State Feedback Stabilization of Stochastic Feedforward Nonlinear Systems with Input Time-delay

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Abstract In this paper, the problem of state feedback stabilization for stochastic feedforward nonlinear systems with input time-delay is considered for the first time. By introducing a variable transformation, skillfully combining the homogeneous domination method, and constructing an appropriate Lyapunov-Krasovskii functional, a state feedback controller is developed to guarantee the closed-loop system globally asymptotically stable in probability.

Key words Stochastic feedforward systems, input time-delay, Lyapunov-Krasovskii functional, state feedback stabilization

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Since the stochastic stability theory was established and improved by [1−3] and other references, in recent years, the study of stochastic lower-triangular/upper-triangular nonlinear systems without time-delay based on backstepping design method has achieved remarkable development.

In the study of stochastic nonlinear time-delay systems based on backstepping method, [4] considered the problem of the fourth-moment exponential output feedback stabilization. Reference [5] laid the theoretical basis for controller design and stability analysis of stochastic nonlinear time-delay systems. For stochastic nonlinear high-order time-delay systems, [6] studied the output-feedback stabilization problem. In [7−8], by introducing the homogeneous domination method first proposed by [9] to stochastic systems, the authors further discussed this problem using conditions on nonlinear terms that are weaker than those in [6].

To our knowledge, for the study of controller design based on backstepping method for stochastic feedforward time-delay systems, [10] was the first paper. Subsequently, [11] improved the result in [10] by relaxing the system order and assumptions on nonlinearities and considered more general stochastic feedforward time-delay systems.

However, all of the aforementioned results only consider stochastic nonlinear systems with time-delay in the nonlinear terms $f_i(\cdot)$ and $g_i(\cdot)$, to our knowledge, there is no result until now for stochastic feedforward nonlinear systems with time-delay in control input. Since input time-delay widely exists in sensors, calculation, information processing or transport[12], etc., how to stabilize stochastic nonlinear systems with input time-delay is a very meaningful

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research problem.

In this paper, we will consider the afore mentioned problem for a class of stochastic feedforward nonlinear systems with input time-delay. By introducing a variable transformation, skillfully combining with the homogeneous domination method, and constructing an appropriate Lyapunov-Krasovskii functional, a state feedback controller is constructed to drive the closed-loop system globally asymptotically stable in probability.

1 Mathematical preliminaries

The following notations, definitions and lemmas are to be used throughout the paper.

For a given vector or matrix X, $\mathrm{tr}\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of vector X. $\mathcal{C}([-d,0];\mathbf{R}^n)$ denotes the space of continuous $\mathbf{R}^n$-value functions on $[-d,0]$; $\mathcal{C}^b_{\mathcal{F}_0}([-d,0];\mathbf{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-d,0];\mathbf{R}^n)$-valued random variables $\boldsymbol{\xi}=\{\boldsymbol{\xi}(\theta):-d\le\theta\le 0\}$. $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathcal{C}^{2,1}(\mathbf{R}^n\times[-d,\infty);\mathbf{R}_+)$ denotes the family of all nonnegative functions $V(\boldsymbol{x},t)$ on $\mathbf{R}^n\times[-d,\infty)$ which are $\mathcal{C}^2$ in $\boldsymbol{x}$ and $\mathcal{C}^1$ in t. Sometimes, we denote $\chi(s)$ as χ to simplify the procedure, where χ and s represent some variables.

Consider the following stochastic time-delay system

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t)=\,&\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-d(t)))\mathrm{d}t+\\&g^{\mathrm{T}}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-d(t)))\mathrm{d}\boldsymbol{\omega}(t),\quad\forall t\ge 0\end{aligned}\tag{1}$$

with initial data $\{\boldsymbol{x}(\theta):-d\le\theta\le 0\}=\boldsymbol{\xi}\in\mathcal{C}^b_{\mathcal{F}_0}([-d,0];\mathbf{R}^n)$, where $d(t):\mathbf{R}_+\to[0,d]$ is a Borel measurable function, $\boldsymbol{\omega}(t)$ is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega,\mathcal{F},P)$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and P being a probability measure. $\boldsymbol{f}:\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ and $g:\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^{m\times n}$ are locally Lipschitz with $\boldsymbol{f}(t,0,0)\equiv 0$ and $g(t,0,0)\equiv 0$.

