Optimal Control of Stochastic System with Markovian Jumping and Multiplicative Noises

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Abstract An optimization problem for a stochastic system of N players is presented. An optimal Pareto controller of the stochastic system with Markovian jumping and multiplicative white noises is designed in infinite time horizon. The optimal Pareto solution is obtained by using the generalized Lyapunov equation approach and solving stochastic generalized Riccati algebraic equations (SGRAEs). It is proved that the controller is a stabilizing feedback control and the solution of SGRAEs is minimal associated with the optimal control.

Key words Pareto solution, stochastic system, Markovian jumping, multiplicative noises, minimal solution

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Stochastic system has attracted a lot of researchers from mathematics and control communities with the appearance of random phenomena in physical, engineering, biological, and social processes. The study of stochastic systems has a long history. Two distinct classes of the systems have been drawing much attention in the control literature, namely the stochastic systems subjected to white noise perturbations and Markovian jumping. At the same time, a lot of remarkable progress in recent decades in control and mathematic theory of deterministic dynamic systems strongly influences the research effort in the stochastic area.

The stochastic systems with Markovian jumping have been focused because they are fit for describing practical systems with random abrupt changes in their structures such as components failures or repairs, sudden environment disturbance, interconnections changing and operating in different conditions of a nonlinear plant. References [1−2] deal with the linear quadratic optimization problems of such systems. For more details of the linear quadratic (LQ) optimization problems or what has been done to the different related problems, readers can also refer to [3−4]. For Itô-type stochastic systems only with white noise perturbation, stochastic LQ and H_∞ control, stochastic stability and stabilization of Itô differential system, and stochastic algebraic Riccati equation have been investigated by many researchers for several decades[5−11]. In particular, the stochastic H_2/H_∞ control problems have received much attention in theory[9−11].

Based on the work of the two kinds of stochastic systems, researchers set to consider a kind of mathematical models described by stochastic differential equations subjected to white noise perturbation and Markovian jumping. In fact, for Itô-type stochastic systems with white noise perturbations and Markovian jumping, Wonham emphasizes the importance of the differential equations subjected to the white noise perturbation and Markovian jumping for control problems[12]. Stability and control problems of the stochastic differential equations depending on the white noise and Markovian jumping are considered[13−14]. Analysis and design methods are developed for advanced control problems on the class of systems as linear-quadratic control, robust stabilization, and disturbance attenuation problems[15−16]. During the recent decades, differential games have been extensively studied to analyze many problems in areas such as industrial control and economics. Stochastic Nash games have been tackled for their deterministic disturbance and stochastic uncertainty[17]. A guaranteed cost control problem for uncertain stochastic systems with N players is discussed and a cost bound is given in [18].

We pay our attention to an optimal control of N players of a stochastic system with white noise perturbation and Markovian jumping. Nash games approach is applied to generalize linear quadratic control problem of the systems with white noise perturbations in [11, 18]. Then, we consider N controls of players in stochastic system with the white noise perturbations and Markovian jumping. The main aim of the paper is to obtain an optimal Pareto solution associated with the stochastic generalized Riccati algebraic equations (SGRAEs) by developing mathematical analytical method. For the system with N controls, we obtain a cooperative analytical optimal Pareto controller and show that the minimal solution of the SGRAEs corresponds to the feedback gain of the optimal controller. The optimal controller is formulated by the minimal solution of the SGRAEs rather than the maximal solution[15−16].

The rest of the paper is organized as follows. Preliminaries are given Section 1. The optimization Pareto problem of stochastic system with Markovian jumping and white noise perturbations is formulated for N decision makers in Section 2. An optimal Pareto solution to the optimization problem is derived explicitly by the minimal solution of SGRAEs in Section 3. A numerical example is presented to illustrate the obtained results in Section 4. Some conclusions are drawn in Section 5.

1 Preliminaries

1.1 Notations

We make use of the following notation in the paper: M^{T} is the transpose of a vector or a matrix. $M > 0$ means that $M = M^{\mathrm{T}}$ and M is positive definite. $\mathbf{R}^n$ is the n-dimensional real Euclidean space, $\mathbf{R}^{m\times n}$ is the set of all $m \times n$ matrices. Let $\mathcal{S}_n \subset \mathbf{R}^{n\times n}$ be the subspace of $n \times n$ symmetric matrices.

$\mathcal{S}_n^d$ and $\mathcal{M}_{n,m}^d$ are finite-dimensional Banach spaces de-

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noted by the direct products

$$\mathcal{S}_n^d = \underbrace{\mathcal{S}_n \times \cdots \times \mathcal{S}_n}_{d}$$
$$\mathcal{M}_{n,m}^d = \underbrace{\mathbf{R}^{n\times m} \times \cdots \times \mathbf{R}^{n\times m}}_{d}$$

$S \in \mathcal{S}_n^d$ if and only if $S = (S(1), \cdots, S(d))$ and $\mathcal{M}_{n,m}^d = \{M; M = (M(1), \cdots, M(d))\}$.

1.2 Definitions and lemma

We introduce the definition of exponentially stable in mean square (ESMS) and the related facts[15].

