Neural Network-based Adaptive State-feedback Control for High-order Stochastic Nonlinear Systems

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Abstract This paper focuses on investigating the issue of adaptive state-feedback control based on neural networks (NNs) for a class of high-order stochastic uncertain systems with unknown nonlinearities. By introducing the radial basis function neural network (RBFNN) approximation method, utilizing the backstepping method and choosing an approximate Lyapunov function, we construct an adaptive state-feedback controller which assures the closed-loop system to be mean square semi-global-uniformly ultimately bounded (M-SGUUB). A simulation example is shown to illustrate the effectiveness of the design scheme.

Key words High-order stochastic nonlinear systems, state-feedback control, neural networks, backstepping

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It is well known that most practical systems are complicated because of nonlinearity and the existence of uncertainties. How to reasonably deal with these nonlinear uncertainties is the major obstruction of the controller design of uncertain systems. Several useful control design methodologies such as Lyapunov approach and typical input-output methods, especially adaptive backstepping technique have obtained globally stable, output tracking results for various nonlinear systems with parameterized uncertainties. Recently, neural networks (NNs) have been frequently introduced to solve these uncertainties owing to their inherent approximation properties and have made great progress[1−5].

While for stochastic nonlinear systems case, with the help of the basic stochastic stability theory in [6−7] and the study of control problems based on the backstepping for stochastic systems in [8] and references therein, the method of combining backstepping with NN approximation has been successfully developed to guarantee stability for stochastic nonlinear systems[9]. However, the control procedures that we mentioned above are all based on the assumptions $p_i = 1$ and $d_i(\cdot) = 1$ for the following system:

$$\begin{cases} \mathrm{d}x_i = d_i(t,\boldsymbol{x},u)x_{i+1}^{p_i}\mathrm{d}t + f_i(\bar{\boldsymbol{x}}_i)\mathrm{d}t + \boldsymbol{g}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i)\mathrm{d}\boldsymbol{\omega}, \\ \qquad\qquad i = 1,\cdots,n-1 \\ \mathrm{d}x_n = d_n(t,\boldsymbol{x},u)u^{p_n}\mathrm{d}t + f_n(\boldsymbol{x})\mathrm{d}t + \boldsymbol{g}_n^{\mathrm{T}}(\boldsymbol{x})\mathrm{d}\boldsymbol{\omega} \end{cases} \tag{1}$$

where $u \in \mathbf{R}$ and $\boldsymbol{x} = (x_1,\cdots,x_n)^{\mathrm{T}} \in \mathbf{R}^n$ are the control input and the measurable state, respectively. For each $1 \le i \le n$, $\bar{\boldsymbol{x}}_i$ denote $(x_1,\cdots,x_i)^{\mathrm{T}}$, $p_i \ge 1$ are the odd integers. $d_i(t,\boldsymbol{x},u)$ is a continuous differential real-valued function of its variables, $f_i(\cdot)$ represents the unknown smooth function with $f_i(\mathbf{0}) = \mathbf{0}$, and $\boldsymbol{g}_i(\cdot)$ is a vector-valued unknown smooth function with $\boldsymbol{g}_i(\mathbf{0}) = \mathbf{0}$. When $p_i \ge 1$, system (1) is said to be high-order stochastic nonlinear system if there exists at least one $p_i > 1$ $(1 \le i \le n)$, p_i is called high-order. $\boldsymbol{\omega}$ is an m-dimensional standard Wiener process defined on a probability space $\{\Omega,\mathcal{F},P\}$, where Ω is a sample space, $\mathcal{F}$ is a σ-field, and P is the probability measure.

When $p_i > 1$, motivated by the certain results in [10−11] and the related papers, [12] firstly considered a class of high-order nonlinear systems with stochastic inverse dynamic. In the succeeding work, the state-feedback control problems and output-feedback control problems for more general systems were further studied, respectively. However, all of the above achievements were obtained through imposing strong restrictions on system nonlinearities. Then in [13−14], the control problems were solved under weaker restrictions.

Naturally, how to apply NN approximation method and backstepping scheme to design the adaptive state-feedback controller for high-order stochastic nonlinear systems without restrictions on unknown nonlinearities?

In the paper, we will solve this problem by constructing an adaptive neural network state-feedback controller for system (1).

1 Mathematical preliminaries

Notations. $\mathbf{R}^+$ represents the set of all non-negative real numbers, $\mathbf{R}^n$ is the n-dimensional Euclidean space, C^2 denotes the family of all the functions with continuous second partial derivations. $\boldsymbol{X}^{\mathrm{T}}$ denotes its transpose, when $\boldsymbol{X}$ is a given vector or matrix, $\mathrm{tr}\{\boldsymbol{X}\}$ is the trace of $\boldsymbol{X}$ where $\boldsymbol{X}$ is a square matrix, $|\boldsymbol{X}|$ is the Euclidean norm of vector $\boldsymbol{X}$, $\|\boldsymbol{X}\|$ is the two-norm of square matrix $\boldsymbol{X}$.