Definition 1[13]**.** For any given $V(\boldsymbol{x}(t),t)\in\mathcal{C}^{2,1}$ associated with system (1), the differential operator $\mathcal{L}$ is defined as $\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial\boldsymbol{x}}\boldsymbol{f}+\frac{1}{2}\mathrm{tr}\{g\frac{\partial^2V}{\partial\boldsymbol{x}^2}g^{\mathrm{T}}\}$, where $\frac{1}{2}\mathrm{tr}\{g\frac{\partial^2V}{\partial\boldsymbol{x}^2}g^{\mathrm{T}}\}$ is called the Hessian term of $\mathcal{L}$.

Lemma 1[13]**.** For system (1), if there exist a function $V(\boldsymbol{x}(t),t)\in\mathcal{C}^{2,1}(\mathbf{R}^n\times[-d,\infty);\mathbf{R}_+)$, two class $\mathcal{K}_\infty$ functions α_1, α_2 and a class $\mathcal{K}$ function α_3 such that $\alpha_1(|\boldsymbol{x}(t)|)\le V(\boldsymbol{x}(t),t)\le\alpha_2\left(\sup_{-d\le s\le 0}|\boldsymbol{x}(t+s)|\right)$ and $\mathcal{L}V(\boldsymbol{x}(t),t)\le-\alpha_3(|\boldsymbol{x}(t)|)$, then there exists a unique solution on $[-d,\infty)$ for (1), and the equilibrium $\boldsymbol{x}(t)=0$ is globally asymptotically stable in probability.

Lemma 2. Given real variables x,y and positive real numbers a,m,n, there exists constant $b>0$ such that $ax^my^n\le b|x|^{m+n}+\frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}}b^{-\frac{m}{n}}a^{\frac{m+n}{n}}|y|^{m+n}$.

Proof. Lemma 2 can be easily proved by Young′s inequality. □

2 Design of state feedback controller

2.1 Problem formulation

Consider the following stochastic nonlinear systems with input time-delay:

$$\begin{aligned}\mathrm{d}x_i(t)=\,&x_{i+1}(t)\mathrm{d}t+f_i(t,\boldsymbol{x}(t),u(t-d))\mathrm{d}t+\\&\boldsymbol{g}_i^{\mathrm{T}}(t,\boldsymbol{x}(t),u(t-d))\mathrm{d}\boldsymbol{\omega}(t),\ i=1,\cdots,n-1\\\mathrm{d}x_n(t)=\,&u(t-d)\mathrm{d}t\end{aligned}\tag{2}$$

where $\boldsymbol{x}(t)=(x_1(t),\cdots,x_n(t))^{\mathrm{T}}\in\mathbf{R}^n$ and $u(t)\in\mathbf{R}$ are system state and control input, respectively, constant d is time-delay. $\boldsymbol{\omega}(t)$ is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega,\mathcal{F},P)$. The nonlinear functions $f_i:\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ and $\boldsymbol{g}_i:\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^m$, $i=1,\cdots,n-1$, are assumed to be $\mathcal{C}^1$ with $f_i(t,0,0)=0$ and $\boldsymbol{g}_i(t,0,0)=0$.

The purpose of this paper is to design a state feedback controller for system (2) under the following assumption such that the closed-loop system is globally asymptotically stable in probability.

