Improved Stability Criteria for Lurie Type Systems with Time-varying Delay

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Abstract In this technical note, we present a new stability analysis procedure for ascertaining the delay-dependent stability of a class of Lurie systems with time-varying delay and sector-bounded nonlinearity using Lyapunov-Krasovskii (LK) functional approach. The proposed analysis, owing to the candidate LK functional and tighter bounding of its time-derivative, yields less conservative absolute and robust stability criteria for nominal and uncertain systems respectively. The effectiveness of the proposed criteria over some of the recently reported results is demonstrated using a numerical example.

Key words Lurie systems, absolute stability, robust stability, Lyapunov-Krasovskii functional, linear matrix inequality

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The problem of absolute stability of Lurie systems has received considerable attention in control community, and many valuable results, such as Popov criterion, circle criterion, and Kalman-Yakubovih-Popov lemma have been reported in the past[1−3]. Many typical nonlinear systems, such as Chua′s circuit and the Lorenz system, can be classified into this type[4]. Recently, reported results on master-slave synchronization for Lurie systems using time-delayed feedback control[5−7] has rekindled the interest on stability studies of the system, and has given a fresh impetus to the problem. As time-delay is often encountered in physical systems like communication systems, aircraft stabilization, nuclear reactor, process systems, population dynamics, etc., and is a source of poor performance and instability, the problem of absolute and robust stability of Lurie systems with time-delay attains considerable significance and interest in control literature. Depending upon whether or not the stability criteria for Lurie system contain the time-delay information, the criteria can be classified respectively into delay-dependent stability criteria and delay-independent stability criteria. In general, the delay-dependent criteria are less conservative than the delay-independent ones if delay size is very small. Hence, delay-dependent stability studies for Lurie systems with constant[8−10] and time-varying delay[11−13] have been receiving increasing attention of the control community in recent years.

In this paper, we research the problem of delay-dependent stability of Lurie system with time-varying delay and sector-bounded nonlinearity using Lyapunov-Krasovskii (LK) functional approach. Subsequently, absolute and robust stability criteria are derived respectively for nominal and uncertain Lurie systems in terms of linear matrix inequalities (LMIs). To make the proposed criteria less conservative than the recently reported results[11−13], a candidate LK functional is used in the delay-dependent stability analysis, and the cross-terms that emerge from the time-derivative of the functional are bounded tightly without neglecting any useful terms using minimal number of slack matrix variables. The proposed analysis, eventually, culminates into a stability condition in convex LMI framework, and is solved non-conservatively at boundary conditions using standard numerical packages[14]. Finally, a numerical example is employed to demonstrate the effectiveness of the proposed criteria.

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Notations. $\mathbf{R}^n$ denotes the n-dimensional Euclidian space, $\mathbf{R}^{n\times m}$ is the set of $n\times m$ real matrices, I and 0 represents the identity matrix and null matrix of appropriate dimensions; the superscript "T" stands for the matrix transposition; $X>0$ (respectively $X\geq 0$), for $X\in\mathbf{R}^{n\times m}$ means that the matrix is real symmetric positive definite (respectively, positive semi definite); $\mathbf{R}$ and $\mathbf{Z}$ denote the set of real numbers and integers; The "$*$" represents the symmetric elements in a symmetric matrix.

1 System description and problem statement

Consider a class of Lurie-type system with time-varying delay given by

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bx(t-h(t))+Dw(t)\\ z(t) &= Mx(t)+Nx(t-h(t))\\ w(t) &= -\varphi(t,z(t))\end{aligned}\tag{1}$$

with

$$x(t)=\phi(t),\ t\in[-h,0]\tag{2}$$

where $x(t)\in\mathbf{R}^n$, $w(t)\in\mathbf{R}^p$, and $z(t)\in\mathbf{R}^q$ are the state, input, and output vectors of the system, respectively; A, B, D, M, and N are constant matrices of appropriate dimensions and the initial condition $\phi(t)$ is a continuous vector valued function; $\varphi(t,z(t))\in\mathbf{R}^p$ is a nonlinear function that is piecewise continuous in t, and globally Lipschitz in $z(t)$; $\varphi(t,0)=0$, and satisfies the following sector condition $\forall t\geq 0,\ \forall z(t)\in\mathbf{R}^p$:

$$[\varphi(t,z(t))-K_1z(t)]^{\mathrm{T}}[\varphi(t,z(t))-K_2z(t)]\leq 0\tag{3}$$

where K_1 and K_2 are real constant matrices of appropriate dimensions, and $K=K_2-K_1$ is a symmetric positive-definite matrix. In other words, the nonlinear function $\varphi(t,z(t))$ is said to belong to the sector $[K_1,\ K_2]$. On the other hand, if the nonlinear function $\varphi(t,z(t))$ belongs to the sector $[0,\ K]$, then, we have the following sector condition $\forall t\geq 0,\ \forall z(t)\in\mathbf{R}^p$:

$$\varphi^{\mathrm{T}}(t,z(t))[\varphi(t,z(t))-Kz(t)]\leq 0\tag{4}$$

The time-varying delay $h(t)$ is a continuous-time function satisfying the following conditions:

$$0\leq h(t)\leq h,\ \dot{h}(t)\leq h_d,\ \forall t\geq 0\tag{5}$$

In addition to absolute stability criterion, in this paper, robust stability criterion is also presented for the following class of uncertain Lurie systems:

$$\begin{aligned}\dot{x}(t) =&\ (A+\Delta A(t))x(t)+(B+\Delta B(t))x(t-h(t))+\\ &\ (D+\Delta D(t))w(t)\\ z(t) =&\ Mx(t)+Nx(t-h(t))\\ w(t) =&\ -\varphi(t,z(t))\end{aligned}\tag{6}$$

where the time-varying uncertainties are of the form:

$$\begin{bmatrix}\Delta A(t) & \Delta B(t) & \Delta D(t)\end{bmatrix}=GF(t)\begin{bmatrix}E_a & E_b & E_d\end{bmatrix}\tag{7}$$

where G, E_a, E_b, and E_d are known constant matrices of appropriate dimensions, and $F(t)$ is an unknown real time-varying matrix satisfying

$$F^{\mathrm{T}}(t)F(t) \le I, \ \forall t \tag{8}$$

The following lemmas are indispensable in deriving the proposed stability criterion, and they are stated below:

Lemma 1[15]**.** For any constant matrix $W \in \mathbf{R}^{n\times n}$, a scalar $\gamma > 0$, and vector function $\dot{x} : [-\gamma, 0] \mapsto \mathbf{R}^n$ such that the integration $\int_{t-\gamma}^{t} \dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s$ is well defined, then

$$-\gamma \int_{t-\gamma}^{t} \dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s \le \delta_\gamma^{\mathrm{T}}(t)\Pi_W\delta_\gamma(t) \tag{9}$$

where

$$\delta_\gamma(t) = \left[\begin{array}{c} x(t) \\ x(t-\gamma) \end{array}\right], \ \Pi_W = \left[\begin{array}{cc} -W & W \\ * & -W \end{array}\right]$$

Lemma 2. Suppose $r_1 \le r(t) \le r_2$, where $r(\cdot) : \mathbf{R}_+$ (or $\mathbf{Z}_+$) $\to \mathbf{R}_+$ (or $\mathbf{Z}_+$). Then, for any $R = R^{\mathrm{T}} > 0$, free matrices T and Y, the following integral inequality holds:

$$\begin{aligned} &-\int_{t-r_2}^{t-r_1} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \le \delta^{\mathrm{T}}(t)[(r_2 - r(t))TR^{-1}T^{\mathrm{T}}+ \\ &(r(t)-r_1)YR^{-1}Y^{\mathrm{T}} + [Y \quad -Y+T \quad -T]+ \\ &[Y \quad -Y+T \quad -T]^{\mathrm{T}}]\delta(t) \end{aligned} \tag{10}$$

where $\delta(t) = [x^{\mathrm{T}}(t-r_1) \quad x^{\mathrm{T}}(t-r(t)) \quad x^{\mathrm{T}}(t-r_2)]^{\mathrm{T}}$, $T = [T_1^{\mathrm{T}} \ \ T_2^{\mathrm{T}} \ \ T_3^{\mathrm{T}}]^{\mathrm{T}}$, and $Y = [Y_1^{\mathrm{T}} \ \ Y_2^{\mathrm{T}} \ \ Y_3^{\mathrm{T}}]^{\mathrm{T}}$.