Let $(\Omega, \mathcal{F}_t, \mathcal{P})$ be a given filtered probability space with an r-dimensional standard Wiener process $\boldsymbol{w}(t) = \{w_1(t), \cdots, w_r(t)\}^{\mathrm{T}}$ and a standard homogeneous Markov chain $\eta(t)$ on $[0, +\infty)$. For each $t \geq 0$, $\mathcal{F}_t$ denotes the family of σ-algebra $\sigma(w_i(t), t \geq 0, 1 \leq i \leq r)$ and $\sigma(\eta(t), t \geq 0)$, with respect to which all functions $w_i(t)$ and $\eta(t)$, $0 \leq i \leq r$, $t \geq 0$ are measurable. $\boldsymbol{w}(0) = \boldsymbol{0}$, the state space set of $\eta(t)$ is $\mathcal{D} = \{1, 2, \cdots, d\}$ with a probability transition matrix $\mathcal{P}(t) = [p_{lj}(t)] = \mathrm{e}^{Qt}, t \geq 0$, $Q = [q_{lj}]$ is a constant matrix with $q_{lj} \geq 0$ if $l \neq j$ and $\sum_{j=1}^d q_{lj} = 0$, $l, j \in \mathcal{D}$. We assume that the Markov chain is prior known (the probability transition matrix $\mathcal{P}(t) = [p_{lj}(t)]$ is prior known), $\boldsymbol{w}(t)$ and $\eta(t)$ are independent stochastic processes for every $t \geq 0$, and $\mathcal{P}\{\eta(0) = l\} > 0, l \in \mathcal{D}$.

Denote $L_{\boldsymbol{w},\eta}^2\{[a, b], \mathbf{R}^m\}$ be the space of measurable stochastic process $\phi : [a, b] \times \Omega \rightarrow \mathbf{R}^m$, $\phi(t)$ is measurable for $\forall\, t \in [a, b]$, $\mathrm{E}\int_a^b |\phi(t)|^2 \mathrm{d}t < +\infty$, where E denotes expectation and $[a, b] \subset [0, +\infty)$ is a compact interval.

Let us consider the following stochastic systems described by Itô equations:

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= A(t, \eta(t))\boldsymbol{x}(t)\mathrm{d}t + \sum_{k=1}^r C_k(t, \eta(t))\boldsymbol{x}(t)\mathrm{d}w_k(t) \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= [A(t, \eta(t))\boldsymbol{x}(t) + B(t, \eta(t))\boldsymbol{u}(t)]\,\mathrm{d}t + \\ &\quad \sum_{k=1}^r C_k(t, \eta(t))\boldsymbol{x}(t)\mathrm{d}w_k(t) \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0 \end{aligned} \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is state vector, $\boldsymbol{u}(t) \in L_{\boldsymbol{w},\eta}^2\{[0, \mathrm{T}], \mathbf{R}^m\}$ is control input for all $T \geq 0$, $\boldsymbol{w}(t) = \{w_1(t), \cdots, w_r(t)\}^{\mathrm{T}}$ and $\eta(t), t \geq 0$ are the stochastic processes defined as above. $t \rightarrow A(t, l) : [0, +\infty) \rightarrow \mathbf{R}^{n\times n}$, $t \rightarrow B(t, l) : [0, +\infty) \rightarrow \mathbf{R}^{n\times m}$, and $t \rightarrow C_k(t, l) : [0, +\infty) \rightarrow \mathbf{R}^{s_k\times n}$, $k = 1, \cdots, r$, $l \in \mathcal{D}$ are bounded and continuous matrix-valued functions. $\boldsymbol{x}_0 \in \mathbf{R}^n$ is the value of the initial state $\boldsymbol{x}(0), t_0 = 0$.

Definition 1[15]**.** Equation (1) is ESMS if there exist $\beta \geq 1$ and $\alpha > 0$ such that

$$\mathrm{E}[|\Phi(t, 0)\boldsymbol{x}(0)|^2 | \eta(0) = l] \leq \beta\mathrm{e}^{-\alpha(t)}|\boldsymbol{x}(0)|^2$$

for all $t \geq t_0 = 0$, $l \in \mathcal{D}$, $\boldsymbol{x}(0) \in \mathbf{R}^n$, where $\Phi(t, 0)$ is the fundamental matrix solution of (1).

Lemma 1[15]**.** The followings are equivalent:

1) Equation (1) is ESMS.

2) There exists a bounded uniform positive continuous function $H : [0, +\infty) \rightarrow \mathcal{S}_n^d$, $H(t) = (H(t, 1),\ H(t, 2), \cdots, H(t, d))$, such that the system of linear differential equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}K(t, l) + A^{\mathrm{T}}(t, l)K(t, l) + K(t, l)A(t, l) + H(t, l) + \\ &\sum_{k=1}^r C_k^{\mathrm{T}}(t, l)K(t, l)C_k(t, l) + \sum_{j=1}^d q_{lj}K(t, j) = 0 \end{aligned}$$

$l \in \mathcal{D}$, has a bounded and uniform positive solution $K(t) = (K(t, 1), K(t, 2), \cdots, K(t, d))$.