Assumption 1. For $1\le i\le n-1$, there exist positive constants a_1 and a_2 such that

$$\begin{aligned}|f_i|&\le a_1\left(|x_{i+2}(t)|+\cdots+|x_n(t)|+|u(t-d)|\right)\\|\boldsymbol{g}_i|&\le a_2\left(|x_{i+2}(t)|+\cdots+|x_n(t)|+|u(t-d)|\right)\end{aligned}$$

Remark 1. Obviously, system (2) satisfying Assumption 1 is a stochastic feedforward nonlinear system. As we discussed in [10] and [11], Assumption 1 is a frequently-used condition. □

2.2 State feedback controller design

Part 1). Change of coordinates

Motivated by [14] ∼ [16], we introduce a variable transformation

$$\tilde{x}_n(t)=x_n(t)+\int_{t-d}^{t}u(s)\mathrm{d}s\tag{3}$$

and a set of coordinate transformations

$$\eta_i=\frac{x_i}{L^{i-1}},\ \eta_n=\frac{\tilde{x}_n}{L^{n-1}},\ v=\frac{u}{L^n},\ i=1,\cdots,n-1\tag{4}$$

where $0<L<1$ is a gain to be determined. By (3) and (4), system (2) can be reinterpreted as

$$\begin{aligned}\mathrm{d}\eta_i(t)=\,&L\eta_{i+1}(t)\mathrm{d}t+\tilde{f}_i(t,\boldsymbol{\eta}(t),v(t-d))\mathrm{d}t+\\&\tilde{\boldsymbol{g}}_i^{\mathrm{T}}(t,\boldsymbol{\eta}(t),v(t-d))\mathrm{d}\boldsymbol{\omega}(t),\ i=1,\cdots,n-1\\\mathrm{d}\eta_n(t)=\,&Lv(t)\mathrm{d}t\end{aligned}\tag{5}$$

where $\tilde{f}_i=\frac{1}{L^{i-1}}f_i(t,x_1(t),\cdots,x_{n-1}(t),\tilde{x}_n-\int_{t-d}^{t}u(s)\mathrm{d}s,u(t-d))$, $i=1,\cdots,n-2$, $\tilde{f}_{n-1}=\frac{1}{L^{n-2}}f_{n-1}(t,x_1(t),\cdots,x_{n-1}(t),\tilde{x}_n-\int_{t-d}^{t}u(s)\mathrm{d}s,u(t-d))-\int_{t-d}^{t}u(s)\mathrm{d}s$, $\tilde{\boldsymbol{g}}_i=\frac{1}{L^{i-1}}\boldsymbol{g}_i(t,x_1(t),\cdots,x_{n-1}(t),\tilde{x}_n-\int_{t-d}^{t}u(s)\mathrm{d}s,u(t-d))$, $i=1,\cdots,n-1$.

Part 2) State feedback controller design of system (5)

In what follows, we design the state feedback controller for system (5) by the homogeneous domination method.

Step 1. Introduce $\xi_1=\eta_1$ and choose $V_1=\frac{1}{4}\xi_1^4$. From Definition 1 and (5), it follows that $\mathcal{L}V_1=L\xi_1^3\eta_2+F_1+G_1$, where $F_1=\frac{\partial V_1}{\partial\eta_1}\tilde{f}_1$, and $G_1=\frac{1}{2}\mathrm{tr}\{\tilde{\boldsymbol{g}}_1\frac{\partial^2V_1}{\partial\eta_1^2}\tilde{\boldsymbol{g}}_1^{\mathrm{T}}\}$. The virtual controller $\eta_2^*=-\lambda_1\xi_1$, $\lambda_1=c_{11}>0$ leads to

$$\mathcal{L}V_1\le-Lc_{11}\xi_1^4+L\xi_1^3(\eta_2-\eta_2^*)+F_1+G_1\tag{6}$$

***Step i* (*i*=2, ···, *n*).** We present this step by the following proposition.

Proposition 1. Suppose that at step $i-1$, there exist a $\mathcal{C}^2$, positive definite and proper Lyapunov function $V_{i-1}=\frac{1}{4}\sum_{j=1}^{i-1}\xi_j^4$ and a series of virtual controllers $\eta_1^*,\cdots,\eta_i^*$:

$$\eta_1^*=0,\ \eta_j^*=-\lambda_{j-1}\xi_{j-1},\ \xi_{j-1}=\eta_{j-1}-\eta_{j-1}^*\tag{7}$$

