H_∞ Design for Sampled-Data Systems *via* Lifting Technique: Conditions and Limitation[1)]

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Abstract The initial motivation of the lifting technique is to solve the H_∞ control problems. However, the conventional weighted H_∞ design does not meet the conditions required by lifting, so the result often leads to a misjudgement of the design. Two conditions required by using the lifting technique are presented based on the basic formulae of the lifting. It is pointed out that only the H_∞ disturbance attenuation problem with no weighting functions can meet these conditions, hence, the application of the lifting technique is quite limited.

Key words Sampled-data control system, lifting technique, design limitation, H_∞ synthesis

1 Introduction

Because the intersample behavior can be taken into account and the L_2-induced norm of the system can be obtained, the lifting technique now becomes the main tool for the H_∞ design of sampled-data systems. However, conditions that must be met when using the lifting technique have generally been ignored. This is because the lifting computation deals mostly with matrix exponentials, and if these conditions are not met, the process of computation can still be continued smoothly. But the performance resulted from the H_∞ synthesis will not be the same as expected if these conditions are violated. And it may lead to a misjudgement of H_∞ design. In this paper we will give a detailed analysis of the conditions required by the H_∞ synthesis *via* lifting.

2 Lifting computation for sampled-data systems

This section briefly reviews the main steps of the lifting computation, and the formulas listed below will be used in the discussion.

The lifting can be visualized as breaking up at each sampling time the continuous-time signal $f(t)$ into an infinite number of consecutive pieces $\hat{f}_k(t)$, *i.e.*,

$$\hat{f}_k(t) = f(\tau k + t), \quad 0 \leqslant t \leqslant \tau, k = 0, 1, 2, \cdots$$

The sequences $\{\hat{f}_k\}$ are also discrete-time signals which take values in the function space $L_2[0,\tau)$. Let $\{\hat{f}_k\} \in l^2_{L_2[0,\tau)}$. That is an $L_2[0,\tau)$-valued sequence whose norm sequence is square summable[1]:

$$\sum_{k=0}^{\infty} \|\hat{f}_k\|^2_{L_2[0,\tau)} < \infty$$

Consider a time-continuous system as follows:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1 w(t), \quad z(t) = C_1 \boldsymbol{x}(t) \tag{1}$$

The behavior of the state $\boldsymbol{x} \in \mathbb{R}^n$ under the lifted input $\{\hat{w}_k\}$ is given by

$$\boldsymbol{x}(k\tau + t) = e^{At}\boldsymbol{x}(k\tau) + \int_0^t e^{A(t-s)} B_1 \hat{w}_k(s) \mathrm{d}s, \quad 0 \leqslant t < \tau$$

Defining the discrete-time state $x_k := x(k\tau)$, then we have

$$\boldsymbol{x}_{k+1} = e^{A\tau}\boldsymbol{x}_k + \int_0^\tau e^{A(\tau-s)} B_1 \hat{w}_k(s) \mathrm{d}s \tag{2}$$

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which can be rewritten in the operator form as[1]

$$\boldsymbol{x}_{k+1} = e^{A\tau}\boldsymbol{x}_k + \Phi_b \hat{w}_k \tag{3}$$

where Φ_b is an operator, $\Phi_b : L_2[0,\tau) \to \mathbb{R}^n$. For $\hat{w}_k \in L_2[0,\tau)$, we have

$$\Phi_b \hat{w}_k = \int_0^\tau e^{A(\tau-s)} B_1 \hat{w}_k(s)\mathrm{d}s$$

The output equation of the system is

$$\hat{z}_k(t) = C_1 e^{At}\boldsymbol{x}_k + C_1 \int_0^t e^{A(t-s)} B_1 \hat{w}_k(s)\mathrm{d}s = \Phi_c \boldsymbol{x}_k + \Phi_{11}\hat{w}_k \tag{4}$$

where $\Phi_{11} : L_2[0,\tau) \to L_2[0,\tau)$ is the convolution operator and $\Phi_c : \mathbb{R}^n \to L_2[0,\tau)$ is the state transition operator. For details about the operator equations, please refer to [2].

Notice that, for the generalized plant of sampled-data control systems there are a second input, namely the control input u, and a second output y (see Fig. 1). They are connected with the discrete-time controller K_d *via* the hold H and the sampler S. The relationship between the input u_k and output y_k can be established by the general discretization method. By adding this second relationship to (3) (4), now the operator-valued transfer function of the generalized plant is of the following form[1]:

$$\tilde{G} = \left[\begin{array}{c|cc} e^{A\tau} & \Phi_b & B_{2d} \\ \hline \Phi_c & \Phi_{11} & \Phi_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{5}$$

where B_{2d} corresponding to the second input $\boldsymbol{u} \in \mathbb{R}^m$ is the input matrix formed by discretization, the operator $\Phi_{12} : \mathbb{R}^m \to L_2[0,\tau)$, and for $u(k\tau + t) = u_k$, $0 \leqslant t \leqslant \tau$, we have

