A New Approach to Robust Stability Analysis of Sampled-data Control Systems[1)]

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Abstract The lifting technique is now the most popular tool for dealing with sampled-data control systems. However, for the robust stability problem the system norm is not preserved by the lifting as expected. And the result is generally conservative under the small gain condition. The reason for the norm difference by the lifting is that the state transition operator in the lifted system is zero in this case. A new approach to the robust stability analysis is proposed. It is to use an equivalent discrete-time uncertainty to replace the continuous-time uncertainty. Then the general discretized method can be used for the robust stability problem, and it is not conservative. Examples are given in the paper.

Key words Sampled-data system, lifting technique, robust stability, small gain theorem

1 Introduction

The controller in a sampled-data system is discrete, but the input and output signals of the plant are continuous-time signals. So the lifting technique has become the first choice for sampled-data system analysis and design in recent years. As for the robust stability problem, since the perturbation of the continuous-time plant is also continuous-time, the system norm between the corresponding continuous-time signals should be considered when using the small gain theorem. The robust stability problem is also one of the two reasons for introducing the lifting technique[1]. But it is just this robust stability problem where the system norm is not preserved by the lifting. In this paper the problems and the conservativeness of the lifting technique are discussed, and a new and not conservative method is proposed.

2 Problems of lifting design

The lifting can be visualized as breaking up at each sampling time the continuous-time signal $f(t)$ into an infinite number of consecutive pieces $\hat{f}_k(t)$

$$\hat{f}_k(t) = f(\tau k + t), \quad 0 \leqslant t \leqslant \tau$$

The sequences $\{\hat{f}_k\}$ are discrete-time signals which take values in the function space $L_2[0,\tau]$. Let $\{\hat{f}_k\} \in l^2_{L_2[0,\tau]}$, *i.e.*, an $L_2[0,\tau]$-valued sequence whose norm sequence is square integrable[1],

$$\sum_{k=0}^{\infty} \|\hat{f}_k\|^2_{L_2[0,\tau]} < \infty$$

Consider a time-continuous system as follows.

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1 w(t) + B_2 u(t), \quad z(t) = C_1\boldsymbol{x}(t), \quad y(t) = C_2\boldsymbol{x}(t) \tag{1}$$

The relationship between the discrete-time state of the system and the lifted signals should be described by operator equations. Realization of the lifted system in operator form is given as

$$\hat{G} = \left[\begin{array}{c|cc} e^{A\tau} & \varPhi_b & B_{2d} \\ \hline \varPhi_c & \varPhi_{11} & \varPhi_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{2}$$

The following equation involving $\varPhi_{11}$ is given here as an example of these operator equations. Consider only the input w. The relationship between the lifted output $\{\hat{z}_k\}$ of system (1) and the lifted input $\{\hat{w}_k\}$ can be given as

$$\hat{z}_k(t) = C_1 e^{At}\boldsymbol{x}_k + C_1\int_0^t e^{A(t-s)}B_1\hat{w}_k(s)\mathrm{d}s$$

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or, in operator notation[1,2]

$$\hat{z}_k(t) = \Phi_c \boldsymbol{x}_k + \Phi_{11}\hat{w}_k \tag{3}$$

where Φ_{11} is the convolution operator, $\Phi_{11} : L_2[0,\tau] \to L_2[0,\tau]$, and Φ_c is the state transition operator, $\Phi_c : \mathbb{R}^x \to L_2[0,\tau]$. (The notation $\mathbb{R}^x$ means x is the dimension of the signal x).

Φ_b and Φ_{12} in (2) are also the related operators, and B_{2d} maps $u(k)$ to the discrete-time state x_k, where $u(k)$ is produced by the zero-order hold. It is the B_2 matrix that is formed from the discretization with hold, and is written with a subscript d.

In control problems, (1) represents the generalized plant, the second input of the generalized plant u_k and the second (sampled) output y_k are connected with the discrete controller $K(Z)$. Let $K(z) = C_k(zI - A_k)^{-1}B_k$; then the corresponding closed-loop system formed from $\hat{G}$ and K is

$$F_l(\hat{G}, K) = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_d & B_{2d}C_k & \Phi_b \\ B_kC_2 & A_k & 0 \\ \hline \Phi_c & \Phi_{12}C_k & 0 \end{array}\right] \tag{4}$$

where A_{cl} is a matrix, $A_d = e^{A\tau}$ is the discretized state matrix of the plant, and B_{cl} and C_{cl} are operators.

Notice that in forming (4), the operator Φ_{11} is assumed to be zero. It is true for the robust stability problem (see later).

(4) shows that in the lifted system, the operator Φ_b maps the lifted input signal $\{\hat{w}_k\}$ to the discrete-time state $\boldsymbol{x}_g(k\tau)$ of the plant $\hat{G}$. This discrete-time state and the discrete-time output of the controller u_k are mapped to the lifted output $\{\hat{z}_k\}$ by operators Φ_c and Φ_{12}, respectively.