with $j=2,\cdots,i$, such that

$$\mathcal{L}V_{i-1}\le -L\sum_{j=1}^{i-1}c_{i-1,j}\xi_j^4+L\xi_{i-1}^3(\eta_i-\eta_i^*)+F_{i-1}+G_{i-1} \tag{8}$$

where λ_j, $c_{i-1,j}$, $j=1,\cdots,i-1$, are positive constants, $F_{i-1}=\sum_{j=1}^{i-1}\frac{\partial V_{i-1}}{\partial\eta_j}\tilde{f}_j$, $G_{i-1}=\sum_{p,q=1}^{i-1}\frac{1}{2}\mathrm{tr}\{\tilde{\boldsymbol{g}}_p\frac{\partial^2V_{i-1}}{\partial\eta_p\partial\eta_q}\tilde{\boldsymbol{g}}_q^{\mathrm{T}}\}$. Then the ith Lyapunov function

$$V_i=V_{i-1}+U_i,\quad U_i=\frac{1}{4}\xi_i^4 \tag{9}$$

is $\mathcal{C}^2$, positive definite and proper, and there is $\eta_{i+1}^*=-\lambda_i\xi_i$ such that

$$\mathcal{L}V_i\le -L\sum_{j=1}^{i}c_{ij}\xi_j^4+L\xi_i^3(\eta_{i+1}-\eta_{i+1}^*)+F_i+G_i \tag{10}$$

with $F_i=\sum_{j=1}^{i}\frac{\partial V_i}{\partial\eta_j}\tilde{f}_j$ and $G_i=\sum_{p,q=1}^{i}\frac{1}{2}\mathrm{tr}\{\tilde{\boldsymbol{g}}_p\frac{\partial^2V_i}{\partial\eta_p\partial\eta_q}\tilde{\boldsymbol{g}}_q^{\mathrm{T}}\}$.

Proof. See Appendix. □

Hence, at step n, by choosing $V_n=\frac{1}{4}\sum_{i=1}^{n}\xi_i^4$, there exists a control law

$$v=\eta_{n+1}^*=-\lambda_n\xi_n=-(\bar{\lambda}_n\eta_n+\cdots+\bar{\lambda}_2\eta_2+\bar{\lambda}_1\eta_1) \tag{11}$$

such that

$$\mathcal{L}V_n\le -L\sum_{i=1}^{n}c_{ni}\xi_i^4+F_n+G_n \tag{12}$$

where $F_n=\sum_{j=1}^{n-1}\frac{\partial V_n}{\partial\eta_j}\tilde{f}_j$, $G_n=\sum_{p,q=1}^{n-1}\frac{1}{2}\mathrm{tr}\{\tilde{\boldsymbol{g}}_p\frac{\partial^2V_n}{\partial\eta_p\partial\eta_q}\tilde{\boldsymbol{g}}_q^{\mathrm{T}}\}$, $\bar{\lambda}_i=\lambda_n\cdots\lambda_i$, c_{ni}, $i=1,\cdots,n$, are positive constants. Next, we estimate F_n+G_n on the right-hand side of (12).

Proposition 2. There exist positive constants a_{i1}, b_{i1}, $\tilde{a}_{n1}$, $\tilde{b}_{n1}$, $\hat{a}_{n1}$ and $\hat{b}_{n1}$, such that

$$|F_n+G_n|\le L^2\sum_{i=1}^{n}(a_{i1}+b_{i1})\xi_i^4+L^2(\tilde{a}_{n1}+\tilde{b}_{n1})\xi_n^4(t-d)+ L^2(\hat{a}_{n1}+\hat{b}_{n1})\int_{t-d}^{t}\xi_n^4(s)\mathrm{d}s$$

Proof. See Appendix. □

Substituting Proposition 2 into (12) yields

$$\mathcal{L}V_n\le -L\sum_{i=1}^{n}(c_{ni}-(a_{i1}+b_{i1})L)\xi_i^4+L^2(\tilde{a}_{n1}+\tilde{b}_{n1})\times \xi_n^4(t-d)+L^2(\hat{a}_{n1}+\hat{b}_{n1})\int_{t-d}^{t}\xi_n^4(s)\mathrm{d}s \tag{13}$$