Lemma 1 (Young′s inequality). For $\forall(x,y) \in \mathbf{R}^2$, $xy \le \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q$ holds, where $\varepsilon > 0$, $p,q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2[14]. For real variables $\boldsymbol{x} \ge 0$ and $\boldsymbol{y} > 0$, $\boldsymbol{x} \le \boldsymbol{y} + (\frac{\boldsymbol{x}}{m})^m(\frac{m-1}{\boldsymbol{y}})^{m-1}$, where $m \ge 1$ is a real number.

Lemma 3[14]. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are real variables, then for any real numbers $m,n,b > 0$ and continuous function $a(\cdot) \ge 0$, $a(\cdot)\boldsymbol{x}^m\boldsymbol{y}^n \le b|\boldsymbol{x}|^{m+n} + \frac{n}{m+n}(\frac{m+n}{m})^{-\frac{m}{n}}a(\cdot)^{\frac{m+n}{n}}b^{-\frac{m}{n}}|\boldsymbol{y}|^{m+n}$.

Consider the stochastic system with the form:

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})\mathrm{d}t + \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x})\mathrm{d}\boldsymbol{\omega} \tag{2}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ is the system state, $\boldsymbol{\omega}$ is an m-dimensional standard Wiener process defined as (1), $\boldsymbol{f}(\cdot) \in \mathbf{R}^n \to \mathbf{R}$ and $\boldsymbol{g}(\cdot) \in \mathbf{R}^n \to \mathbf{R}^m$ are locally Lipschitz functions and satisfy $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, $\boldsymbol{g}(\mathbf{0}) = \mathbf{0}$. For any given $V(\boldsymbol{x}) \in C^2$, according to system (2), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}\Big\{\boldsymbol{g}(\boldsymbol{x})\frac{\partial^2 V(\boldsymbol{x})}{\partial \boldsymbol{x}^2}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x})\Big\} \tag{3}$$

Definition 1[5]. The solution process $\{x(t), t \ge t_0\}$ of the nonlinear stochastic system (2) with initial condition $x_0 \in S_0$ (the compact set with the origin in it) is called to be mean square semi-global-uniformly ultimately bounded

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(M-SGUUB) if for any ideal escape risk ε $(0 < \varepsilon < 1)$, it is bounded with probability $1-\varepsilon$ in some compact set $S_\varepsilon \supset S_0$, i.e., $\inf_{x_0 \in S_0} P\{\tau_{S_\varepsilon} = \infty\} \geq 1-\varepsilon$. The hitting time τ_{S_ε} is the first time the trajectory of the state variable reaches the boundary of S_ε.

Lemma 4[5]**.** For system (2), if there are a positive-definite radially unbounded, twice continuously differential Lyapunov function $V: \mathbf{R}^n \to \mathbf{R}$, constants $a_1 > 0$, $a_2 > 0$ and $r_0 > \frac{a_2}{a_1}$ such that for some $0 < \varepsilon < 1$ and $x_0 \in S_0 := \{\boldsymbol{x} \in \mathbf{R}^n | V(\boldsymbol{x}) \leq r_0\}$,

$$\mathcal{L}V(\boldsymbol{x}) \leq -a_1 V(\boldsymbol{x}) + a_2, \ \ \boldsymbol{x} \in S_\varepsilon = \{\boldsymbol{x} \in \mathbf{R}^n | V(\boldsymbol{x}) \leq \tfrac{r_0}{\varepsilon}\}$$

holds, then for $\forall\, t \in [t_0, \tau_{S_\varepsilon}]$, 1) there is a unique solution to system (2); 2) the nonlinear system is bounded with probability $1-\varepsilon$ in S_ε with initial condition $x_0 \in S_0$, i.e., the solution to the system is M-SGUUB.

In the following, RBF NN will be used to estimate the unknown nonlinear functions. For any continuous unknown function $f(\boldsymbol{x})$ over a compact set $S_{\boldsymbol{x}} \subset \mathbf{R}^q$, there is ${\boldsymbol{W}^*}^{\mathrm{T}} \boldsymbol{S}(\boldsymbol{x})$ so that for an ideal level of accuracy ε,

$$f(\boldsymbol{x}) = {\boldsymbol{W}^*}^{\mathrm{T}} \boldsymbol{S}(\boldsymbol{x}) + \delta(\boldsymbol{x}), \ |\delta(\boldsymbol{x})| \leq \varepsilon \tag{4}$$

where $\delta(\boldsymbol{x})$ is the approximation error, the known function vector is $\boldsymbol{S}(\boldsymbol{x}) = [s_1(\boldsymbol{x}), \cdots, s_N(\boldsymbol{x})]^{\mathrm{T}}$ with $N > 1$ being the number of RBF NN nodes. Function $s_i(\boldsymbol{x})$, $1 \leq i \leq N$ is chosen as the commonly utilized Gaussian function as $s_i(\boldsymbol{x}) = \exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{b}_i)^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{b}_i)}{\varsigma^2}\right]$, where ς is the width of the above Gaussian function, $\boldsymbol{b}_i = [b_{i1}, \cdots, b_{in}]^{\mathrm{T}}$ is the center of the receptive field and $\boldsymbol{W}^*$ is the ideal constant weight vector and given by $\boldsymbol{W}^* = \arg\min_{\boldsymbol{W} \in \mathbf{R}^N}\{\sup_{\boldsymbol{x} \in \boldsymbol{S_x}} |f(\boldsymbol{x}) - \boldsymbol{W}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{x})|\}$, where arg min is the value of variable W when the objective function $\sup_{\boldsymbol{x} \in \boldsymbol{S_x}} |f(\boldsymbol{x}) - \boldsymbol{W}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{x})|$ is minimum, $\boldsymbol{W} = [w_1, \cdots, w_N]^{\mathrm{T}}$ is a weight vector.