$$\Phi_{12} u_k = C_1 (\int_0^t e^{A(t-s)}\mathrm{d}s) B_2 u_k \tag{6}$$

Φ_{11} is a convolution operator (see (4)), which will have a direct effect on conditions discussed in this paper. This is because Φ_{11} in (5) must be removed in the lifting computation, and the main lifting algorithm is developed for this purpose. [1] used the concept of Loop-shifting[3] proposed by Safonov to remove the operator Φ_{11}. The essential of the loop-shifting approach is to replace the effect of the feed through operator Φ_{11} by modifying the input/output operators Φ_b and Φ_c. The last step in the lifting computation is to transform the operator-valued transfer function into a finite-dimensional matrix-valued transfer function G_d, *i.e.*,

$$\tilde{G} = \left[\begin{array}{c|cc} A_d & B_{1d} & B_{2d} \\ \hline C_{1d} & 0 & D_{12d} \\ C_{2d} & 0 & 0 \end{array}\right] \tag{7}$$

The resultant G_d is an equivalent discrete-time plant, and the general H_∞ synthesis approach can then be used in the design. The algorithm for the lifting computation now is standard and available for use[4,5].

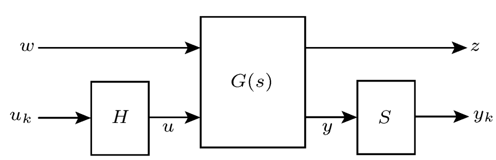


Fig. 1 The plant with a sampler and a hold

3 Conditions for applying the lifting technique

In [1] it was proved that the L_2-induced norm of a sampled-dada control system is preserved after lifting. But this does not indicate any conditions for norm preserving. Only assumed was the following assumption A1) for discussion of the computation formulas.

A1) (A, B_1) is controllable, (C_1, A) is observable.

In fact, this A1) is just one of the conditions for norm equivalence. This is because the lifting transformation involves operations on operators, some of the operators map the input to the state, while others map the state to the output. If condition of A1) is not met, some states of the plant certainly will not be included in the mapping chain of the operator transformation, and the resulting input-output relationship will be incorrect. This assumption has generally been ignored. It is because the lifting computation deals mostly with matrix exponentials[4], if A1) is violated then only some (matrix) zeros will appear in the system′s realization, and the process of computation can still be continued smoothly. However this condition of A1) usually can not be met by most H_∞ design problems, so we must re-emphasise it here. Notice that the matrix A of the generalized plant (see (1)) certainly includes the antialiasing filter of the system. But the output of the filter is connected to the discrete-time controller K_d, so the states of the filter are not observable from the system (performance) output z. This means that the observability condition of (C_1, A) is not met in all these cases. But the time constants of the filter are pretty small, and neglecting them will have little effect on the final result. So the antialiasing filter will be excluded in the verification of the condition A1).

As an example of application of the condition A1), let us consider a typical H_∞ design problem — robust stability problem (Fig. 2), where P is the nominal plant, W is the weighting function of the multiplicative uncertainty, F is the antialiasing filter, and K_d is the discrete controller. The remaining part of Fig. 2 with the exclusion of K_d and Δ is the generalized plant of the problem, where inputs and outputs of the plant are w, u_k, and z, y_k, respectively (see also Fig. 1). As can be seen from Fig. 2, the states of P and W are not controllable from the first input w of the generalized plant, *i.e.*, (A, B_1) is not controllable. Because this robust stability problem does not meet the condition of A1), so the norm of the system is not preserved under lifting. Indeed, the result of H_∞ robust stability design by lifting is incorrect, as has been shown by using the small gain theorem[6].

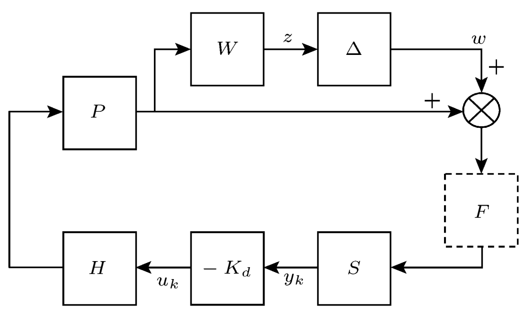


Fig. 2 Robust stability problem of the sampled-data system

In addition to the condition A1), there is a second condition that must be considered for norm equivalence. It is as follows.

A2) (C_2, A) is observable.

Where C_2 is the output matrix associated with the second output y. The observability of (C_2, A) is problematic for the system with weighting functions. It is because the weighting functions are located outside the closed-loop and are not observable from the second output y, *i.e.*, (C_2, A) is not observable. This means that the states of the weighting functions are not fully included in the lifting transformation process, and the result of the H_∞ lifting design is questionable. This problem can further be illustrated with the following examples.

For the disturbance attenuation problem (Fig. 3(a)), the states of the weighting function W_1 are not observable from y. Suppose that the plant and the weighting function in this example are

$$P(s)=\frac{20-s}{(5s+1)(s+20)},\quad W_1(s)=\frac{0.01\pi}{s+0.01\pi}$$

Let the sampling period $\tau=0.1$ sec. Lifting the generalized plant of the system, an equivalent discrete-time plant G_d (see (7)) can be obtained as follows.