Let the system operator be $\Sigma : L_2[0,\infty) \to L_2[0,\infty)$, and its lifted be $\hat{\Sigma} : l^2_{L_2[0,\tau]} \to l^2_{L_2[0,\tau]}$. It is proved[1] that the system norm is preserved after lifting, *i.e.*, $\|\Sigma\| = \|\hat{\Sigma}\|$.

The last step of the lifting design is to transform the operator realization $\hat{G}$ into a matrix one G_{dd}, which is also called the H_∞ discretization (it means that the H_∞ norm is preserved)[3]. Then the general method for discrete-time system can be used to design the system.

Now consider the robust stability problem of Fig. 1, where P is the plant, K is the discrete controller, H is the zero-order hold, S is the sampler, F is the antialiasing filter, and W is the weighting function of the multiplicative uncertainty.

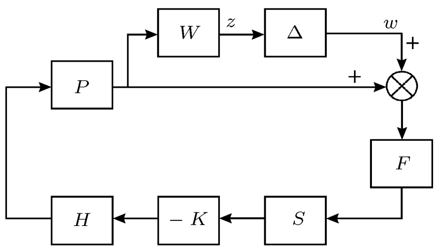

Fig. 1 Robust stability problem

The state space realization of the generalized plant (continuous-time system) in Fig. 1 is given by

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right] \tag{5}$$

where

$$A = \begin{bmatrix} A_f & B_fC_p & 0 \\ 0 & A_p & 0 \\ 0 & B_wC_p & A_w \end{bmatrix}, \ B_1 = \begin{bmatrix} B_f \\ 0 \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ B_p \\ 0 \end{bmatrix}$$

$$C_1 = [0 \quad D_wC_p \quad C_w], \quad C_2 = [C_f \quad 0 \quad 0]$$

where the subscript shows the corresponding part in Fig. 1.

Now, lift the plant G to get $\hat{G}$. Note that in the robust stability problem, $B_1^{\mathrm{T}} = [B_f^{\mathrm{T}} \quad 0 \quad 0]$, $C_1 = [0 \quad D_w C_p \quad C_w]$, and the corresponding zero blocks in matrix A all make the operator $\Phi_{11} = 0$ [see (3)]. And the closed-loop system corresponding to (4) is:

$$F_l(\hat{G}, K) = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & 0 \end{array}\right] = \left[\begin{array}{c:ccc|c} A_{fd} & \multicolumn{3}{c|}{\begin{pmatrix} B_f C_p & 0 & 0 \\ A_p & 0 & B_p C_k \\ B_w C_p & A_w & 0 \end{pmatrix}_d} & \begin{matrix} \Phi_{bf} \\ 0 \\ 0 \end{matrix} \\ \hdashline B_k C_f & 0 & 0 & A_k & 0 \\ \hline 0 & 0 & \Phi_w & \Phi_{12} C_k & 0 \end{array}\right] \tag{6}$$

where A_{cl} is a matrix, *i.e.*, the state matrix of the closed-loop system, B_{cl} and C_{cl} are operators. The block subscripted with d stands for the discretization of the overall continuous-time part from the output of the discrete controller (through C_k) to the plant P connecting with the weight W and the filter F. According to (4), the state matrix A_{cl} can be divided into four blocks with dashed lines, where block (1,1) including A_{fd} corresponds to the plant A_d of (4). Notice that the operator Φ_c in (4) equals zero in the robust stability problem (6). This Φ_c is just the operator mapping the discrete-time states of the plant A_d to the lifted output $\{\hat{z}_k\}$, and is an important ingredient of the output of a lifted system, but it is absent now. Because of this, norm equivalence $\|\Sigma\| = \|\hat{\Sigma}\|$ will not hold. This default problem usually may not catch any attention, because the lifting computation deals mostly with matrix exponentials[1,4]. If there are some zero blocks in the matrix, the process of computation can still continue smoothly. But because the norm equivalence does not hold, the resulting norm of the system will not be correct. The reason leading to this problem is that in the robust stability problem, the states of the plant P and the weight W in the generalized plant are not controllable from the input w, and the states of the filter F are not observable from the output z [see the (A, B_1, C_1) in (5)].

Example 1. Consider the system of Fig. 1. Let

$$P(s) = \frac{2-s}{(s+2)(10s+1)}, \ F(s) = \frac{3.14}{s+31.4}, \ W(s) = \frac{2.895(s+0.1)}{(0.1s+1)}, \ K(z) = -\Big(2.5 + \frac{0.5\tau}{z-1}\Big)$$

where τ is the sampling period, $\tau = 0.1$.