2 Controller design

To obtain the control goal, we impose the following assumption on system (1).

Assumption 1. For $i = 1, \cdots, n$, there are known real numbers $\lambda_i, \mu_i > 0$ such that $0 < \lambda_i \leq d_i(t, \boldsymbol{x}, u) \leq \mu_i$.

To simplify the design process, we define a constant $p = \max_{i=1,\cdots,n}\{p_i\}$ and

$$\theta = \max\{N_{ij}|\boldsymbol{W}_{ij}^*|^2, i = 1, \cdots, n, j = 1, 2\} \tag{5}$$

where N_{ij} is the number of RBFNN nodes, and $\boldsymbol{W}_{ij}^*$ is the ideal constant weight vector.

Firstly, we introduce the following coordinate change:

$$z_1 = x_1, \ z_i = x_i - \alpha_i(\bar{\boldsymbol{x}}_{i-1}, \hat{\theta}), \ i = 2, \cdots, n \tag{6}$$

where $\hat{\theta}$ is the estimation of θ, and $\alpha_i(\bar{\boldsymbol{x}}_{i-1}, \hat{\theta})$ is the virtual control law to be designed later.

Step 1. Using (1), (3) and (6) and choosing $V_1(z_1, \hat{\theta}) = \frac{k_1}{p-p_1+4} z_1^{p-p_1+4} + \frac{1}{2\Gamma}\tilde{\theta}^2$, it follows that

$$\mathcal{L}V_1 = k_1 z_1^{p-p_1+3}(d_1(t, \boldsymbol{x}, u)x_2^{p_1} + F_1(x_1)) + \frac{p-p_1+3}{2} \times$$
$$k_1 z_1^{p-p_1+2} \mathrm{tr}\{\boldsymbol{G}_1(x_1)\boldsymbol{G}_1^{\mathrm{T}}(x_1)\} - \frac{\tilde{\theta}}{\Gamma}\dot{\hat{\theta}} \tag{7}$$

where $\Gamma, k_1 > 0$ are constants, $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimation error, $F_1(x_1) = f_1(x_1)$ and $\boldsymbol{G}_1(x_1) = \boldsymbol{g}_1(x_1)$.

By (4), for any given $0 < \varepsilon_{11} < 1, 0 < \varepsilon_{12} < 1$, there exist ${\boldsymbol{W}_{11}^*}^{\mathrm{T}} \boldsymbol{S}_{11}(x_1)$ and ${\boldsymbol{W}_{12}^*}^{\mathrm{T}} \boldsymbol{S}_{12}(x_1)$ such that

$$\begin{aligned} F_1(x_1) &= {\boldsymbol{W}_{11}^*}^{\mathrm{T}} \boldsymbol{S}_{11}(x_1) + \delta_{11}(x_1) \\ \boldsymbol{G}_1^{\mathrm{T}}(x_1)\boldsymbol{G}_1(x_1) &= {\boldsymbol{W}_{12}^*}^{\mathrm{T}} \boldsymbol{S}_{12}(x_1) + \delta_{12}(x_1) \\ |\delta_{11}(x_1)| &\leq \varepsilon_{11}, \ |\delta_{12}(x_1)| \leq \varepsilon_{12} \end{aligned} \tag{8}$$

where $x_1 \in S_{x_1} = \{x_1 | x_1 \in S_{\boldsymbol{x}}\}$ and $S_{\boldsymbol{x}}$ is a defined compact set by which the state trajectories may pass. According to $\boldsymbol{S}_{1j}^{\mathrm{T}}\boldsymbol{S}_{1j} \leq N_{1j}$ and (5), we have

$$|{\boldsymbol{W}_{1j}^*}^{\mathrm{T}}|^2 |\boldsymbol{S}_{1j}|^2 \leq |{\boldsymbol{W}_{1j}^*}^{\mathrm{T}}|^2 N_{1j} \leq \theta, \ j = 1, 2 \tag{9}$$

By Lemmas 1$\sim$2, $\mathrm{tr}\{\boldsymbol{X}\} \leq n\|\boldsymbol{X}\|_\infty \leq n\sqrt{n}\|\boldsymbol{X}\|$ ($\boldsymbol{X} \in \mathbf{R}^{n\times n}$ is a matrix), (8) and (9), there always exist positive numbers ξ_{11}, ξ_{12} and nonnegative smooth functions $\Psi_{11}(\hat{\theta})$, $\Phi_{11}(\hat{\theta})$, $\Psi_{12}(\hat{\theta})$ and $\Phi_{12}(\hat{\theta})$ such that

$$\begin{aligned} & k_1 z_1^{p-p_1+3} F_1(x_1) \leq \\ & \quad z_1^{p-p_1+3}\Psi_{11}(\hat{\theta}) + \frac{1}{2} z_1^{p-p_1+3}\tilde{\theta} \leq \\ & \quad \xi_{11} + z_1^{p+3}\Phi_{11}(\hat{\theta}) + \frac{1}{2} z_1^{p-p_1+3}\tilde{\theta} \end{aligned} \tag{10}$$