Proof. For any vectors z, y, and symmetric, positive definite matrix X, the following inequality holds:

$$-2z^{\mathrm{T}}y \le z^{\mathrm{T}}X^{-1}z + y^{\mathrm{T}}Xy$$

Substituting $z = T^{\mathrm{T}}\delta(t)$, $y = \int_{t-r_2}^{t-r(t)} \dot{x}(s)\mathrm{d}s$, $X = \frac{R}{r_2 - r(t)}$, and subsequently using Jenson integral inequality, we get

$$\begin{aligned} &-2\delta^{\mathrm{T}}(t)T\int_{t-r_2}^{t-r(t)} \dot{x}(s)\mathrm{d}s \le (r_2 - r(t))\times \\ &\delta^{\mathrm{T}}(t)TR^{-1}T^{\mathrm{T}}\delta(t) + \int_{t-r_2}^{t-r(t)} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \end{aligned}$$

which, by Newton-Leibniz formula, is expressed as follows:

$$\begin{aligned} &-\int_{t-r_2}^{t-r(t)} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \le \\ &\delta^{\mathrm{T}}(t)\Big[(r_2 - r(t))TR^{-1}T^{\mathrm{T}}+ \\ &[0 \ \ T \ \ -T] + [0 \ \ T \ \ -T]^{\mathrm{T}}\Big]\delta(t) \end{aligned}$$

Similarly, we can deduce the following inequality as well:

$$\begin{aligned} &-\int_{t-r(t)}^{t-r_1} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \le \delta^{\mathrm{T}}(t)\Big[(r(t)-r_1)YR^{-1}Y^{\mathrm{T}}+ \\ &[Y \ \ -Y \ \ 0] + [Y \ \ -Y \ \ 0]^{\mathrm{T}}\Big]\delta(t) \end{aligned}$$

Summation of the last two equations completes the proof of the Lemma 2. □

Lemma 3[16]**.** Suppose $\gamma_1 \le \gamma(t) \le \gamma_2$, where $\gamma(\cdot) : \mathbf{R}_+$ (or $\mathbf{Z}_+$) $\to \mathbf{R}_+$ (or $\mathbf{Z}_+$). Then, for any constant matrices Ξ_1, Ξ_2, and Ω, the inequality

$$\Omega + (\gamma(t) - \gamma_1)\Xi_1 + (\gamma_2 - \gamma(t))\Xi_2 < 0$$

holds if and only if the following boundary conditions hold:

$$\begin{aligned} &\Omega + (\gamma_2 - \gamma_1)\Xi_1 < 0 \\ &\Omega + (\gamma_2 - \gamma_1)\Xi_2 < 0 \end{aligned}$$

Lemma 4[17]**.** Given matrices $Q = Q^{\mathrm{T}}$, H, E, and $R = R^{\mathrm{T}}$ of appropriate dimensions

$$Q + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0$$

for all F satisfying $F^{\mathrm{T}}F \le R$ holds, if and only if there exists some scalar, $\epsilon > 0$ such that

$$Q + \epsilon HH^{\mathrm{T}} + \epsilon^{-1}E^{\mathrm{T}}RE < 0$$

2 The proposed absolute stability criterion

The proposed absolute stability criterion for the system (1) satisfying (5) is stated below for $\varphi(t, z(t)) \in [0, \ K]$:

Theorem 1. The system (1) satisfying (5) with $\varphi(t, z(t)) \in [0, \ K]$ is absolutely stable for a given value of h and h_d, if there exist real symmetric positive definite matrices P, Q, Z_j, $j = 1, \ 2$; matrices Q_{11}, Q_{12}, Q_{22} and slack matrices T_i, Y_i, M_i, and N_i, $i = \ 1, \ 2, \ 3$ of appropriate dimensions such that the following LMIs hold:

$$\left[\begin{array}{cc} Q_{11} & Q_{12} \\ * & Q_{22} \end{array}\right] > 0$$

$$\left[\begin{array}{ccc} \Pi + \Pi_1 + \Pi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}}U & \frac{h}{2}T_a \\ * & -U & 0 \\ * & * & -\frac{h}{2}Z_1 \end{array}\right] < 0 \tag{11}$$

$$\left[\begin{array}{ccc} \Pi + \Pi_1 + \Pi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}}U & \frac{h}{2}Y_a \\ * & -U & 0 \\ * & * & -\frac{h}{2}Z_1 \end{array}\right] < 0 \tag{12}$$

$$\left[\begin{array}{ccc} \Phi + \Phi_1 + \Phi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}}U & \frac{h}{2}M_a \\ * & -U & 0 \\ * & * & -\frac{h}{2}Z_2 \end{array}\right] < 0 \tag{13}$$

$$\left[\begin{array}{ccc} \Phi + \Phi_1 + \Phi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}}U & \frac{h}{2}N_a \\ * & -U & 0 \\ * & * & -\frac{h}{2}Z_2 \end{array}\right] < 0 \tag{14}$$