Construct the Lyapunov-Krasovskii functional:

$$V(\boldsymbol{\eta}(t))=V_n(\boldsymbol{\eta}(t))+L^2(\tilde{a}_{n1}+\tilde{b}_{n1})\int_{t-d}^{t}\xi_n^4(s)\mathrm{d}s+ L^2(\hat{a}_{n1}+\hat{b}_{n1})\int_{-d}^{0}\int_{\theta+t}^{t}\xi_n^4(s)\mathrm{d}s\mathrm{d}\theta \tag{14}$$

which together with (13) yields

$$\mathcal{L}V\le -L\sum_{i=1}^{n-1}(c_{ni}-L(a_{i1}+b_{i1}))\xi_i^4-L(c_{nn}-L(a_{n1}+ b_{n1}+\tilde{a}_{n1}+\tilde{b}_{n1}+\hat{a}_{n1}d+\hat{b}_{n1}d))\xi_n^4(t) \tag{15}$$

Defining $L^*=\min_{1\le i\le n-1}\{\frac{c_{nn}}{a_{n1}+b_{n1}+\tilde{a}_{n1}+\tilde{b}_{n1}+(\hat{a}_{n1}+\hat{b}_{n1})d},1,\frac{c_{ni}}{a_{i1}+b_{i1}},\}$ and choosing $0<L<L^*$, one has

$$\mathcal{L}V(\boldsymbol{\eta}(t))\le -\mu\sum_{i=1}^{n}\xi_i^4(t) \tag{16}$$

where $\mu>0$ is a constant.

Part 3) State feedback controller design of system (2).

From (3) and (4), it follows that the state feedback controller of system (2) is

$$u(t)=-L^n\bar{\lambda}_1x_1(t)-L^{n-1}\bar{\lambda}_2x_2(t)-\cdots-L\bar{\lambda}_nx_n(t)- L\bar{\lambda}_n\int_{t-d}^{t}u(s)\mathrm{d}s \tag{17}$$

3 Stability analysis

We now present the main result of this paper.

Theorem 1. If Assumption 1 holds for system (2), then under the state feedback controller (17), the closed-loop system has a unique solution on $[-d,\infty)$, and the equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability.

Proof. Considering the Lyapunov-Krasovskii functional V given in (14), it is obvious that V is $\mathcal{C}^2$. Since V_n is $\mathcal{C}^2$, positive definite and proper, by Lemma 4.3 in [17], there exist $\mathcal{K}_\infty$ functions α_1, α_{21} such that

$$\alpha_1(|\boldsymbol{\eta}(t)|)\le V_n(\boldsymbol{\eta}(t))\le\alpha_{21}(|\boldsymbol{\eta}(t)|) \tag{18}$$

By the first integral mean value theorem, one gets

$$\begin{aligned}&L^2(\tilde{a}_{n1}+\tilde{b}_{n1})\int_{t-d}^{t}\xi_n^4(s)\mathrm{d}s+L^2\int_{-d}^{0}\int_{\theta+t}^{t}\xi_n^4(s)\mathrm{d}s\mathrm{d}\theta\times\\ &\quad(\hat{a}_{n1}+\hat{b}_{n1})\le\\ &\quad L^2(\tilde{a}_{n1}+\tilde{b}_{n1}+\hat{a}_{n1}d+\hat{b}_{n1}d)\int_{t-d}^{t}\xi_n^4(s)\mathrm{d}s\le\\ &\quad c_{01}\int_{t-d}^{t}\alpha_{22}(|\boldsymbol{\eta}(\sigma)|)\mathrm{d}\sigma\overset{\sigma=s+t}{=}\\ &\quad c_{01}\int_{-d}^{0}\alpha_{22}(|\boldsymbol{\eta}(s+t)|)\mathrm{d}(s+t)\le\\ &\quad c_{02}\sup_{-d\le s\le 0}\alpha_{22}(|\boldsymbol{\eta}(s+t)|)\le\\ &\quad\bar{\alpha}_{22}\left(\sup_{-d\le s\le 0}|\boldsymbol{\eta}(s+t)|\right)\end{aligned} \tag{19}$$