$$\begin{aligned} & \frac{p-p_1+3}{2} k_1 z_1^{p-p_1+2} \mathrm{tr}\{\boldsymbol{G}_1(x_1)\boldsymbol{G}_1^{\mathrm{T}}(x_1)\} \leq \\ & \quad \frac{p-p_1+3}{2} z_1^{p-p_1+2}\Psi_{12}(\hat{\theta}) + \frac{p-p_1+3}{4} z_1^{p-p_1+2}\tilde{\theta} \leq \\ & \quad \xi_{12} + z_1^{p+3}\Phi_{12}(\hat{\theta}) + \frac{p-p_1+3}{4} z_1^{p-p_1+2}\tilde{\theta} \end{aligned} \tag{11}$$

where $\Psi_{11}(\hat{\theta}) = k_1^2 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{1}{2}\varepsilon_{11}^2$ and functions $\Phi_{11}(\hat{\theta}) \geq \left(\frac{p-p_1+3}{p+3}\Psi_{11}\right)^{\frac{p+3}{p-p_1+3}} \left(\frac{p_1}{\xi_{11}(p-p_1+3)}\right)^{\frac{p_1}{p-p_1+3}}$, and $\Psi_{12}(\hat{\theta}) = k_1^2 n^3 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{1}{2}\varepsilon_{12}^2$, $\Phi_{12}(\hat{\theta}) \geq \left(\frac{(p-p_1+3)(p-p_1+2)}{2(p+3)}\Psi_{12}(\hat{\theta})\right)^{\frac{p+3}{p-p_1+2}} \left(\frac{p_1+1}{\xi_{12}(p-p_1+2)}\right)^{\frac{p_1+1}{p-p_1+2}}$.

Choosing the 1st virtual control law

$$\alpha_2(x_1, \hat{\theta}) = -z_1\left(\frac{c_1 + \Phi_{11}(\hat{\theta}) + \Phi_{12}(\hat{\theta})}{k_1\lambda_1}\right)^{\frac{1}{p_1}} \tag{12}$$

using Assumption 1 and substituting (10)$\sim$(12) into (7) yield

$$\begin{aligned} \mathcal{L}V_1 \leq & -c_1 z_1^{p+3} + k_1 d_1(t, \boldsymbol{x}, u) z_1^{p-p_1+3}(x_2^{p_1} - \alpha_2^{p_1}) + \\ & \xi_1 - \frac{\tilde{\theta}}{\Gamma}(\dot{\hat{\theta}} - \tau_1) \end{aligned} \tag{13}$$

where $c_1 > 0$ is a positive constant, $\xi_1 = \xi_{11} + \xi_{12}$ and $\tau_1 = \frac{1}{2}\Gamma z_1^{p-p_1+3} + \frac{p-p_1+3}{4}\Gamma z_1^{p-p_1+2}$.

Step ($\boldsymbol{i = 2 \cdots n}$). At this step, the design procedure is similar to step 1 and shown by the following proposition.

Proposition 1. For the ith Lyapunov function candidate $V_i(\bar{\boldsymbol{z}}_i, \hat{\theta}) = \sum_{j=1}^{i} \frac{k_j}{p-p_j+4} z_j^{p-p_j+4} + \frac{1}{2\Gamma}\tilde{\theta}^2$, there exists the virtual control law $\alpha_{i+1}(\bar{\boldsymbol{x}}_i, \hat{\theta})$ with the form

$$\alpha_{i+1}(\cdot) = -z_i\left(\frac{c_i + \Phi_{i0}(\hat{\theta}) + \Phi_{i1}(\hat{\theta}) + \Phi_{i2}(\hat{\theta})}{k_i\lambda_i}\right)^{\frac{1}{p_i}} \tag{14}$$

such that

$$\mathcal{L}V_i \le -\sum_{j=1}^{i-1}(c_j-\gamma_j)z_j^{p+3} - c_i z_i^{p+3} + k_i d_i(t,\boldsymbol{x},u)\times$$
$$z_i^{p-p_i+3}(x_{i+1}^{p_i}-\alpha_{i+1}^{p_i}) + \xi_i - \frac{\tilde{\theta}}{\Gamma}(\dot{\hat{\theta}}-\tau_i) \qquad (15)$$

where $k_i, c_i > 0$, $\gamma_1, \cdots, \gamma_{i-1}$ are positive parameters, $\xi_i = \xi_{i-1}+\xi_{i1}+\xi_{i2}$, and $\tau_i = \tau_{i-1}+\frac{1}{2}\Gamma z_i^{p-p_i+3}+\frac{p-p_i+3}{4}\Gamma z_i^{p-p_i+2}$.