where Π and Φ are given on the top of the next page, and

$$\begin{aligned} \Pi_1 &= \left[\begin{array}{ccccc} Y_a & -Y_a + T_a & -T_a & 0 & 0 \end{array}\right] \\ Y_a &= \left[\begin{array}{ccccc} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & Y_3^{\mathrm{T}} & 0 & 0 \end{array}\right]^{\mathrm{T}} \\ T_a &= \left[\begin{array}{ccccc} T_1^{\mathrm{T}} & T_2^{\mathrm{T}} & T_3^{\mathrm{T}} & 0 & 0 \end{array}\right]^{\mathrm{T}} \\ \Phi_1 &= \left[\begin{array}{ccccc} 0 & -N_a + M_a & N_a & -M_a & 0 \end{array}\right] \\ M_a &= \left[\begin{array}{ccccc} 0 & M_1^{\mathrm{T}} & M_2^{\mathrm{T}} & M_3^{\mathrm{T}} & 0 \end{array}\right]^{\mathrm{T}} \\ N_a &= \left[\begin{array}{ccccc} 0 & N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & 0 \end{array}\right]^{\mathrm{T}} \\ \bar{A} &= \left[\begin{array}{ccccc} A & B & 0 & 0 & D \end{array}\right] \\ U &= \frac{h}{2}(Z_1 + Z_2) \end{aligned}$$

$$\Pi = \begin{bmatrix} A^{\mathrm{T}}P + PA + Q + Q_{11} & PB & Q_{12} & 0 & PD - M^{\mathrm{T}}K^{\mathrm{T}} \\ * & -(1-h_D)Q & 0 & 0 & -N^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & Q_{22} - Q_{11} - \dfrac{2}{h}Z_2 & -Q_{12} + \dfrac{2}{h}Z_2 & 0 \\ * & * & * & -Q_{22} - \dfrac{2}{h}Z_2 & 0 \\ * & * & * & * & -2I \end{bmatrix}$$

$$\Phi = \begin{bmatrix} A^{\mathrm{T}}P + PA + Q + Q_{11} - \dfrac{2}{h}Z_1 & PB & Q_{12} + \dfrac{2}{h}Z_1 & 0 & PD - M^{\mathrm{T}}K^{\mathrm{T}} \\ * & -(1-h_D)Q & 0 & 0 & -N^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & Q_{22} - Q_{11} - \dfrac{2}{h}Z_1 & -Q_{12} & 0 \\ * & * & * & -Q_{22} & 0 \\ * & * & * & * & -2I \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} A^{\mathrm{T}}P + PA + Q + Q_{11} & PB & Q_{12} & 0 & PD - M^{\mathrm{T}}K^{\mathrm{T}} \\ * & -(1-h_D)Q & 0 & 0 & -N^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & Q_{22} - Q_{11} & -Q_{12} & 0 \\ * & * & * & -Q_{22} & 0 \\ * & * & * & * & -2I \end{bmatrix}$$

Proof. Consider the following LK functional candidate:

$$\begin{aligned} V(x_t,t) = {} & x^{\mathrm{T}}(t)Px(t) + \int_{t-h(t)}^{t} x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s + \\ & \int_{t-\frac{h}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h}{2}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{h}{2}) \end{bmatrix} \mathrm{d}s + \\ & \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{15}$$

The time-derivative of the LK functional (15) along the trajectory of (1) is given by

$$\begin{aligned} \dot{V}(x_t,t) = {} & 2x^{\mathrm{T}}(t)P(Ax(t) + Bx(t-h(t)) + Dw(t)) + \\ & \delta^{\mathrm{T}}(t)\Upsilon\delta(t) + x^{\mathrm{T}}(t)Qx(t) - \\ & (1-\dot{h}(t))x^{\mathrm{T}}(t-h(t))Qx(t-h(t)) + \\ & \dot{x}^{\mathrm{T}}(t)U\dot{x}(t) - \int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s - \\ & \int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s \end{aligned} \tag{16}$$

where

$$\delta(t) = \begin{bmatrix} x(t) \\ x(t-\dfrac{h}{2}) \\ x(t-h) \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ * & Q_{22} - Q_{11} & -Q_{12} \\ * & * & -Q_{22} \end{bmatrix}$$

Now, using the bound on the delay-derivative, and the sector condition defined in (4), we transform (16) into the following inequality:

$$\begin{aligned} \dot{V}(x_t,t) \le {} & 2x^{\mathrm{T}}(t)P(Ax(t) + Bx(t-h(t)) + Dw(t)) + \\ & \delta^{\mathrm{T}}(t)\Upsilon\delta(t) + x^{\mathrm{T}}(t)Qx(t) - \\ & (1-h_d)x^{\mathrm{T}}(t-h(t))Qx(t-h(t)) + \\ & \dot{x}^{\mathrm{T}}(t)U\dot{x}(t) - \int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s - \\ & \int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s - 2w^{\mathrm{T}}(t)w(t) - \\ & 2w^{\mathrm{T}}(t)K(Mx(t) + Nx(t-h(t))) \end{aligned} \tag{17}$$