Definition 2[15−16]**.** Equation (2) is stochastically stabilizable if there exists $F : [0, +\infty) \rightarrow \mathcal{M}_{m,n}^d$ is bounded and continuous function such that the following system:

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= [A(t, \eta(t)) + B(t, \eta(t))F(t, \eta(t))]\,\boldsymbol{x}(t)\mathrm{d}t + \\ &\quad \sum_{k=1}^r C_k(t, \eta(t))\boldsymbol{x}(t)\mathrm{d}w_k(t) \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0 \end{aligned} \tag{3}$$

obtained by taking $\boldsymbol{u}(t) = F(t, \eta(t))\boldsymbol{x}(t), t \geq 0$ is ESMS. $F(t) = (F(t, 1), F(t, 2), \cdots, F(t, d))$ is termed stabilizing feedback gain and the feedback control $\boldsymbol{u}(t) = F(t, \eta(t))\boldsymbol{x}(t)$ is stabilizing.

2 Problem formulation

In this section, we will present the Pareto optimization problem.

Given a stochastic system described by Itô equation

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= \left[A(t, \eta(t))\boldsymbol{x}(t) + \sum_{i=1}^N B_i(t, \eta(t))\boldsymbol{u}_i(t)\right]\mathrm{d}t + \\ &\quad \sum_{k=1}^r C_k(t, \eta(t))\boldsymbol{x}(t)\mathrm{d}w_k(t) \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0 \end{aligned} \tag{4}$$

where $t \rightarrow B_i(t, l) : [0, +\infty) \rightarrow \mathbf{R}^{n\times m}$, $i = 1, \cdots, N$, N is the number of control input. $l \in \mathcal{D}$ are bounded and continuous matrix-valued functions, others are the same as in the above section.

Denote

$$B(t, \eta(t)) = [B_1(t, \eta(t)), B_2(t, \eta(t)), \cdots, B_N(t, \eta(t))]$$
$$\boldsymbol{u}(t) = [\boldsymbol{u}_1^{\mathrm{T}}(t), \boldsymbol{u}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{u}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$$

where $B(t, \eta(t))$ and $\boldsymbol{u}(t)$ are block matrices built from $B_i(t, \eta(t))$ and $\boldsymbol{u}_i(t)$ as blocks $(i = 1, 2, \cdots, N)$, then the system (4) can be changed into the same form as (2).

For each given $\boldsymbol{x}_0 \in \mathbf{R}^n$, we define the corresponding set of admissible controls $U^{\boldsymbol{x}_0}$ as the subset of stochastic process $\boldsymbol{u}(\cdot)$ for

$$U^{\boldsymbol{x}_0} = \left\{\boldsymbol{u}(t) | \boldsymbol{u}_i(t) \in L_{\boldsymbol{w},\eta}^2\{[0, T], \mathbf{R}^m\}\right\}$$

and

$$\lim_{t\rightarrow+\infty} \mathrm{E}[|\boldsymbol{x}(t, \boldsymbol{x}_0)|^2 | \eta(0) = l] = 0,\ l \in \mathcal{D}$$

where $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is the corresponding state of (4) with the initial state $\boldsymbol{x}_0$. We call $\boldsymbol{x}(\cdot)$ and $\boldsymbol{u}(\cdot)$ formulated above an admissible pair of the system (4), and denote $(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)) \in \mathbf{R}^n \times U^{\boldsymbol{x}_0}$.

For each $(\boldsymbol{x}_0, \boldsymbol{u}(\cdot)) \in \mathbf{R}^n \times U^{\boldsymbol{x}_0}$, the associated performance criterion for the system (4) is given by

$$J_i(\boldsymbol{x}_0, \boldsymbol{u}_i(\cdot)) = \mathrm{E}\int_0^{\infty} \Big[\boldsymbol{x}^{\mathrm{T}}(t)Q_i(t,\eta(t))\boldsymbol{x}(t) + \boldsymbol{u}_i^{\mathrm{T}}(t)R_i(t,\eta(t))\boldsymbol{u}_i(t)\Big]\,\mathrm{d}t \tag{5}$$

where $t \to Q_i(t,l) : [0,+\infty) \to \mathbf{R}^{n\times n}$ and $t \to R_i(t,l) : [0,+\infty) \to \mathbf{R}^{m\times m}$, $i \in \mathcal{D}$ are bounded and continuous functions and $Q_i(t,l) = Q_i^{\mathrm{T}}(t,l) > 0$, $R_i(t,l) = R_i^{\mathrm{T}}(t,l) > 0$, $i = 1, \cdots, N$.

Due to the presence of N inputs in (4) and (5), we consider a Nash game of N players and a Nash equilibrum solution. On one hand, the i-th players will design his control strategy based on the state information, and his design specification will be expressed in terms of a cost function J_i. On the other hand, N players decide their strategies through mutual cooperation. The solution to such a problem is found and is called a Pareto solution[18].

Definition 3[18]. A Pareto solution is a set $\{\boldsymbol{u} \in \mathbf{U}^{\boldsymbol{x}_0} | \boldsymbol{u}(\cdot) = (\boldsymbol{u}_1^{\mathrm{T}}(\cdot), \boldsymbol{u}_2^{\mathrm{T}}(\cdot), \cdots, \boldsymbol{u}_N^{\mathrm{T}}(\cdot))^{\mathrm{T}}\}$, which minimizes

$$J(\boldsymbol{x}_0, \boldsymbol{u}(\cdot)) = \sum_{i=1}^{N} \gamma_i J_i(\boldsymbol{x}_0, \boldsymbol{u}_i(\cdot)),\ \ \sum_{i=1}^{N}\gamma_i = 1 \tag{6}$$

for a given initial condition $\boldsymbol{x}_0 \in \mathbf{R}^n$, $l_0 \in \mathcal{D}$ and some $0 < \gamma_i < 1, i = 1, 2, \cdots, N$.