Proof. See Appendix. □

Hence at step n, choosing Lyapunov function

$$V_n(\boldsymbol{z},\hat{\theta}) = \sum_{i=1}^{n}\frac{k_i}{p-p_i+4}z_i^{p-p_i+4} + \frac{1}{2\Gamma}\tilde{\theta}^2 \qquad (16)$$

and constructing the controller and adaptive law as

$$u = -z_n\left(\frac{c_n+\Phi_{n0}(\hat{\theta})+\Phi_{n1}(\hat{\theta})+\Phi_{n2}(\hat{\theta})}{k_n\lambda_n}\right)^{\frac{1}{p_n}} \qquad (17)$$

$$\dot{\hat{\theta}} = \sum_{i=1}^{n}\Gamma\Big(\frac{1}{2}z_i^{p-p_i+3} + \frac{p-p_i+3}{4}z_i^{p-p_i+2}\Big) - \hat{\theta} \qquad (18)$$

yield

$$\mathcal{L}V_n \le -\sum_{i=1}^{n}(c_i-\gamma_i)z_i^{p+3} + \xi_n + \frac{1}{\Gamma}\tilde{\theta}\hat{\theta} \qquad (19)$$

where $\xi_n = \sum_{i=1}^{n}(\xi_{i1}+\xi_{i2})$, $c_i-\gamma_i>0$, $\gamma_n = 0$ and $c_n > 0$.

3 Controller analysis

Now we give the major result of the paper.

Theorem 1. For system (1) satisfying Assumption 1, the control laws chosen as (12), (14), (17) and the adaptive law $\hat{\theta}$ chosen as (18), when constants a_1 and a_2 satisfy $r_0 > \frac{a_2}{a_1}$ and are given by

$$a_1 = \min\Big\{\frac{c_i-\gamma_i}{\beta_i}, 1\Big\},\ a_2 = \xi_n + \frac{1}{2\Gamma}\theta^2 + a_1\xi \qquad (20)$$

$\beta_i > 0$, the closed-loop system (1), (6), (12), (14), (17) and (18) can be ensured to be M-SGUUB with probability $1-\varepsilon$ in $S_\varepsilon = \{\Xi | V_n \le \frac{r_0}{\varepsilon}\}$, where $\Xi = (\boldsymbol{z},\tilde{\theta})^{\rm T}$ is the closed-loop state variable with the initial condition $\Xi_0 = (\boldsymbol{z}(t_0),\tilde{\theta}(t_0))^{\rm T}$, and ε is a positive constant.

Proof. Firstly, define the initial state compact set $S_0 = \{\Xi | V_n \le r_0\}$ and the approximation region $S_{\boldsymbol{X}_n} = \{\boldsymbol{X}_n | \sum_{i=1}^{n}\frac{k_i}{p-p_i+4}z_i^{p-p_i+4} \le \frac{r_0}{\varepsilon}\}$, where $\varepsilon\,(0<\varepsilon<1)$ is a design parameter, and $\boldsymbol{X}_n = (\boldsymbol{x},\hat{\theta})$. When $r_0 > \frac{a_2}{a_1}$, according to the definitions of S_0 and S_ε, if $\Xi \in S_\varepsilon$, then $\boldsymbol{X}_n \in S_{\boldsymbol{X}_n}$. This means $\tau_\varepsilon \le \tau_{\boldsymbol{X}_n}$, where τ_ε is the first time the trajectory of the new state Ξ reaches the boundary of S_ε and $\tau_{\boldsymbol{x}}$ is the first time the trajectory of the state variable $\boldsymbol{X}_n$ arrives at the boundary of $S_{\boldsymbol{X}_n}$.

According to (19) and $\tilde{\theta}\hat{\theta} \le -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2$, one has $\mathcal{L}V_n \le -\sum_{i=1}^{n}(c_i-\gamma_i)z_i^{p+3} + \bar{\xi}_n - \frac{\tilde{\theta}^2}{2\Gamma}$, where $\bar{\xi}_n = \xi_n + \frac{1}{2\Gamma}\theta^2$. Using Lemma 2 and (16), one can find a positive constant ξ_{n+1} such that $V_n(\boldsymbol{z},\hat{\theta}) \le \sum_{i=1}^{n}\beta_i z_i^{p+3} + \xi + \frac{1}{2\Gamma}\tilde{\theta}^2$, where $\beta_i = \frac{k_i}{p-p_i+4}\big(\frac{p-p_i+4}{p+3}\big)^{\frac{p+3}{p-p_i+4}}\big(\frac{p_i-1}{\xi_{n+1}(p-p_i+4)}\big)^{\frac{p_i-1}{p-p_i+4}}$ and $\xi = \sum_{i=1}^{n}\frac{k_i}{p-p_i+4}\xi_{n+1}$.

By combining (20), it is easy to get

$$\mathcal{L}V_n \le -a_1V_n + a_2, \qquad \Xi \in S_\varepsilon \qquad (21)$$

Since $r_0 > \frac{a_2}{a_1}$ and $\tau_\varepsilon \le \tau_{\boldsymbol{X}_n}$, it can be summarized from Lemma 4 that the closed-loop system (1), (6), (12), (14), (17) and (18) is ensured to be M-SGUUB . □

Remark 1. The analysis process in Theorem 1 is rigorous and reasonable. For a certain system, it has a finite solution set. We firstly give a sufficiently large S_{NN} which contains the solution set and determines θ, then all the closed-loop signals are enabled to stay in the compact set $S_{\boldsymbol{\Xi}}$ and converge to S_ε by regulating parameters c_i, β_i, γ_i, ξ_{i1}, ξ_{i2}, ξ_{n+1}, k_i and Γ appropriately. The relationships of these compact sets are $S_0 \subseteq S_\varepsilon \subseteq S_{\boldsymbol{\Xi}} \subseteq S_{NN}$. Apparently, a larger S_{NN} leads to a more relaxed S_0.