By defining an augmented vector $\zeta(t) = [x^{\mathrm{T}}(t) \;\; x^{\mathrm{T}}(t-h(t)) \;\; x^{\mathrm{T}}(t-\frac{h}{2}) \;\; x^{\mathrm{T}}(t-h) \;\; w^{\mathrm{T}}(t)]^{\mathrm{T}}$, we can express (17) as follows:

$$\begin{aligned} \dot{V}(x_t,t) \le {} & \zeta^{\mathrm{T}}(t)(\Gamma + \bar{A}^{\mathrm{T}}U\bar{A})\zeta(t) - \\ & \int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s - \\ & \int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s \end{aligned} \tag{18}$$

where Γ is given on the top of this page. Now, when $0 \le h(t) \le h/2$, the cross-terms $-\int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s$ and $-\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s$ are dealt using Lemmas 1 and 2 respectively as follows:

$$\begin{aligned} - & \int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s \le \delta_1^{\mathrm{T}}(t)\Big[\Big(\frac{h}{2} - h(t)\Big)TZ_1^{-1}T^{\mathrm{T}} + \\ & h(t)YZ_1^{-1}Y^{\mathrm{T}} + \begin{bmatrix} Y & -Y+T & -T \end{bmatrix} + \\ & \begin{bmatrix} Y & -Y+T & -T \end{bmatrix}^{\mathrm{T}}\Big]\delta_1(t) \end{aligned} \tag{19}$$

and

$$-\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s) Z_2 \dot{x}(s) \mathrm{d}s \leq \begin{bmatrix} x(t-\frac{h}{2}) \\ x(t-h) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{2}{h}Z_2 & \frac{2}{h}Z_2 \\ * & -\frac{2}{h}Z_2 \end{bmatrix} \begin{bmatrix} x(t-\frac{h}{2}) \\ x(t-h) \end{bmatrix} \quad (20)$$

where

$$\delta_1(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-h(t)) & x^{\mathrm{T}}(t-\frac{h}{2}) \end{bmatrix}^{\mathrm{T}}$$
$$T = \begin{bmatrix} T_1^{\mathrm{T}} & T_2^{\mathrm{T}} & T_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$
$$Y = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & Y_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

In a similar manner, when $h/2 \leq h(t) \leq h$, the cross-terms $-\int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s) Z_1 \dot{x}(s) \mathrm{d}s$ and $-\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s) Z_2 \dot{x}(s) \mathrm{d}s$ are dealt using Lemmas 1 and 2 respectively as follows:

$$-\int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(s) Z_1 \dot{x}(s) \mathrm{d}s \leq \begin{bmatrix} x(t) \\ x(t-\frac{h}{2}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{2}{h}Z_1 & \frac{2}{h}Z_1 \\ * & -\frac{2}{h}Z_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{h}{2}) \end{bmatrix} \quad (21)$$

and

$$-\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s) Z_2 \dot{x}(s) \mathrm{d}s \leq \delta_2^{\mathrm{T}}(t)\Big[(h-h(t)) M Z_2^{-1} M^{\mathrm{T}} + \left(h(t)-\frac{h}{2}\right) N Z_2^{-1} N^{\mathrm{T}} + \begin{bmatrix} -N+M & N & -M \end{bmatrix} + \begin{bmatrix} -N+M & N & -M \end{bmatrix}^{\mathrm{T}}\Big]\delta_2(t) \quad (22)$$

where

$$\delta_2(t) = \begin{bmatrix} x^{\mathrm{T}}(t-h(t)) & x^{\mathrm{T}}(t-\frac{h}{2}) & x^{\mathrm{T}}(t-h) \end{bmatrix}^{\mathrm{T}}$$
$$M = \begin{bmatrix} M_1^{\mathrm{T}} & M_2^{\mathrm{T}} & M_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$
$$N = \begin{bmatrix} N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