where c_{01} and c_{02} are positive constants, α_{22} and $\bar{\alpha}_{22}$ are class $\mathcal{K}_\infty$ functions. Note that $\alpha_{21}(|\boldsymbol{\eta}(t)|)\le\alpha_{21}(\sup_{-d\le s\le 0}|\boldsymbol{\eta}(s+t)|)$. Setting $\alpha_2=\alpha_{21}+\bar{\alpha}_{22}$, by (14), (18) and (19), one gets

$$\alpha_1(|\boldsymbol{\eta}(t)|)\le V(\boldsymbol{\eta}(t))\le\alpha_2\left(\sup_{-d\le s\le 0}|\boldsymbol{\eta}(s+t)|\right) \tag{20}$$

From (16) and (18), it follows that

$$\mathcal{L}V(\boldsymbol{\eta}(t))\le -4\mu\alpha_1(|\boldsymbol{\eta}(t)|) \tag{21}$$

By (20) and (21), the conditions of Lemma 1 are satisfied. Then, the closed-loop system (5) and (11) has a unique solution on $[-d,\infty)$, and $\boldsymbol{\eta}(t)=0$ is globally asymptotically stable in probability.

Note that (4) is an equivalent transformation, and that when $(x_1, \cdots, x_{n-1}, \tilde{x}_n)$ and u converge to zero asymptotically as $t \to \infty$, by (3), $(x_1, \cdots, x_{n-1}, x_n)$ converges to zero asymptotically as $t \to \infty$. Hence, the closed-loop system consisting of (2) and (17) has a unique solution on $[-d, \infty)$, and the equilibrium $\boldsymbol{x}(t) = 0$ is globally asymptotically stable in probability. □

Remark 2. In this paper, the homogeneous domination idea is generalized to the stochastic feedforward nonlinear systems with input time-delay for the first time. The underlying philosophy of this approach is that the state feedback controller is first constructed without dealing with the nonlinear terms, and then a scaling gain L in (4) whose value range is given in (16) is introduced to the state feedback controller to dominate the nonlinearities.

4 A simulation example

Consider the stochastic feedforward nonlinear system:

$$\begin{aligned} \mathrm{d}x_1(t) &= x_2(t)\mathrm{d}t + u(t-0.3)\mathrm{d}t + 0.2u(t-0.3)\mathrm{d}\omega(t) \\ \mathrm{d}x_2(t) &= u(t-0.3)\mathrm{d}t \end{aligned} \tag{22}$$

It is obvious that Assumption 1 holds with $a_1 = 1$ and $a_2 = 0.2$. Following the design procedure as in Section 3, the state feedback controller is designed as

$$u(t) = -L^2\bar{\lambda}_1 x_1(t) - L\bar{\lambda}_2 x_2(t) - L\bar{\lambda}_2 \int_{t-0.3}^{t} u(s)\mathrm{d}s \tag{23}$$

In the simulation, $L = 0.05$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 4.9449$, the initial values $x_1(0) = -3$, $x_2(0) = 0.3$. Fig. 1 demonstrates the effectiveness of the control scheme.