4 A simulation example

Consider a high-order stochastic nonlinear system:

$$\begin{cases} {\rm d}x_1 = x_2{\rm d}t + f_1(x_1){\rm d}t + \boldsymbol{g}_1(x_1){\rm d}\boldsymbol{\omega} \\ {\rm d}x_2 = u^3{\rm d}t + f_2(x_1,x_2){\rm d}t + \boldsymbol{g}_2(x_1,x_2){\rm d}\boldsymbol{\omega} \end{cases} \qquad (22)$$

where $d_1 = d_2 = 1, p_1 = 1, p_2 = 3$, $p=3$, $f_1 = x_1^2 - 2x_1$, $f_2 = x_1x_2^2$, $\boldsymbol{g}_1 = \sin(x_1)$ and $\boldsymbol{g}_2 = x_1^2\sin x_2$.

The adaptive controller is designed as

$$\begin{cases} \alpha_2(x_1,\hat{\theta}) = -z_1\frac{c_1+\Phi_{11}(\hat{\theta})+\Phi_{12}(\hat{\theta})}{k_1} \\ u(\bar{\boldsymbol{x}}_2,\hat{\theta}) = -z_2\Big(\frac{c_2+\Phi_{20}(\hat{\theta})+\Phi_{21}(\hat{\theta})+\Phi_{22}(\hat{\theta})}{k_2}\Big)^{\frac{1}{3}} \\ \dot{\hat{\theta}} = \frac{1}{2}\Gamma z_1^5 + \frac{5}{4}\Gamma z_1^4 + \frac{1}{2}\Gamma z_2^3 + \frac{3}{4}\Gamma z_2^2 - \hat{\theta} \end{cases} \qquad (23)$$

where $z_1 = x_1$, $z_2 = x_2 - \alpha_2(x_1,\hat{\theta})$, $\bar{\boldsymbol{x}}_2 = (x_1,x_2)^{\rm T}$, $c_1>0$, $c_2>0$, $\Phi_{11}(\hat{\theta}) = \big(\frac{5}{6}(k_1^2 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{1}{2}\varepsilon_{11}^2)\big)^{6/5}(1/(5\xi_{11}))^{1/5}$, $\Phi_{12}(\hat{\theta}) = \big(\frac{5}{3}(k_1^2n^3 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{1}{2}\varepsilon_{12}^2)\big)^{3/2}(1/(2\xi_{12}))^{1/2}$ and $\Phi_{20}(\hat{\theta}) = \frac{1}{6}(\frac{6}{5})^{-5}k_1^6\gamma_1^{-5}$, $\Phi_{21}(\hat{\theta}) = \frac{1}{4\xi_{21}}\big(k_2^2 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{1}{2}\varepsilon_{21}^2\big)^2$, and $\Phi_{22}(\hat{\theta}) = \frac{1}{2\xi_{22}^2}\big(k_2^2n^3 + \frac{\sqrt{1+\hat{\theta}^2}}{2} + \frac{\varepsilon_{22}^2}{2}\big)^3$.

In simulation, $k_1 = 0.5$, $k_2 = 0.5$, $c_1 = 3$, $c_2 = 1.5$, $\varepsilon_{11} = 0.1$, $\varepsilon_{12} = 0.1$, $\varepsilon_{21} = 0.1$, $\varepsilon_{22} = 0.1$, $\xi_{11} = 5$, $\xi_{12} = 5$, $\xi_{21} = 5$, $\xi_{22} = 5$, $n = 2$, $\gamma_1 = 2$, and $\Gamma = 100$. By choosing the appropriate initial values as $\hat{\theta}(0) = 0$, $x_1(0) = -0.5$ and $x_2(0) = 3$, Fig. 1 verifies the effectiveness of the control scheme.

5 Concluding remarks

The paper investigates the problem of state-feedback control for a class of high-order stochastic uncertain nonlinear systems with the aid of neural network and the designed controller guarantees the closed-loop system to be M-SGUUB.

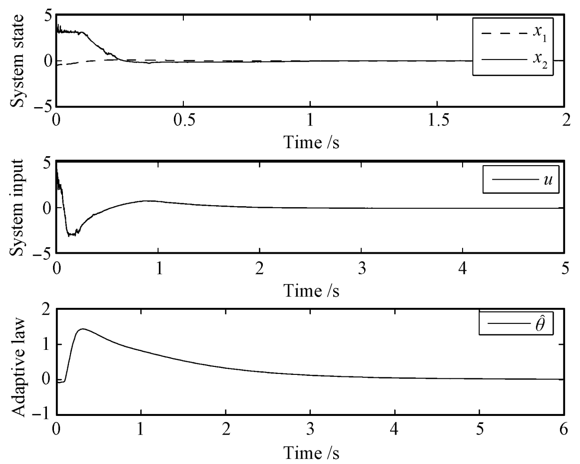

Fig. 1 The responses of closed-loop system (22) and (23)

There are still many problems to be solved: 1) When $d_i(t,\boldsymbol{x},u)$ is nonzero but of unknown control direction, how to investigate the state-feedback controller? 2) How to design an adaptive state-feedback controller for system (1) with time-delays based on neural network? 3) How to design an output-feedback controller for this class of systems based on neural network?