We will consider an optimal control problem, which can now be stated as (Q): for a given initial condition $\boldsymbol{x}_0 \in \mathbf{R}^n$ of (4), determine an optimal state-feedback control strategy

$$\boldsymbol{u}(t) \quad = \quad F(t,\eta(t))\boldsymbol{x}(t)$$

such that (4) is stochastically stabilizable and the cost function (6) is minimized over $\{\boldsymbol{u}(t) \in U^{\boldsymbol{x}_0}\}$, where $F(t,l) = (F_1^{\mathrm{T}}(t,l), F_2^{\mathrm{T}}(t,l), \cdots, F_N^{\mathrm{T}}(t,l))^{\mathrm{T}}$, $l \in \mathcal{D}$. By invoking of Definition 3, we call the state-feedback control strategy $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t)$ optimal Pareto control or optimal Pareto solution.

It can be obtained from the theory of linear matrices that for any $\boldsymbol{u}_j(t) \in \boldsymbol{u}(t)$, there exists matrix $p_i(t) \in \mathbf{R}^{m\times m}$ subjected to $\boldsymbol{u}_i(t) = p_i(t)\boldsymbol{u}_j(t), i = 1, 2, \cdots, N, p_j(t) = I$, I is a unit matrix, because $\boldsymbol{u}(t)$ is a linear feedback of $\boldsymbol{x}(t)$. Without loss of generality, we assume that $\boldsymbol{u}_i(t) = p_i(t)\boldsymbol{u}_j(t), p_j(t) = I$, $p_i(t), i = 1, 2, \cdots, N$ is the weight matrix of $\boldsymbol{u}_i(t)$. It is explained that when N controls are acting on a system, every control law is designed according to its attention ($p_i(t)$) paid by workers considering all controls effects.

In the following section, we shall seek for an optimal Pareto control $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t)$ of (Q) for a given initial condition $\boldsymbol{x}_0 \in \mathbf{R}^n$ over $\{\boldsymbol{u}(t) \in U^{\boldsymbol{x}_0}\}$.

3 Optimal Pareto control

In this section, we shall present solutions of the optimization problem stated in the above section.

First, we give several results which will be used in subsequent developments. For each quadruple $(0, \tau, \boldsymbol{x}_0, l), 0 < \tau < \infty, \boldsymbol{x}_0 \in \mathbf{R}^n, l \in \mathcal{D}$, we define an auxiliary cost function $J'(0, \tau, \boldsymbol{x}_0, l, \cdot) : L^2_{\boldsymbol{w},\eta}([t_0, \tau], \mathbf{R}^m) \to \mathbf{R}$ by

$$J'(0,\tau,\boldsymbol{x}_0,l;\boldsymbol{u}) = \sum_{i=1}^{N}\gamma_i \mathrm{E}\left[\int_0^{\tau}(\boldsymbol{x}^{\mathrm{T}}(t)Q_i(t,\eta(t))\boldsymbol{x}(t) + \boldsymbol{u}_i^{\mathrm{T}}(t)R_i(t,\eta(t))\boldsymbol{u}_i(t)\mathrm{d}t)|\eta(0) = l\right]$$

Applying the Itô-type formula we obtain the following results.

Lemma 2. If $t \to K(t,l) : [0,+\infty) \to \mathcal{S}_n, l \in \mathcal{D}$, are C^1-functions, then for any control $\boldsymbol{u}_j(t)$ in any admissible pair $(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot))$ of (4), we have

$$\begin{aligned}&J'(0,\tau,\boldsymbol{x}_0,l;\boldsymbol{u}(\cdot)) = \\ &\quad \boldsymbol{x}_0^{\mathrm{T}}K(0,l)\boldsymbol{x}_0 - \mathrm{E}[\boldsymbol{x}^{\mathrm{T}}(\tau)K(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0) = l] + \\ &\mathrm{E}\left[\int_{t_0}^{\tau}({\boldsymbol{u}_j}^{\mathrm{T}}(t), \boldsymbol{x}^{\mathrm{T}}(t))\mathcal{M}(t,\eta(t))\begin{bmatrix}\boldsymbol{u}_j(t)\\ \boldsymbol{x}(t)\end{bmatrix}\mathrm{d}t|\eta(t_0) = l\right]\end{aligned}$$

where $0 < \tau < \infty, \boldsymbol{x}_0 \in \mathbf{R}^n, l \in \mathcal{D}$, and

$$\mathcal{M}(t,l) = \begin{bmatrix} R(t,l) & b_1^{\mathrm{T}}(t,l)K(t,l) \\ K(t,l)b_1(t,l) & \Delta(t,l)\end{bmatrix}$$

with

$$\begin{aligned}R(t,l) =& \sum_{i=1}^{N}\gamma_i p_i(t)^{\mathrm{T}}R_i(t,l)p_i(t)\\ b_1(t,l) =& \sum_{i=1}^{N}B_i(t,l)p_i(t)\\ \Delta(t,l) =& \frac{\mathrm{d}}{\mathrm{d}t}K(t,l) + K(t,l)A(t,l) + A^{\mathrm{T}}(t,l)K(t,l) + \\ &\sum_{k=1}^{r}C_k^{\mathrm{T}}(t,l)K(t,l)C_k(t,l) + \sum_{i=1}^{N}\gamma_i Q_i(t,l) + \\ &\sum_{j=1}^{d}q_{lj}K(t,j)\end{aligned}$$

and $\boldsymbol{u}_i(t) = p_i(t)\boldsymbol{u}_j(t), i = 1, \cdots, N, p_j(t) = I$.