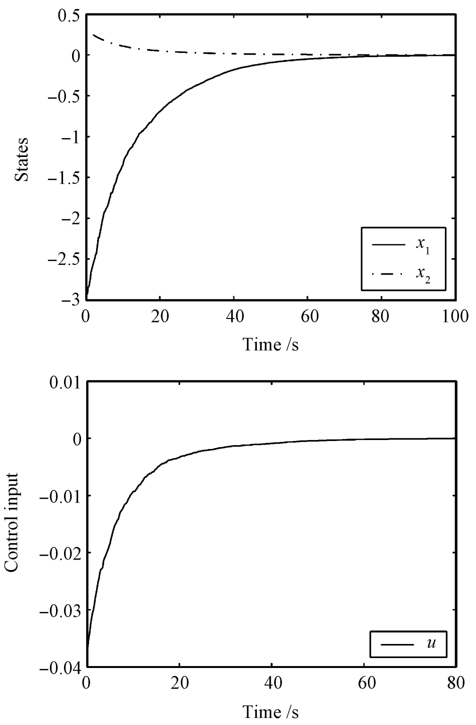

Fig. 1 The responses of closed-loop system (22) and (23)

5 Conclusions

In this paper, we make an initial attempt to construct a state feedback controller for stochastic feedforward nonlinear systems with input time-delay. Our future work is to explore more deeply the properties and control problems of stochastic feedforward nonlinear time-delay systems, as we have done on stochastic nonlinear systems [10−11].

Appendix

Proof of Proposition 1. By (9), it is easy to verify that V_i is $\mathcal{C}^2$, positive definite and proper. Next, we prove inequality (10). From (7) ~ (9), it follows that

$$\begin{aligned} \mathcal{L}V_i \le & -L\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^4 + L\xi_{i-1}^3(\eta_i - \eta_i^*) + L\xi_i^3\eta_{i+1} + \\ & \left(F_{i-1} + \sum_{j=1}^{i} \frac{\partial U_i}{\partial \eta_j}\frac{\tilde{f}_j}{L^{j-1}}\right) - L\xi_i^3\sum_{j=1}^{i-1}\frac{\partial \eta_i^*}{\partial \eta_j}\eta_{j+1} + \\ & \left(G_{i-1} + \sum_{p,q=1}^{i}\frac{1}{2}\mathrm{tr}\left\{\frac{\tilde{\boldsymbol{g}}_p}{L^{p-1}}\frac{\partial^2 U_i}{\partial\eta_p\partial\eta_q}\frac{\tilde{\boldsymbol{g}}_q^{\mathrm{T}}}{L^{q-1}}\right\}\right) \le \\ & -L\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^4 + L\xi_i^3(\eta_{i+1} - \eta_{i+1}^*) + L\xi_i^3\eta_{i+1}^* + F_i + \\ & G_i + L\xi_{i-1}^3(\eta_i - \eta_i^*) + L\xi_i^3\sum_{j=1}^{i-1}\lambda_{i-1}\cdots\lambda_j\eta_{j+1} \end{aligned} \tag{A1}$$

By (7) and Lemma 2, one obtains

$$\begin{aligned} & |\xi_{i-1}^3(\eta_i - \eta_i^*)| \le l_{i,i-1,1}\xi_{i-1}^4 + \sigma_{i1}\xi_i^4 \\ & \left|\xi_i^3\sum_{j=1}^{i-1}\lambda_{i-1}\cdots\lambda_j\eta_{j+1}\right| \le \sum_{j=1}^{i-1} l_{ij2}\xi_j^4 + \sigma_{i2}\xi_i^4 \end{aligned} \tag{A2}$$

Choosing $c_{ij} = \begin{cases} c_{i-1,j} - l_{ij2} > 0, j = 1, \cdots, i-2, \\ c_{i-1,i-1} - l_{i,i-1,1} - l_{i,i-1,2} > 0, j = i-1, \end{cases}$ $\eta_{i+1}^* = -\lambda_i\xi_i$, $\lambda_i = c_{ii} + \sigma_{i1} + \sigma_{i2}$, $c_{ii} > 0$, and substituting (A2) into (A1), one gets the result. □

Proof of Proposition 2. For $i = 1, \cdots, n-2$, by Assumption 1, (4) and $0 < L < 1$, one has

$$\begin{aligned} |\tilde{f}_i| \le & \frac{a_1}{L^{i-1}}\Bigg(L^{i+1}|\eta_{i+2}| + \cdots + L^{n-1}|\eta_n| + \\ & L^n\int_{t-d}^{t}|v(s)|\mathrm{d}s + L^n|v(t-d)|\Bigg) \le \\ & a_1L^2\left(\sum_{j=i+2}^{n}|\eta_j| + \int_{t-d}^{t}|v(s)|\mathrm{d}s + |v(t-d)|\right) \end{aligned} \tag{A3}$$