Appendix

Proof of Proposition 1. We prove the proposition by induction. Assume that at step $i-1$, there are a series of control laws $\alpha_2(\cdot)=-z_1\left(\frac{c_1+\Phi_{11}(\hat\theta)+\Phi_{12}(\hat\theta)}{k_1\lambda_1}\right)^{\frac{1}{p_1}},\cdots$, $\alpha_i(\cdot)=-z_{i-1}\eta_{i-1}(\hat\theta)=\left(\frac{c_{i-1}+\Phi_{i-1,0}+\Phi_{i-1,1}+\Phi_{i-1,2}}{k_{i-1}\lambda_{i-1}}\right)^{\frac{1}{p_{i-1}}}$, such that $V_{i-1}(\cdot)=\sum_{j=1}^{i-1}\frac{k_j}{p-p_j+4}z_j^{p-p_j+4}+\frac{1}{2\Gamma}\tilde\theta^2$ satisfies

$$\mathcal{L}V_{i-1}\le-\sum_{j=1}^{i-2}(c_j-\gamma_j)z_j^{p+3}-c_{i-1}z_{i-1}^{p+3}+k_{i-1}\times z_{i-1}^{p-p_{i-1}+3}d_{i-1}(t,\boldsymbol{x},u)(x_i^{p_{i-1}}-\alpha_i^{p_{i-1}})+\xi_{i-1}-\frac{\tilde\theta}{\Gamma}\left(\dot{\hat\theta}-\tau_{i-1}\right)\tag{A1}$$

We will verify that (A1) holds for the ith Lyapunov function. By (1), (6), (A1) and Itô rule, one has

$$\mathcal{L}V_i\le-\sum_{j=1}^{i-2}(c_j-\gamma_j)z_j^{p+3}-c_{i-1}z_{i-1}^{p+3}+k_{i-1}\times d_{i-1}(t,\boldsymbol{x},u)z_{i-1}^{p-p_{i-1}+3}\left(x_i^{p_{i-1}}-\alpha_i^{p_{i-1}}\right)+\xi_{i-1}-\frac{\tilde\theta}{\Gamma}(\dot{\hat\theta}-\tau_{i-1})+k_iz_i^{p-p_i+3}\Bigg(d_i(t,\boldsymbol{x},u)\times x_{i+1}^{p_i}+F_i(\bar{\boldsymbol{x}}_i,\hat\theta)\Bigg)+\frac{p-p_i+3}{2}k_iz_i^{p-p_i+2}\times \mathrm{tr}\{\boldsymbol{G}_i(\bar{\boldsymbol{x}}_i,\hat\theta)\boldsymbol{G}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i,\hat\theta)\}\tag{A2}$$

where $F_i=f_i(\bar{\boldsymbol{x}}_i)-\sum_{j=1}^{i-1}\frac{\partial\alpha_i}{\partial x_j}(d_j(t,\boldsymbol{x},u)x_{j+1}^{p_j}+f_j(\bar{\boldsymbol{x}}_j))-\frac{\partial\alpha_i}{\partial\hat\theta}\dot{\hat\theta}-\frac{1}{2}\sum_{j,l=1}^{i-1}\frac{\partial^2\alpha_i}{\partial x_j\partial x_l}\boldsymbol{g}_j^{\mathrm{T}}(\bar{\boldsymbol{x}}_j)\boldsymbol{g}_l(\bar{\boldsymbol{x}}_l)$, and $\boldsymbol{G}_i=\boldsymbol{g}_i(\bar{\boldsymbol{x}}_i)-\sum_{j=1}^{i-1}\frac{\partial\alpha_i}{\partial x_j}\boldsymbol{g}_j(\bar{\boldsymbol{x}}_j)$.

In terms of RBF NN approximation (4), for any given $0<\varepsilon_{i1}<1$, and $0<\varepsilon_{i2}<1$, there exist RBF NN $\boldsymbol{W}_{i1}^{*\mathrm{T}}\boldsymbol{S}_{i1}(\bar{\boldsymbol{x}}_i,\hat\theta),\boldsymbol{W}_{i2}^{*\mathrm{T}}\boldsymbol{S}_{i2}(\bar{\boldsymbol{x}}_i,\hat\theta)$ such that

$$\begin{aligned}F_i(\bar{\boldsymbol{x}}_i,\hat{\dot\theta})&=\boldsymbol{W}_{i1}^{*\mathrm{T}}\boldsymbol{S}_{i1}(\bar{\boldsymbol{x}}_i,\hat\theta)+\delta_{i1}(\bar{\boldsymbol{x}}_i,\hat\theta)\\ \boldsymbol{G}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i,\hat\theta)\boldsymbol{G}_i(\bar{\boldsymbol{x}}_i,\hat\theta)&=\boldsymbol{W}_{i2}^{*\mathrm{T}}\boldsymbol{S}_{i2}(\bar{\boldsymbol{x}}_i,\hat\theta)+\delta_{i2}(\bar{\boldsymbol{x}}_i,\hat\theta)\\ |\delta_{i1}(\bar{\boldsymbol{x}}_i,\hat\theta)|\le\varepsilon_{i1},&\ |\delta_{i2}(\bar{\boldsymbol{x}}_i,\hat\theta)|\le\varepsilon_{i2}\end{aligned}\tag{A3}$$