Hence, for $0 \leq h(t) \leq h/2$, we use the relationships given in (19) and (20) in (18) and obtain the following inequality:

$$\dot{V}(x_t,t) \leq \zeta^{\mathrm{T}}(t)\Big[\Omega_1 + \left(\frac{h}{2}-h(t)\right) T_a Z_1^{-1} T_a^{\mathrm{T}} + h(t) Y_a Z_1^{-1} Y_a^{\mathrm{T}}\Big]\zeta(t) \quad (23)$$

similarly, for $h/2 \leq h(t) \leq h$, we use (21) and (22) in (18) and obtain the following inequality condition:

$$\dot{V}(x_t,t) \leq \zeta^{\mathrm{T}}(t)\Big[\Omega_2 + (h-h(t)) M_a Z_2^{-1} M_a^{\mathrm{T}} + \left(h(t)-\frac{h}{2}\right) N_a Z_2^{-1} N_a^{\mathrm{T}}\Big]\zeta(t) \quad (24)$$

where $\Omega_1 = \Pi + \Pi_1 + \Pi_1^{\mathrm{T}} + \bar{A}^{\mathrm{T}} U \bar{A}$ and $\Omega_2 = \Phi + \Phi_1 + \Phi_1^{\mathrm{T}} + \bar{A}^{\mathrm{T}} U \bar{A}$. By Lyapunov-Krasovskii stability theorem[18], the system (1) is absolutely stable, if $\dot{V}(x_t,t) \leq -\lambda \|x(t)\|^2$ for some $\lambda > 0$, if, for $0 \leq h(t) \leq h/2$,

$$\Omega_1 + \left(\frac{h}{2}-h(t)\right) T_a Z_1^{-1} T_a^{\mathrm{T}} + h(t) Y_a Z_1^{-1} Y_a^{\mathrm{T}} < 0 \quad (25)$$

and for $h/2 \leq h(t) \leq h$,

$$\Omega_2 + (h-h(t)) M_a Z_2^{-1} M_a^{\mathrm{T}} + \left(h(t)-\frac{h}{2}\right) N_a Z_2^{-1} N_a^{\mathrm{T}} < 0 \quad (26)$$

By applying Lemma 3 and Schur complement[19] successively to (25) and (26), we deduce the LMIs stated in Theorem 1. □

3 Robust stability criterion

The proposed robust stability criterion for the uncertain system (6) satisfying the time-varying delay (5) is stated below for $\varphi(t, z(t)) \in [0,\ K]$:

Theorem 2. The uncertain system (6) satisfying (5) with $\varphi(t, z(t)) \in [0,\ K]$ is robustly absolutely stable for a given value of h and h_d, if there exist real symmetric positive definite matrices P, Q, Z_j, $j = 1,\ 2$; matrices Q_{11}, Q_{12}, Q_{22} and slack matrices T_i, Y_i, M_i and N_i, $i = 1,\ 2,\ 3$ of appropriate dimensions, and scalars $\mu_i > 0$, $i = 1$ to 4 such that the following LMIs hold:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0$$

$$\begin{bmatrix} \Pi + \Pi_1 + \Pi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}} U & \bar{P} G & \mu_1 \bar{E}^{\mathrm{T}} & \frac{h}{2} T_a \\ * & -U & UG & 0 & 0 \\ * & * & -\mu_1 I & 0 & 0 \\ * & * & * & -\mu_1 I & 0 \\ * & * & * & * & -\frac{h}{2} Z_1 \end{bmatrix} < 0 \quad (27)$$

$$\begin{bmatrix} \Pi + \Pi_1 + \Pi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}} U & \bar{P} G & \mu_2 \bar{E}^{\mathrm{T}} & \frac{h}{2} Y_a \\ * & -U & UG & 0 & 0 \\ * & * & -\mu_2 I & 0 & 0 \\ * & * & * & -\mu_2 I & 0 \\ * & * & * & * & -\frac{h}{2} Z_1 \end{bmatrix} < 0 \quad (28)$$

$$\begin{bmatrix} \Phi + \Phi_1 + \Phi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}} U & \bar{P} G & \mu_3 \bar{E}^{\mathrm{T}} & \frac{h}{2} M_a \\ * & -U & UG & 0 & 0 \\ * & * & -\mu_3 I & 0 & 0 \\ * & * & * & -\mu_3 I & 0 \\ * & * & * & * & -\frac{h}{2} Z_2 \end{bmatrix} < 0 \quad (29)$$

$$\begin{bmatrix} \Phi + \Phi_1 + \Phi_1^{\mathrm{T}} & \bar{A}^{\mathrm{T}} U & \bar{P} G & \mu_4 \bar{E}^{\mathrm{T}} & \frac{h}{2} N_a \\ * & -U & UG & 0 & 0 \\ * & * & -\mu_4 I & 0 & 0 \\ * & * & * & -\mu_4 I & 0 \\ * & * & * & * & -\frac{h}{2} Z_2 \end{bmatrix} < 0 \quad (30)$$

where

$$\bar{P} = \begin{bmatrix} P^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$
$$\bar{E} = \begin{bmatrix} E_a & E_b & 0 & 0 & E_d \end{bmatrix}$$