The lifted generalized plant G_{dd} can be found by using the algorithms given in [4]. The resulting H_∞-norm of the system formed from this G_{dd} and $K(z)$ is 1.3214. It is also the L_2-induced norm of this sampled-data system from w to z.

The bandwidth in this example is $\omega_b \approx 0.314$rad/sec, and the corresponding period is $T = 2\pi/\omega_b =$ 20sec. Here we take such a narrow bandwidth on purpose to make the sampled-data system close to a continuous-time one. And the H_∞-norm of the continuous-time system can then be used to check the correctness of the lifting technique. In fact the continuous-time counterpart of $K(z)$ is $K(s) = -(2.5 + 0.5/s)$, and the H_∞-norm of the corresponding continuous-time system $\|T_{zw}\|_\infty = 0.9993$. It indicates that the L_2-induced norm given by lifting quite differs from the real value.

3 Discretized analysis of the robust stability

Here a discretized method is proposed, which is not conservative for robust stability analysis. The essential of the method is to replace the uncertainty Δ with a discrete uncertainty Δ_d plus ZOH (zero-order hold), as indicated by the dashed line in Fig. 2. Two samplers are used here to emphasize the discrete nature of the uncertainty:

$$\Delta_d = \{\Delta_k\}_{k=0}^{\infty}, \quad \bar{\sigma}(\Delta_k) \leqslant 1 \tag{7}$$

If we extract the gain

$$|H(j\omega)| = \left|\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right| \tag{8}$$

from the ZOH and add it to the weighting function W, then the uncertainty enclosed by the dashed line satisfies the same norm bounded condition $\|\Delta\|_\infty \leqslant 1$ as in the original system (Fig. 1), so we can use Δ_d plus ZOH instead of the original Δ in the robust stability analysis.

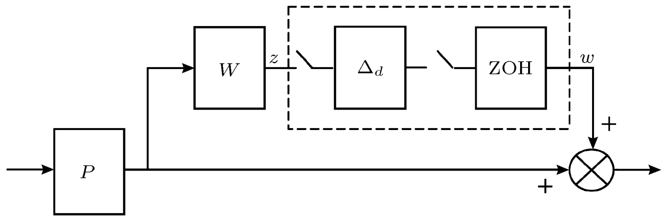

Fig. 2 Discretized uncertainty

Because for the sampled-data system only the frequencies of primary band, *i.e.*, $\omega \in [-\omega_s/2, \omega_s/2]$ need to be considered, adding ZOH gain to the weight W has only a little effect on it.

As an example, suppose that the unmodeled dynamics of the plant is

$$U(s) = \frac{1}{(1 + T_u s/3)^3} \tag{9}$$

and also suppose that the time constant is known only to the extent that it lies in the interval $0 \leqslant T_u \leqslant 0.1$. The unmodeled dynamics can be treated as a multiplicative perturbation as in Fig. 1, where the weighting function W satisfies[5]

$$|W(j\omega)| \geqslant |U(j\omega) - 1| \tag{10}$$

Let $T_u = 0.1$sec (the worst case). Then we can find

$$W(s) = \frac{24(s + 0.24)}{(s + 240)} \tag{11}$$

Fig. 3 shows the Bode magnitude plots of the weight $W(j\omega)$(solid) and $|U(j\omega) - 1|$(dashed) with $T_u = 0.1$. The $|W(j\omega)|$ is really the upper bound of the multiplicative uncertainty. The dotted curve is the weight W multiplied by the ZOH gain (8). It can be seen that addition of ZOH gain has little effect on the weight, the latter is still the bounding function of the uncertainty.

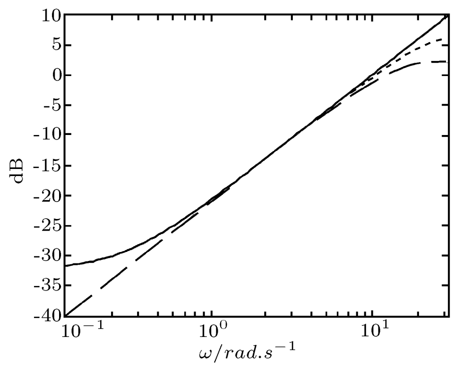

Fig. 3 The weighting function of the uncertainty

Hence, we can use the discrete uncertainty Δ_d instead of the continuous-time Δ as in Fig. 2, and the input signal to the system now is produced by the hold, which is

$$w(k\tau + t) = w(k\tau), \quad 0 \leqslant t \leqslant \tau$$

Notice that the output signal in the robust stability analysis is also a sampled signal, $z(k\tau)$. Thus the robust stability analysis of the sampled-data system can be carried out by the general discretization method with the well-known sufficient and necessary condition—the small gain theorem.