where $(\bar{\boldsymbol{x}}_i,\hat\theta)^{\mathrm{T}}\in S(\bar{\boldsymbol{x}}_i,\hat\theta)^{\mathrm{T}}=\{(\bar{\boldsymbol{x}}_i,\hat\theta)^{\mathrm{T}}|(\bar{\boldsymbol{x}}_i,\hat\theta)^{\mathrm{T}}\in S_{\boldsymbol{x}}\}$. According to $\boldsymbol{S}_{ij}^{\mathrm{T}}\boldsymbol{S}_{ij}\le N_{ij}$ and (5), one can get

$$|\boldsymbol{W}_{ij}^{*\mathrm{T}}|^2|\boldsymbol{S}_{ij}|^2\le|\boldsymbol{W}_{ij}^{*\mathrm{T}}|^2N_{ij}\le\theta,\ j=1,2\tag{A4}$$

By Lemmas 1$\sim$3, (A3), (A4), $\mathrm{tr}\{\boldsymbol{X}\}\le n\|\boldsymbol{X}\|_\infty\le n\sqrt{n}\|\boldsymbol{X}\|$ and $(a+b)^n=\sum_{i=0}^nC_n^ia^{n-i}b^i$, there must exist positive real numbers $\gamma_{i-1},\gamma_{i-1,j}(j=0,\cdots,(p_{i-1}-1)/2),\xi_{i1},\xi_{i2}$ and nonnegative functions $\Phi_{i0}(\hat\theta)$, $\Psi_{i1}(\hat\theta)$, $\Phi_{i1}(\hat\theta)$, $\Psi_{i2}(\hat\theta)$ and $\Phi_{i2}(\hat\theta)$ such that

$$\begin{aligned}&k_{i-1}d_{i-1}(t,\boldsymbol{x},u)z_{i-1}^{p-p_{i-1}+3}(x_i^{p_{i-1}}-\alpha_i^{p_{i-1}})\le\\&\quad\gamma_{i-1}z_{i-1}^{p+3}+\Phi_{i0}(\hat\theta)z_i^{p+3}\end{aligned}\tag{A5}$$

$$\begin{aligned}&k_iz_i^{p-p_i+3}F_i(\bar{\boldsymbol{x}}_i,\hat\theta)\le\\&\quad\xi_{i1}+z_i^{p+3}\Phi_{i1}(\hat\theta)+\frac{1}{2}z_i^{p-p_i+3}\tilde\theta\end{aligned}\tag{A6}$$

$$\begin{aligned}&\frac{p-p_i+3}{2}k_iz_i^{p-p_i+2}\mathrm{tr}\{\boldsymbol{G}_i(\bar{x}_i,\hat\theta)\boldsymbol{G}_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i,\hat\theta)\}\le\\&\quad\xi_{i2}+z_i^{p+3}\Phi_{i2}(\hat\theta)+\frac{p-p_i+3}{4}z_i^{p-p_i+2}\tilde\theta\end{aligned}\tag{A7}$$

where $\gamma_{i-1}=\sum_{j=0}^{(p_{i-1}-1)/2}\gamma_{i-1,j}$, $\Phi_{i0}(\hat\theta)\ge\sum_{j=0}^{(p_{i-1}-1)/2}\frac{p_{i-1}-2j}{p+3}(\frac{(p+3)\gamma_{i-1,j}}{p-p_{i-1}+3+2j})^{-\frac{p-p_{i-1}+3+2j}{p_{i-1}-2j}}(k_{i-1}\mu_{i-1}\cdot C_{p_{i-1}}^{2j}\sqrt{1+\eta_{i-1}^{4j}})^{\frac{p+3}{p_{i-1}-2j}}$, $\Psi_{i1}(\hat\theta)=k_i^2+\frac{\sqrt{1+\hat\theta^2}}{2}+\frac{1}{2}\varepsilon_{i1}^2$, $\Phi_{i1}(\hat\theta)\ge(\frac{p-p_i+3}{p+3}\Psi_{i1}(\hat\theta))^{\frac{p+3}{p-p_i+3}}(\frac{p_i}{\xi_{i1}(p-p_i+3)})^{\frac{p_i}{p-p_i+3}}$, $\Psi_{i2}(\hat\theta)=k_i^2n^3+\frac{\sqrt{1+\hat\theta^2}}{2}+\frac{1}{2}\varepsilon_{i2}^2$ and $\Phi_{i2}(\hat\theta)\ge(\frac{(p-p_i+3)(p-p_i+2)}{2(p+3)}\Psi_{i2}(\hat\theta))^{\frac{p+3}{p-p_i+2}}(\frac{p_i+1}{\xi_{i2}(p-p_i+2)})^{\frac{p_i+1}{p-p_i+2}}$.

Choosing the ith virtual control law as (14), using Assumption 1 and substituting (A5)$\sim$(A7) into (A2), one has (15). This completes the proof. □

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