Proof. Applying Itô-type formula to $\boldsymbol{x}^{\mathrm{T}}(t)K(t,l)\boldsymbol{x}(t)$, integrating from t_0 to τ, and taking expectations, we easily obtain the desired result. □

Let us give a system of matrix differential equations of the following form:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}X(t,l) + A^{\mathrm{T}}(t,l)X(t,l) + X(t,l)A(t,l) + \sum_{i=1}^{N}\gamma_i Q_i(t,l) + \\ &\quad \sum_{k=1}^{r}C_k^{\mathrm{T}}(t,l)X(t,l)C_k(t,l) + \sum_{j=1}^{d}q_{lj}X(t,j) - \\ &\quad X(t,l)b_1(t,l)R(t,l)^{-1}b_1^{\mathrm{T}}(t,l)X(t,l) = 0,\ t > 0,\ l \in \mathcal{D}\end{aligned} \tag{7}$$

where the parameters are defined as above. It is SGRAEs.

Definition 4[15]. A C^1 function $X : [0,+\infty) \to \mathcal{S}_n^d$, $X(t) = (X(t,1), X(t,2), \cdots, X(t,d))$ is said to be a solution of (7) if for every $t \in [0,+\infty)$ and $l \in \mathcal{D}$, the matrix $R(t,l)$ is invertible and the relations (7) hold for all $t \in [0,+\infty)$. A solution X of (7) is a minimal solution if $0 \le X(t) \le \hat{X}(t)$ for arbitrary $\hat{X}(\cdot)$, where $X(t)$ and $\hat{X}(t)$ are bounded and uniform positive solutions.

Corollary 1. If $X(t) = (X(t,1), X(t,2), \cdots, X(t,d))$ is a solution of the system (7) defined on $[t_0, \tau]$, then

$$\begin{aligned} &J'(t_0,\tau,\boldsymbol{x}_0,l;\boldsymbol{u}(\cdot)) = \\ &\boldsymbol{x}_0^{\mathrm{T}} X(t_0,l)\boldsymbol{x}_0 - \mathrm{E}[\boldsymbol{x}^{\mathrm{T}}(\tau)X(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(t_0)=l] + \\ &\mathrm{E}\Bigg[\int_{t_0}^{\tau}(\boldsymbol{u}_j(t)-F_j(t,\eta(t))\boldsymbol{x}(t))^{\mathrm{T}}R(t,\eta(t)) \times \\ &(\boldsymbol{u}_j(t)-F_j(t,\eta(t))\boldsymbol{x}(t))\mathrm{d}t|\eta(t_0)=l\Bigg] \end{aligned} \tag{8}$$

$\forall \boldsymbol{x}_0 \in \mathbf{R}^n, l \in \mathcal{D}, \boldsymbol{u}_j(t) \in L^2_{\boldsymbol{w},\eta}([t_0,\tau],\mathbf{R}^m)$, where

$$F_j(t,l) = -R(t,l)^{-1}b_1^{\mathrm{T}}(t,l)X(t,l) \tag{9}$$

Proof. It is easily obtained from Lemma 2. □

Lemma 3. Assume that $X(t)$ is a bounded and uniform positive solution of (7). Then, (4) is stochastically stabilizable and there exists a stabilizing feedback control

$$\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t), \quad \boldsymbol{u}_i(t) = F_i(t,l)\boldsymbol{x}(t) \tag{10}$$

where $F_i(t,l) = -p_i(R(t,l))^{-1}b_1^{\mathrm{T}}(t,l)X(t,l) = p_iF_j(t,l)$, $p_j = I$, $\eta(t) = l$, $l \in \mathcal{D}$.

Proof. From Lemma 2 and Corollary 1, taking $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t)$, $\boldsymbol{u}_i(t) = p_i(t)\boldsymbol{u}_j(t)$, $\boldsymbol{u}_j(t) = F_j(t,l)\boldsymbol{x}(t)$, $F_j(t,l) = -(R(t,l))^{-1}b_1^{\mathrm{T}}(t,l)X(t,l)$, $i = 1,\cdots,N, p_j(t) = I$, $t \in [0,+\infty)$, and changing (7) into the following formulation:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}X(t,l) + \left(A(t,l)+\sum_{i=1}^{N}B_i(t,l)F_i(t,l)\right)^{\mathrm{T}}X(t,l) + \\ &X(t,l)\left(A(t,l)+\sum_{i=1}^{N}B_i(t,l)F_i(t,l)\right) + \\ &\sum_{k=1}^{r}C_k^{\mathrm{T}}(t,l)X(t,l)C_k(t,l) + \sum_{j=1}^{d}q_{lj}X(t,j) + \\ &\sum_{i=1}^{N}\gamma_iQ_i(t,l) + F_j^{\mathrm{T}}(t,l)R(t,l)F_j(t,l) = 0 \end{aligned} \tag{11}$$