Proof. Replace the system matrices A, B, and D in LMIs (11) ∼ (14) with $(A + \Delta A(t))$, $(B + \Delta B(t))$, and $(D + \Delta D(t))$, respectively. Then, from the definition of norm-bounded uncertainties (7), we obtain the following inequalities:

$$\Omega_1 + (\bar{\bar{P}}GF(t)\bar{\bar{E}}) + (\bar{\bar{P}}GF(t)\bar{\bar{E}})^{\mathrm{T}} + \frac{h}{2}T_b\ Z_1^{-1}\ T_b^{\mathrm{T}} < 0 \tag{31}$$

$$\Omega_1 + (\bar{\bar{P}}GF(t)\bar{\bar{E}}) + (\bar{\bar{P}}GF(t)\bar{\bar{E}})^{\mathrm{T}} + \frac{h}{2}Y_b\ Z_1^{-1}\ Y_b^{\mathrm{T}} < 0 \tag{32}$$

$$\Omega_2 + (\bar{\bar{P}}GF(t)\bar{\bar{E}}) + (\bar{\bar{P}}GF(t)\bar{\bar{E}})^{\mathrm{T}} + \frac{h}{2}M_bZ_2^{-1}M_b^{\mathrm{T}} < 0 \tag{33}$$

$$\Omega_2 + (\bar{\bar{P}}GF(t)\bar{\bar{E}}) + (\bar{\bar{P}}GF(t)\bar{\bar{E}})^{\mathrm{T}} + \frac{h}{2}N_b\ Z_2^{-1}N_b^{\mathrm{T}} < 0 \tag{34}$$

where $\bar{\bar{P}} = [\bar{P}^{\mathrm{T}}\quad U^{\mathrm{T}}]^{\mathrm{T}}$, $\bar{\bar{E}} = [\bar{E}\quad 0]$, $T_b = [T_a^{\mathrm{T}}\quad 0]^{\mathrm{T}}$, and $Y_b = [Y_a^{\mathrm{T}}\quad 0]^{\mathrm{T}}$. Now, there exists $\mu_i > 0$, $i = 1, 2, 3, 4$ such that by Lemma 4, if (31) ∼ (34) hold, the following inequalities hold:

$$\Omega_1 + \mu_1^{-1}(\bar{\bar{P}}G)(\bar{\bar{P}}G)^{\mathrm{T}} + \mu_1\bar{\bar{E}}^{\mathrm{T}}\bar{\bar{E}} + \frac{h}{2}T_b\ Z_1^{-1}\ T_b^{\mathrm{T}} < 0 \tag{35}$$

$$\Omega_1 + \mu_2^{-1}(\bar{\bar{P}}G)(\bar{\bar{P}}G)^{\mathrm{T}} + \mu_2\bar{\bar{E}}^{\mathrm{T}}\bar{\bar{E}} + \frac{h}{2}Y_b\ Z_1^{-1}\ Y_b^{\mathrm{T}} < 0 \tag{36}$$

$$\Omega_2 + \mu_3^{-1}(\bar{\bar{P}}G)(\bar{\bar{P}}G)^{\mathrm{T}} + \mu_3\bar{\bar{E}}^{\mathrm{T}}\bar{\bar{E}} + \frac{h}{2}M_bZ_2^{-1}M_b^{\mathrm{T}} < 0 \tag{37}$$

$$\Omega_2 + \mu_4^{-1}(\bar{\bar{P}}G)(\bar{\bar{P}}G)^{\mathrm{T}} + \mu_4\bar{\bar{E}}^{\mathrm{T}}\bar{\bar{E}} + \frac{h}{2}N_b\ Z_2^{-1}N_b^{\mathrm{T}} < 0 \tag{38}$$

Schur complement to (35) ∼ (38) completes the proof. □

Remark 1. For $\varphi(t, z(t)) \in [K_1,\ K_2]$, by using loop transformation[2], we can obtain the absolute and robust stability criteria by replacing A, B and K in Theorems 1 and 2 with $(A - DK_1M)$, $(B - DK_1N)$, and $(K_2 - K_1)$, respectively.