For $i = n-1$, by the definition of $\tilde{f}_{n-1}$, Assumption 1 and $0 < L < 1$, one has

$$|\tilde{f}_{n-1}| \le a_1L^2\left(|v(t-d)| + \int_{t-d}^{t}|v(s)|\mathrm{d}s\right) \tag{A4}$$

Combining (A3) and (A4), for $i = 1, \cdots, n-1$, one has

$$|\tilde{f}_i| \le a_1L^2\left(\sum_{j=i+2}^{n}|\eta_j| + \int_{t-d}^{t}|v(s)|\mathrm{d}s + |v(t-d)|\right) \tag{A5}$$

Hence, by Lemma 2, (7) and (11), one obtains

$$
\begin{aligned}
|F_n| \le\ & a_1 L^2 \sum_{i=1}^{n-1} \left| \xi_i^3 + \sum_{j=i+1}^{n} \lambda_{j-1} \cdots \lambda_i \xi_j^3 \right| \Bigg(|\lambda_n \xi_n(t-d)| + \\
& \sum_{j=i+2}^{n} |\xi_j - \lambda_{j-1}\xi_{j-1}| + \lambda_n \int_{t-d}^{t} |\xi_n(s)| \mathrm{d}s \Bigg) \le \\
& L^2 \sum_{i=1}^{n} a_{i1} \xi_i^4 + L^2 \hat{a}_{n1} \int_{t-d}^{t} \xi_n^4(s) \mathrm{d}s + \\
& L^2 \tilde{a}_{n1} \xi_n^4(t-d)
\end{aligned} \tag{A6}
$$

where a_{i1}, $\tilde{a}_{n1}$ and $\hat{a}_{n1}$, $i = 1, \cdots, n$ are positive constants.

Similar to (A5), for $i = 1, \cdots, n-1$, one has

$$
|\tilde{\boldsymbol{g}}_i| \le \tilde{a}_2 L^2 \left(\sum_{j=i+1}^{n} |\xi_j| + \int_{t-d}^{t} |\xi_n(s)| \mathrm{d}s + |\xi_n(t-d)| \right) \tag{A7}
$$

with $\tilde{a}_2$ being a positive constant. From (A7), $V_n = \sum_{i=1}^{n} U_i$ and Lemma 2, it follows that

$$
\begin{aligned}
|G_n| \le\ & \sum_{i=1}^{n} \sum_{p,q=1}^{i} \hat{b} \left| \frac{\partial^2 U_i}{\partial \eta_p \partial \eta_q} \right| |\tilde{\boldsymbol{g}}_p| |\tilde{\boldsymbol{g}}_q| \le \\
& L^2 \sum_{i=1}^{n} \sum_{p,q=1}^{i} \breve{b}_{pq} \xi_i^2 \Bigg(\sum_{j=p+1}^{n} |\xi_j| + \int_{t-d}^{t} |\xi_n(s)| \mathrm{d}s + \\
& |\xi_n(t-d)| \Bigg) \Bigg(\sum_{j=q+1}^{n} |\xi_j| + \int_{t-d}^{t} |\xi_n(s)| \mathrm{d}s + \\
& |\xi_n(t-d)| \Bigg) \le \\
& L^2 \sum_{j=1}^{n} b_{j1} \xi_j^4 + L^2 \tilde{b}_{n1} \xi_n^4(t-d) + L^2 \hat{b}_{n1} \int_{t-d}^{t} \xi_n^4(s) \mathrm{d}s
\end{aligned} \tag{A8}
$$

where $\hat{b}$, $\breve{b}_{pq}$, b_{j1}, $\tilde{b}_{n1}$ and $\hat{b}_{n1}$ are positive constants. Combining (A6) and (A8), one gets the desired result. □

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