Since $R(t,l), Q_i(t,l), b_1^{\mathrm{T}}(t,l)$, and $X(t,l)$ are bounded and continuous, $F_j(t,l)$ is bounded and continuous, too. Let

$$\begin{aligned} H(t,l) = &\sum_{i=1}^{N}\gamma_iQ_i(t,l) + F_j^{\mathrm{T}}(t,l)R(t,l))F_j(t,l) \\ H(t) = &(H(t,1),H(t,2),\cdots,H(t,d)) \end{aligned}$$

Obviously, $H: [0,+\infty) \to \mathcal{S}_n^d$ is a bounded uniform positive and continuous function and satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}X(t,l) + \left(A(t,l)+\sum_{i=1}^{N}B_i(t,l)F_i(t,l)\right)^{\mathrm{T}}X(t,l) + \\ &X(t,l)\left(A(t,l)+\sum_{i=1}^{N}B_i(t,l)F_i(t,l)\right) + H(t,l) + \\ &\sum_{k=1}^{r}C_k^{\mathrm{T}}(t,l)X(t,l)C_k(t,l) + \sum_{j=1}^{d}q_{lj}X(t,j) = 0,\ l \in \mathcal{D} \end{aligned}$$

From Lemma 1, we have known (4) is ESMS. That is, there exists a bounded and continuous matrix function $F(t,\eta(t))$ such that $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t)$ and (3) is ESMS. By Definition 3, we obtain (4) is stochastically stabilizable and $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t)$ is a stabilizing feedback control. □

For each $\boldsymbol{x}_0 \in \mathbf{R}^n$, the value function V associated with the above problem is defined as

$$V(\boldsymbol{x}_0) = \inf_{\boldsymbol{u}(\cdot)\in U^{\boldsymbol{x}_0}} J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot))$$

Definition 5. The optimization problem (Q) is called well posed if $V(\boldsymbol{x}_0) < +\infty$, for all $\boldsymbol{x}_0 \in \mathbf{R}^n$.

Lemma 4. Let the feedback control $\boldsymbol{u}(t) = F(t,\eta(t))\boldsymbol{x}(t), \boldsymbol{u}_i(t) = F_i(t,l)\boldsymbol{x}(t,l)$, $F_i(t,l) = -p_i(R(t,l))^{-1}\times b_1^{\mathrm{T}}(t,l)X(t,l) = p_iF_j(t,l)$, $p_j = I$, $\eta(t) = l \in \mathcal{D}$ be stabilizing. Then, its corresponding cost (5) with the initial condition $\boldsymbol{x}_0$ is

$$J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}X(0,l)\boldsymbol{x}_0$$

where $\pi_l(0) = \mathcal{P}(\eta(0) = l)$ and $X(t) = (X(t,1), X(t,2), \cdots, X(t,d))$ is a bounded and uniform positive solution of the system (11).

Proof. Lemma 1 guarantees the existence of the solution $X(t)$ for (11). Applying Corollary 1 and substituting $\boldsymbol{u}_i(t) = F_i(t,l)\boldsymbol{x}(t)$ into (8), we have:

$$\begin{aligned} &J'(0,\tau,\boldsymbol{x}_0,l;\boldsymbol{u}) = \\ &\boldsymbol{x}_0^{\mathrm{T}}X(0,l)\boldsymbol{x}_0 - \mathrm{E}\left[\boldsymbol{x}^{\mathrm{T}}(\tau)X(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0)=l\right] \end{aligned}$$

Letting $t \to +\infty$, we obtain

$$J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}X(0,l)\boldsymbol{x}_0 \qquad \square$$

Remark 1. Under the assumption that (4) is ESMS, from Lemma 4 we know that any bounded and uniform positive solution $X(t) = (X(t,1), X(t,2), \cdots, X(t,d))$ of (11) gives rise to an upper bound of the value function $V(\boldsymbol{x}_0) \le \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}X(0,l)\boldsymbol{x}_0$. Hence $V(\boldsymbol{x}_0) < +\infty$. It is clear that ESMS implies the well-posed.

Theorem 1. Suppose that (4) is stochastically stabilizable, then for the optimization problem (Q) described by (4) and (5), we have:

1) The optimization Pareto problem is well posed.

2) The optimal performance value $V(\boldsymbol{x}_0)$ and optimal control $\boldsymbol{u}(t) = [\boldsymbol{u}_1^{\mathrm{T}}(t), \boldsymbol{u}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{u}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ of the system with the initial condition $\boldsymbol{x}_0$ are respectively given by

$$V(\boldsymbol{x}_0) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0$$

and

$$\boldsymbol{u}_i(t) = F_i(t,l)\boldsymbol{x}(t)$$

where $F_i(t,l) = -p_i(t)R(t,l))^{-1}b_1^{\mathrm{T}}(t,l)\tilde{P}(t,l)$, $\pi_l(0) = \mathcal{P}(\eta(0) = l), p_j(t) = 1$, and $\tilde{P}(t) = (\tilde{P}(t,1),\cdots,\tilde{P}(t,d))$ is a bounded and uniform positive solution of (7).

3) $\tilde{P}(t)$ is the minimal bounded and uniform positive solution of (7).

Proof. 1) From the stochastic stable of (4) and Remark 1, it is easy to obtain the first conclusion.

