sigma_1(t)\sigma_e(t)\rho_{1,3} + 2\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3}\right]$$

$$\begin{aligned} III = {} & \max_{(x,c)} \mathbb{E}\int_0^T e^{-rt}\frac{W^{1-a}(t)}{1-a}\Big[\dot{Q}(t) + ((1-a)(b_1(t) + d_1(t)) - r - \\ & \frac{1}{2}a(1-a)\sigma_1^2(t) + (1-a)\Delta_t)Q(t) + aQ^{1-\frac{1}{a}}(t)\Big]\mathrm{d}t \end{aligned}$$

Now we let $Q(t)$ be the solution to the following ordinary differential equation of Bernoulli type:

$$\begin{aligned} & -\dot{Q}(t) = M(t)Q(t) + aQ^{1-\frac{1}{a}}(t), \quad Q(T) = 1 \\ & M(t) = (1-a)(b_1(t) + d_1(t)) - r - \frac{1}{2}a(1-a)\sigma_1^2(t) + (1-a)\Delta_t \end{aligned} \tag{13}$$

We can get

$$Q(t) = e^{\int_t^T M(s)ds}\left[1 + \int_t^T e^{-\frac{1}{a}\int_s^T M(\tau)d\tau}ds\right]^a, \quad t \in [0,T] \tag{14}$$

Then we can look back at I and II. If we take

$$c^*(t) = (1-a)^{-\frac{1}{a}}Q^{-\frac{1}{a}}(t)W(t), \quad t \in [0,T] \tag{15}$$

where $Q(t)$ is given by (14) and is positive. One can show that I attains its maximum at point $c^*(t)$ and $I=0$. This is also the feed back form of wealth. Then we take

$$x^*(t) = \frac{\Delta_t^u}{\Delta_t^d} \tag{16}$$

where the denominator is positive. One can show $L'(x^*)=0$ and $L''(x^*)<0$. Thus the function $L(x)$ attains its maximum at point x^* and $L(x^*)=0$, $II=0$. Since $b_1(t)$, $d_1(t)$, $\bar{b}_1(t)$, $\bar{\sigma}_1(t)$, $b_e(t)$, $\bar{b}_e$ are all bounded, it is easy to know that $(x^*(\cdot), c^*(\cdot))$ is the admissible strategy. We conclude the above discussion in the following Theorems.

Theorem 1. For the optimal security investment portfolio and consumption choice problem (12) in the CRRA utility function case, the optimal proportion $x^*(t)$ for the investor to invest in foreign stock is denoted by (16), the optimal consumption rate $c^*(t)$ for the investor in home country is denoted by (15), and the optimal value function is

$$J(W_0) = \frac{1}{1-a} W_0^{1-a} e^{\int_0^T M(s)ds} [1 + \int_0^T e^{-\frac{1}{a}\int_s^T M(\tau)d\tau} ds]^a \tag{17}$$

From Theorem 1 and the proof process. Following Theorem 2 is easy to know.

Theorem 2. The Pratt-Arrow measure of relative risk aversion for CRRA case of our optimal international security investment problem is $A=a$, where a is the constant in (12) and $0<a<1$.

Second case: In this part, we consider one simple case. Let $U=0$ and $h(W(T)) = \log W(T)$. We let $c(t)=0$ in (5) and look for the optimal decision $x^*(t)$ to maximize the following problem:

$$J = \max_x \mathbb{E} \log W(T) \tag{18}$$

We study this case and try to get the explicit optimal decision and value function. We apply it is formula to $\log W(T)$ from 0 to T. Then

$$J = \log W_0 + \max_x \mathbb{E} \int_0^T \left[-\frac{\bar{\Delta}_t^d}{2} \left(x(t) - \frac{\bar{\Delta}_t^u}{\bar{\Delta}_t^d} \right)^2 + \frac{(\bar{\Delta}_t^u)^2}{2\bar{\Delta}_t^d} + b_1(t) + d_1(t) - \frac{1}{2}\sigma_1^2(t) \right] dt$$

where

$$\begin{cases} \bar{\Delta}_t^u = b_R(t) + \bar{d}_1(t) - b_1(t) - d_1(t) + \sigma_1^2(t) - \sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - \sigma_1(t)\sigma_e(t)\rho_{1,3} \\ \bar{\Delta}_t^d = \sigma_1^2(t) + \bar{\sigma}_1^2(t) + \sigma_e^2(t) - 2\sigma_1(t)\bar{\sigma}_1(t)\rho_{1,2} - 2\sigma_1(t)\sigma_e(t)\rho_{1,3} + 2\bar{\sigma}_1(t)\sigma_e(t)\rho_{2,3} \end{cases} \tag{19}$$

And it is easy to show that $\bar{\Delta}_t^d$ is positive. So we have the following theorem.

Theorem 3. For the optimal portfolio choice problem of (18), the optimal proportion for the investor to invest in foreign stock is denoted by $x^* = \frac{\bar{\Delta}_t^u}{\bar{\Delta}_t^d}$, where $x^*(t)$ is bounded, $\bar{\Delta}_t^u$ and $\bar{\Delta}_t^d$ are given by (19), the corresponding optimal value function is

$$J = \log W_0 + \int_0^T \left[\frac{(\bar{\Delta}_t^u)^2}{2\bar{\Delta}_t^d} + b_1(t) + d_1(t) - \frac{1}{2}\sigma_1^2(t) \right] dt \tag{20}$$

Similar to Theorem 2, we can also easily get the following theorem.

Theorem 4. For the international security optimal investment portfolio problem (20), the Pratt-Arrow measure of relative risk aversion is $A=1$.