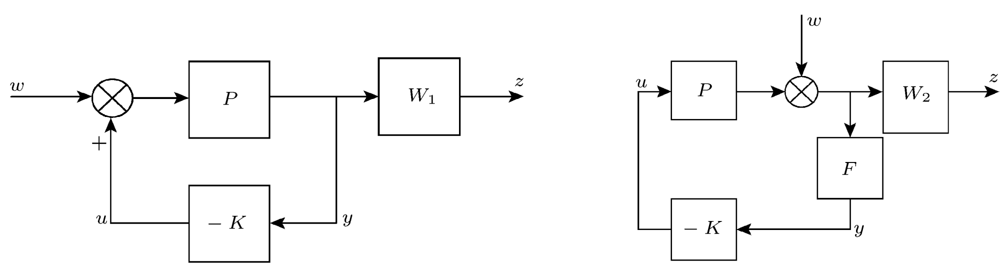
$$G_d=\left[\begin{array}{ccc|cccc} 0.1353 & -0.0002 & -0.1084 & 6.2664 & -0.0000 & 0.0000 & 1.7293 \\ 0.0085 & 0.9802 & -0.0007 & -0.0074 & -0.0309 & -0.0000 & 0.0027 \\ 0.0000 & 0.0031 & 0.9969 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ \hline -0.0000 & -0.0005 & -0.3157 & 0 & 0 & 0 & -9.753e-007 \\ -0.0000 & 0.0003 & -0.0000 & 0 & 0 & 0 & -5.335e-007 \\ -0.0000 & 0.0000 & -0.0000 & 0 & 0 & 0 & -1.566e-007 \\ -0.0000 & 0.0000 & -0.0000 & 0 & 0 & 0 & 0.966e-007 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{8}$$

By using the MATLAB function *dhinflmi* to solve the H_∞ optimization problem, the resulting H_∞ controller is

$$K(z)=\frac{8.3868(z-0.918)(z-0.1246)}{(z-0.9998)(z+0.07344)} \tag{9}$$

The corresponding L_2-induced norm is

$$\gamma=0.0058 \tag{10}$$



(a) (b)

Fig. 3 Typical H_∞ problems: (a) the disturbance attenuation problem, and (b) the sensitivity problem

For verification, notice that the static gain of the controller from (9) is

$$K(z)|_{z=1}=2849.2$$

And the steady-state disturbance attenuation value is

$$\frac{1}{1+K(e^{j0})P(j0)}=\frac{1}{1+2849.2}=0.00035 \tag{11}$$

Equation (11) is just the real disturbance attenuation performance γ of the system, which obviously differs from the lifting result of (10).

As for the sensitivity problem (Fig. 3(b)), the states of the plant P are not controllable from the first input w, *i.e.*, (A, B_1) is not controllable. And (C_2, A) is also not observable as in the first example. However, the lifting computation can still be continued. But the gain of the H_∞ controller obtained is quite low, and the resulting performance of the system is very poor (numerical examples are omitted). Such a design is useless.

4 H_∞ design with no weighting functions

According to the requirements of conditions A1) and A2), the system shown in Fig. 4 is the only sampled-data system that can meet these conditions. Fig. 4 represents the disturbance attenuation problem in H_∞ design but with no weighting functions.

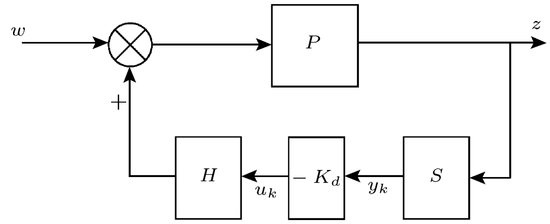


Fig. 4 The sampled-data control system satisfying the lifting conditions

Using weighting functions is the soul of the H_∞ design, which is based on the loop shaping concept. Although the disturbance attenuation problem without the weighting function (Fig. 4) can still be designed, it is not a loop shaping design and can achieve only a mild performance. Because weighting functions can not be used in the lifting design, the lifting technique can not be used in the H_∞ sensitivity problem or the robust stability problem, so the application of the lifting technique is quite limited.

5 Conclusions

1) The conditions A1) and A2) presented in this paper are the conditions required to meet for the H_∞ design by lifting. Because of the peculiarity of the lifting computation (matrix exponential calculations), even if these conditions are not met, the lifting transformation can still be continued, but the design result will not be the result expected by the synthesis, and sometimes the result may be useless.

2) For H_∞ design, only the disturbance attenuation problem without weighting functions can meet all the conditions of A1) and A2). So the limitation of the lifting technique is clear. The lifting technique was originally proposed for the demand of H_∞ design, but in fact it is not suitable for the H_∞ optimal design. For H_∞ design of the sampled-data control system it seems better to start directly from the frequency response of the sampled-data system[7], and this is our next topic.

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