2) Applying Corollary 1 for $X(t,l)$ replaced by $\tilde{P}(t,l)$,

we get

$$\begin{aligned}J'(0,\tau,\boldsymbol{x}_0,l;\boldsymbol{u}(\cdot)) =\\ \boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0 - \mathrm{E}[\boldsymbol{x}^{\mathrm{T}}(\tau)\tilde{P}(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0)=l]+\\ \mathrm{E}\bigg[\int_0^{\tau}(\boldsymbol{u}_j(t)-\tilde{F}_j(t,\eta(t))\boldsymbol{x}(t))^{\mathrm{T}}R(t,\eta(t))\times\\ (\boldsymbol{u}_j(t)-\tilde{F}_j(t,\eta(t))\boldsymbol{x}(t))\mathrm{d}t|\eta(t_0)=l\bigg]\end{aligned} \tag{12}$$

$\forall\boldsymbol{x}_0\in\mathbf{R}^n, l\in\mathcal{D}, \boldsymbol{u}_j(t)\in L^2_{\boldsymbol{w},\eta}([0,\tau],\mathbf{R}^m)$, where

$$\tilde{F}_j(t,l) = -R(t,l)^{-1}b_1^{\mathrm{T}}(t,l)\tilde{P}(t,l)$$

Taking the limit in (12), we get

$$\begin{aligned}J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0 -\\ \lim_{\tau\to\infty}\mathrm{E}[\boldsymbol{x}^{\mathrm{T}}(\tau)\tilde{P}(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0)=l]+\\ \sum_{l\in\mathcal{D}}\mathrm{E}\bigg[\int_0^{\infty}(\boldsymbol{u}_j(t)-\tilde{F}_j(t,\eta(t))\boldsymbol{x}(t))^{\mathrm{T}}\times\\ R(t,\eta(t))(\boldsymbol{u}_j(t)-\tilde{F}_j(t,\eta(t))\boldsymbol{x}(t))\mathrm{d}t|\eta(0)=l\bigg]\end{aligned}$$

for all $\boldsymbol{u}\in U^{\boldsymbol{x}_0}$.

On one hand, $\boldsymbol{u}_j(t) = \tilde{F}_j(t,\eta(t))\boldsymbol{x}(t)$, $\tilde{F}_j(t,l) = -R(t,l)^{-1}b_1^{\mathrm{T}}(t,l)\tilde{P}(t,l)$, from Lemma 4, we have

$$J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0$$

where $\pi_l(0)=\mathcal{P}(\eta(0)=l)$ and $\tilde{P}(t)=(\tilde{P}(t,1),\cdots,\tilde{P}(t,d))$ is a bounded and uniform positive solution of (11). By the property of the value function, we obtain

$$V(\boldsymbol{x}_0)\le J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0 \tag{13}$$

On the other hand, since $\tilde{P}(t)$ is a bounded solution, it follows that there exists $\tilde{c}>0$ such that $|\tilde{P}(t,l)|\le\tilde{c}, \forall(t,l)\in[0,+\infty)\times\mathcal{D}$. Then, from the inequality

$$\left|\mathrm{E}\left[\boldsymbol{x}^{\mathrm{T}}(\tau)\tilde{P}(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0)=l\right]\right|\le\tilde{c}\mathrm{E}[|\boldsymbol{x}(\tau)|^2|\eta(0)=l]$$

from Lemma 3, (4) is stochastically stabilizable, then $\lim_{\tau\to\infty}\mathrm{E}[|\boldsymbol{x}(\tau)|^2|\eta(0)=l]=0$, we obtain

$$\lim_{\tau\to\infty}\left|\mathrm{E}\left[\boldsymbol{x}^{\mathrm{T}}(\tau)\tilde{P}(\tau,\eta(\tau))\boldsymbol{x}(\tau)|\eta(0)=l\right]\right| = 0 \tag{14}$$

Considering $R_i(t,l)>0$ and $\gamma_i>0$, we obtain that $J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot))\ge\sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0$, which lead to

$$V(\boldsymbol{x}_0)\ge\sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0 \tag{15}$$

Combining (13) with (15), we obtain that

$$V(\boldsymbol{x}_0) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0$$

3) Since (4) is stochastically stabilizable, the existence of the minimal solution of (7) guaranteed by Theorem 14 in Chapter 4 of [15] and (14) is verified. We assume that $\tilde{P}(t)$ is not the minimal bounded and uniform positive solution of (7) and $\hat{P}(t)$ is a minimal bounded and uniform positive solution of (7). Noting the following fact: based on Corollary 1 and (14), for the solution $\hat{P}(t)=(\hat{P}(t,1),\hat{P}(t,2),\cdots,\hat{P}(t,d))$ of (7), we have

$$\begin{aligned}J(\boldsymbol{x}_0,\hat{\boldsymbol{u}}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\hat{P}(0,l)\boldsymbol{x}_0+\\ \sum_{l\in\mathcal{D}}\mathrm{E}\bigg[\int_0^{\infty}(\hat{\boldsymbol{u}}_j(t)-F_j^{\hat{P}}(t,\eta(t))\boldsymbol{x}(t))^{\mathrm{T}}\times\\ R(t,\eta(t))(\hat{\boldsymbol{u}}_j(t)-F_j^{\hat{P}}(t,\eta(t))\boldsymbol{x}(t))\mathrm{d}t|\eta(0)=l\bigg]\end{aligned}$$

where $F_j^{\hat{P}}(t,l) = -(R(t,l))^{-1}b_1^{\mathrm{T}}(t,l)\hat{P}(t,l)$.