It needs to be pointed out that we don't consider the time difference of the two countries and the periodic open-close of the markets in this paper. But the idea in our paper can be applied to these cases, the technique is more complex.

4 Simulation results and the sensitivities of the parameters

In the last section we give some simulation results of the optimal choices obtained in Section 3 for the two typical cases. We also consider the influence of the parameters on the optimal choice of the investor, which is called the sensitivities of the parameters. For simplicity, we let all the coefficients in the models be bounded constants in the followings. For some parameters, we first have the following two propositions.

Proposition 2. For constant relative risk aversion(CRRA) case, the higher the expected rates $\bar{b}_1$ and b_e in foreign stock and currency exchange rate, or the dividend rate $\bar{d}_1$ in foreign stock, the bigger the proportion x^* of the investor in foreign stock; the higher the expected rate b_1 and the dividend rate d_1 in home stock, the smaller the proportion x^* of the investor in foreign stock; the higher the discount rate r, the higher the consumption rate c^*.

Proof. From (16) and (15), we can check that

$$\frac{\partial x^*}{\partial \bar{b}_1}>0,\quad \frac{\partial x^*}{\partial b_e}>0\quad \frac{\partial x^*}{\partial \bar{d}_1}>0\quad \frac{\partial x^*}{\partial b_1}<0,\quad \frac{\partial x^*}{\partial d_1}<0,\quad \frac{\partial c^*}{\partial r}>0$$

So the conclusion is obtained. □

And for the second case, problem (18) in Section 3, we also get

Proposition 3. The higher the expected rates $\bar{b}_1$ and b_e in foreign stock and currency exchange rate, or the dividend rate $\bar{d}_1$ in foreign stock, the bigger the proportion x^* of the investor in foreign stock; the higher the expected rate b_1 and the dividend rate d_1 in home stock, the smaller the proportion x^* of the investor in foreign stock.

Now we give some numerical examples to show our results in Section 3.

Example 1.

Suppose $b_1=0.08$, $d_1=0.01$, $\bar{b}_1=0.075$, $\bar{d}_1=0.005$, $b_e=0.006$, $\sigma_1=0.2$, $\bar{\sigma}_1=0.15$, $\sigma_e=0.05$, $\rho_{1,2}=0.8$, $\rho_{1,3}=0.6$, $\rho_{2,3}=0.4$, $a=0.5$, $r=0.05$, $T=30$.

For the CRRA case, we can get $x^*=0.7273$, $c^*(0)=0.6059W_0$ and $J(W_0)=5.1386W_0^{\frac{1}{2}}$.

For the second case in Section 3, we can get $x^*=0.8182$. $J(W_0)=2.21+\log W_0$.

We see that the expected rate and dividend rate of home stock b_1 and d_1 are bigger than those in foreign stock $\bar{b}_1$ and $\bar{d}_1$, but the volatility coefficients $\sigma_1>\bar{\sigma}_1$, most part of the wealth of the investor is invested in foreign stock.

Example 2. Suppose $b_1=0.075$, $d_1=0.012$, $\bar{b}_1=0.08$, $\bar{d}_1=0.005$, $b_e=0.005$, $\sigma_1=0.1$, $\bar{\sigma}_1=0.2$, $\sigma_e=0.01$, $\rho_{1,2}=0.6$, $\rho_{1,3}=0.6$, $\rho_{2,3}=0.6$, $a=0.3$, $r=0.04$, $T=20$.

For the CRRA case, we can get $x^*=0.4176$, $c^*(0)=0.818W_0$, and $J(W_0)=2.1677W_0^{\frac{7}{10}}$.

For the second case in Section 3, we can get $x^*=0.0586$. $J(W_0)=1.64+\log W_0$.

In this example, most part of the wealth of the investor is invested in home stock.

References

1 Choi J B. Corporate international investment. *Journal of International Business Studies*, 1989, **20**(1): 145~155

2 Black F. International capital market equilibrium with investment barriers. *Journal of Financial Economics*, 1974, **1**(3): 337~352

3 Merton R. Continuous-time finance. Oxford: Blackwell Publishers, 1990

4 Duffie D. Security Markets, Stochastic Models. Boston: Academic Press, 1988

5 Duffie D. Dynamic Asset Pricing Theory. Princeton: Princeton University Press, 1992

6 Karatzas I. Optimization problems in the theory of continuous trading. *SIAM. J. Contrl and Optimization*, 1987, **27**(5):1221~1259

7 Bellalah M, Wu Zhen. Corporate international investment and diversification. *Finance India*, 2002, **16**(3):977~989

8 Yong Jiong-Min, Zhou Xun-Yu. Stochastic Controls: Hamiltonian Systems and HJB Equations. New York: Springer-Verlag, 1999

9 Adler M, Dumas B. Exposure to currency risk: definition and measurement. *Financial Management*, 1984, **3**(1): 41~50

10 Dornbusch R. Exchange rate risk and the macroeconomics of exchange rate determination. Cambridge: National Bureau of Economic Research Working Paper, N 493, 1980

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一类国际证券投资组合和消费选择的最优控制问题

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摘　要　首先运用经典动态规划方法，研究股票付息下国际证券市场中一类最优证券投资组合和消费选择问题，并利用投资学理论对投资者只投资两种证券情形的最优组合给出经济分析和解释. 然后，运用非常简单和直接的方法对两种典型的效用函数给出最优解的显式形式，求解的技巧来自解决线性二次最优控制问题的配平方法. 最后，给出一些数值计算例子来展示各模型参数对最优选择的影响.

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