Remark 2. The reason for less conservativeness of the proposed stability criteria over the recently reported results [11−12] is attributed to the candidate LK functional used in the delay-dependent stability analysis. In [11−12], the LK functional is formulated by considering the variation of the delay $h(t)$ in the entire delay range, i.e., $h(t) \in [0,\ h]$; whereas in this paper, we have considered the variation of the delay in two equal segments of the delay-range viz. $h(t) \in [0,\ h/2]$ and $h(t) \in [h/2,\ h]$, and have suitably embedded the segment information into the LK functional. The analysis, subsequently, paves way to stability criteria that yield less conservative stability criteria compared to those of [11−12].

Remark 3. Though the delay-dependent stability analysis presented in [13] also splits the delay-interval $[0,\ h]$ into two equal sub-intervals and defines different energy functions on each interval, the following over-bounding:

$$-\left(h - \frac{h}{2}\right)\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s \leq -\left(h(t) - \frac{h}{2}\right)\int_{t-h(t)}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s$$

neglecting the terms $-(h - h(t))\int_{t-h}^{t-h(t)} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s$, $-(h(t) - \frac{h}{2})\int_{t-h}^{t-h(t)} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s$, and $-(h - h(t)) \times \int_{t-h(t)}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s$ in the stability analysis introduces conservatism in the ensuing stability criteria. On the other hand, in the proposed stability analysis, we have reduced the conservatism of the stability criteria by using a candidate LK functional[20], and have bounded the time-derivative of the functional tightly (without neglecting any useful terms) using minimal number of slack matrix variables that are not redundant. This, in turn, yields less conservative stability criteria compared to those of [13].

Remark 4. In situations where there is no restriction on the derivative of the time-varying delay, the stability criteria can be deduced readily by letting $Q = 0$ in the Theorems 1 and 2.

4 A numerical example

The effectiveness of the proposed stability criteria over [11−12] and [13] is demonstrated on a numerical example in this section.

Example 1. Consider the nominal Lurie system given in (1) with following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$D = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix},\ M = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}$$
$$N = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},\ K_1 = 0.2,\ K_2 = 0.5$$

The maximum allowable delay bounds (MADB) provided by the proposed stability criterion for Example 1 is listed in Table 1 for different values of h_d. Owing to the use of a candidate LK functional (15) and tighter bounding conditions on the cross-terms, the proposed absolute stability criterion yields less conservative delay bounds than the existing results[11−13].

Consider the uncertain Lurie system which has the following parameters in addition to those furnished in Example 1:

$$E_a = E_b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ E_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

The maximum delay bounds provided by the proposed robust stability criterion of Theorem 2 for the uncertain system is listed in Table 2 for different values of h_d. For the uncertain case, the proposed robust stability criterion also yields less conservative delay bounds than those of the existing results.

Table 1 Absolute stability: maximum allowable delay bound h for given h_d

Method	h_d	0	0.3	0.6	0.9	1	> 1
[11]	h	4.6839	2.6634	1.7846	1.1721	0.9798	0.9798
[12]	h	4.6839	2.8688	2.1925	1.8724	1.8517	1.8517
[13][a]	h	5.9964	3.0955	1.8753	1.3149	1.3149	1.3149
Theorem 1	h	5.9964	3.1436	2.3064	2.1595	2.1595	2.1595

Table 2 Robust stability: maximum allowable delay bound h for given h_d

Method	h_d	0	0.3	0.6	0.9	1	> 1
[11]	h	3.3056	2.0787	1.4195	0.9228	0.7638	0.7638
[12]	h	3.3056	2.2262	1.7409	1.4682	1.4383	1.4383
[13][a]	h	4.1077	2.3707	1.4819	1.0346	1.0346	1.0346
Theorem 2	h	4.1077	2.4335	1.8718	1.7077	1.6995	1.6995

[a] These are the true results of Corollaries 1 and 2 of [13].

Remark 5. By solving the LMIs presented in Corollaries 1 and 2 of [13], we have found that the delay bounds claimed in the paper are not true. In fact, the actual delay bounds provided by the corollaries are more conservative than what is actually claimed. These results are presented in Tables 1 and 2.

5 Conclusion

In this note, we have proposed less conservative absolute and robust stability criteria for a class of Lurie systems with time-varying delay and sector-bounded nonlinearity. The delay-dependent stability analysis that yields the stability criteria uses a candidate LK functional, and the time-derivative of the functional is bounded tightly without neglecting any useful terms using minimal number of slack matrix variables. The proposed analysis, subsequently, yields a stability criterion in convex LMI framework and is solved non-conservatively at boundary conditions using standard numerical packages. The effectiveness of the proposed criteria over recently reported results is demonstrated on a standard numerical example. As master-slave synchronization of Lurie systems with time-delayed feedback control is an important practical application, the future research is focused on delay-dependent stabilization of Lurie systems with time-varying delay.

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