Let

$$\hat{\boldsymbol{u}}_j(t) = F_j^{\hat{P}}(t,\eta(t))\boldsymbol{x}(t),\ \hat{\boldsymbol{u}}_i(t) = p_i\hat{\boldsymbol{u}}_j(t)$$

then

$$J(\boldsymbol{x}_0,\hat{\boldsymbol{u}}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\hat{P}(0,l)\boldsymbol{x}_0$$

So, we have

$$\begin{aligned}J(\boldsymbol{x}_0,\hat{\boldsymbol{u}}(\cdot)) = \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\hat{P}(0,l)\boldsymbol{x}_0\le\\ \sum_{l\in\mathcal{D}}\pi_l(0)\boldsymbol{x}_0^{\mathrm{T}}\tilde{P}(0,l)\boldsymbol{x}_0 = V(\boldsymbol{x}_0)\end{aligned}$$

for any $\boldsymbol{x}_0\in\mathbf{R}^n$ and $\pi_l(0)\ge0$, $\hat{P}(0,l)\le\tilde{P}(0,l)$. The above inequality contradicts with the the result of 1). Therefore, $\tilde{P}(t)$ is the minimal solution of (7). □

Remark 2. Theorem 1 shows that the optimal Pareto control of the system with white noise and Markovian jumping corresponds to the minimal bounded and uniform positive solution of (7). It is generalized Lemma 4.1 of [11] to the system adding Markovian jumping.

Remark 3. For the stochastic linear quadratic regulation (LQR) with white noise and Markovian jumping, [15−16] proved that its optimal value corresponds to the maximal bounded and uniform positive solution of SGRAEs when the cost weights are indefinite. When $N=1$ or all controls are same and the matrices of cost weights are positive definite, it is obtained that the optimal value of the stochastic LQR corresponds to the minimal bounded and uniform positive solution of (7). This result is better than those of [15−16].

Remark 4. Nash games of N controls corresponding to N states and cost functions for stochastic Itô system will be a further research topic based on our obtained results and [19].

4 Numerical example

In order to illustrate the above result, we present a numerical example. Consider the stochastic linear system (4) subjected to Markovian jumping and multiplicative noise with $n=2$, $r=1$, $d=1$, $m=1$, and $N=2$. In this case, (4) becomes

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t) =&(A\boldsymbol{x}(t)+B_1u_1(t)+B_2u_2(t))\mathrm{d}t+\\ &C_1\boldsymbol{x}(t)\mathrm{d}w_1(t)+C_2\boldsymbol{x}(t)\mathrm{d}w_2(t)\end{aligned} \tag{16}$$

$$\boldsymbol{x} = \left[\begin{array}{c}x_1\\x_2\end{array}\right]\in\mathbf{R}^2, \boldsymbol{x}_0 = \left[\begin{array}{c}x_{10}\\x_{20}\end{array}\right], u_i(t)\in\mathbf{R}$$

and the coefficient matrices are given as follows:

$$A=\begin{bmatrix}1&0\\0&3\end{bmatrix}, B_1=\begin{bmatrix}1\\1\end{bmatrix}, B_2=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$$

$$C_1=C_2=\frac{1}{2}I_2$$

The cost function is

$$J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot))=\mathrm{E}\int_0^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(t)(\gamma_1 Q_1+\gamma_2 Q_2)\boldsymbol{x}(t)+ \frac{2}{3}u_1^2(t)+\frac{1}{12}u_2^2(t)\Big]\,\mathrm{d}t \tag{17}$$

with

$$Q_1=\ Q_2=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \gamma_1=\frac{2}{3},\ \gamma_2=\frac{1}{3}$$

$$R_i=1,\ R_2=\frac{1}{4},\ p_1=1,\ p_2=2$$

In this case, (7) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t)+\begin{bmatrix}1&0\\0&3\end{bmatrix}X(t)+X(t)\begin{bmatrix}1&0\\0&3\end{bmatrix}+ \begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}X(t)\begin{bmatrix}1&0\\0&1\end{bmatrix}- X(t)\begin{bmatrix}2\\1\end{bmatrix}[2,1]X(t)=0 \tag{18}$$

From Subsection 4.4 of [15], we know that the minimal solution of (18) is

$$\tilde{P}(t)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\geq 0$$

Then the solution of the optimization problem is described by (16), (17), and the set of admissible controls $U^{\boldsymbol{x}_0}$ is constructed with the minimal solution of (18), that is

$$u_1(t)=-(2,1)\tilde{P}(t)\boldsymbol{x}(t)$$
$$u_2(t)=-2(2,1)\tilde{P}(t)\boldsymbol{x}(t)$$

The optimal value is

$$J(\boldsymbol{x}_0,\boldsymbol{u}(\cdot))=[x_{10},x_{20}]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x_{10}\\x_{20}\end{bmatrix}$$

5 Conclusions

In this paper, we discuss an optimization problem with N players of the stochastic systems with multiplicative white noise and Markovian jumping in infinite time horizon. By applying the generalized Lyapunov equation approach and the solution of SGRAEs, we obtain an optimal Pareto solution of the stochastic systems with multiplicative white noises and Markovian jumping. It is proved that the controller is a stabilizing feedback control and the solution of SGRAEs is minimal associated with the optimal control.

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