It should be noted that the controllability and observability performance must not be lost during discretization. The sufficient and necessary condition to ensure this is that the distinct eigenvalues s_i, s_j of the continuous system must lead to distinct eigenvalues $\exp s_i\tau$, $\exp s_j\tau$[7], *i.e.*,

$$\exp s_i\tau \neq \exp s_j\tau, \quad \text{for } s_i \neq s_j \tag{12}$$

where τ is the sampling period. For a pair of complex eigenvalues $s_{1,2} = \sigma_1 \pm j\omega_1$, if $\tau = q\pi/\omega_1$, or when

$$\omega_s/2 = \omega_1/q, \quad q = 1, 2, 3, \cdots$$

condition (12) will be violated, where $\omega_s = 2\pi/\tau$. But in fact ω_s is always larger than the eigenvalues ω_i of the system, so condition (12) can generally be satisfied. Of course, if there really exists a high frequency resonance mode, then condition (12) must be checked.

According to the small gain theorem, the sufficient and necessary condition for robust stability of the discrete-time system is as follows[6].

$$\|T_{zw}\|_\infty = \sup_{0\leqslant\theta\leqslant 2\pi} \sigma_{\max}[T_{zw}(e^{j\theta})] < 1 \tag{13}$$

Here, $T_{zw}(z)$ is the discrete transfer function from w to z, see Fig. 1 and Fig. 2. $T_{zw}(z)$ can be obtained by using classical methods, then the robust stability of the system can be analyzed according to (13). This method is simple and easy, and the result is not conservative.

Example 2. Let

$$P(s) = \frac{24(48-s)}{(s+48)(10s+24)},\ F(s) = \frac{31.4}{s+31.4},\ K(z) = -\Big(1.852 + \frac{8.889\tau}{z-1}\Big),\ \tau = 0.1\text{sec}$$

and let the weighting function be the same as given by (11), *i.e.*,

$$W(s) = \frac{24(s+0.24)}{(s+240)}$$

Now the bandwidth of the system in this example is $\omega_b \approx 6.28$rad/sec. The discrete-time frequency response $T_{zw}(e^{j\omega\tau})$ can be obtained by the common method (Fig.4), and its maximum magnitude is the H_∞ norm, $\|T_{zw}\|_\infty = 0.9949$.

Consider now a worst-case perturbation, $T_u = 0.1$. Fig. 3 shows that this perturbation is close to the upper bound $|W|$. So according to the small gain theorem [see (13)], the system should still be stable under this perturbation, but on the verge of instability.

Fig. 5 is the simulation result of the sampled-data system under this worst-case perturbation. This transient behavior coincides with the above judgement and shows clearly that the result based on the norm obtained by the discretized method is not conservative.

Notice that if the norm of this example is obtained by lifting, it is 1.3244, far beyond the small gain condition, but the perturbed system is still stable.

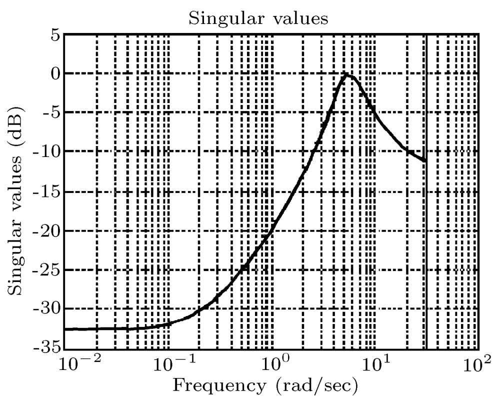

Fig. 5 The Bode plot of example 2

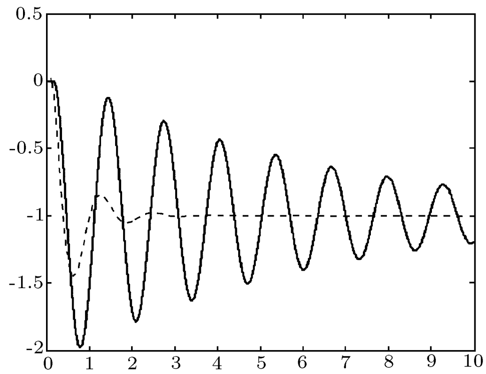

Fig. 6 The step response of the perturbed system (solid) and the nominal system (dashed)

4 Conclusions

1) The lifting technique is generally accepted as a unique tool for sampled-data systems, but in the robust stability problem, the state transition operator is zero, and the norm of the lifted system can not remain unchanged. Generally speaking, the lifting result is conservative under the small gain condition.

2) If an equivalent discrete uncertainty is used instead of the continuous-time uncertainty, then the sampled-data system can be treated as a common discretized system, and the result will not be conservative.

3) The treatment of the robust stability problem proposed in this paper can further be expanded to the mixed sensitivity problem in the H_∞ design of the sampled-data system.

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