Robust Exponential Admissibility of Uncertain Switched Singular Time-delay Systems

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Abstract This paper investigates the problem of robust exponential admissibility for a class of continuous-time uncertain switched singular systems with interval time-varying delay. By defining a properly constructed decay-rate-dependent Lyapunov function and the average dwell time approach, a delay-range-dependent sufficient condition is derived for the nominal system to be regular, impulse free, and exponentially stable. This condition is also extended to uncertain case. The obtained results provide a solution to one of the basic problems in continuous-time switched singular time-delay systems, that is, to identify a switching signal for which the switched singular time-delay system is regular, impulse free, and exponentially stable. Numerical examples are given to demonstrate the effectiveness of the obtained results.

Key words Switched singular system, time-varying delay, exponential stability, average dwell time

DOI 10.3724/SP.J.1004.2010.01773

Switched systems have drawn considerable attention since 1990s, due to their great flexibility in modeling of event-driven systems, logic-based systems, parameter- or structure-varying systems, and so on; for details, see [1−2] and the references therein. Switched systems are a class of hybrid systems, which consist of a collection of continuous- or discrete-time subsystems and a switching rule that specifies the switching between them. It is commonly recognized that there are three basic problems in stability analysis and design of switched systems[3]: 1) find conditions for stability under arbitrary switching; 2) identify the limited but useful class of stabilizing switching signals; and 3) construct a stabilizing switching signal. Many effective methods have been presented to tackle these three basic problems, such as the multiple Lyapunov function approach[4], the piecewise Lyapunov function approach[5], the switched Lyapunov function approach[6], and the dwell-time or average dwell-time scheme[3, 7−10]. On the other hand, time-delay is commonly encountered in many practical systems and is frequently a source of instability and poor performance[11]. Therefore, the stability analysis of switched time-delay systems has received more and more attention in recent years[12−20].

As far as we know, singular systems also provide a natural framework for modeling of dynamic systems and describe a larger class of systems than state-space models[21]. Recently, many efforts have been done to the study of switched singular systems and a great number of results on stability and stabilization[22−25], reachability[26], and filtering problem[27] have been obtained. For switched singular time-delay (SSTD) systems, due to the coupling between the switching and the time-delay and because of the algebraic constraints in singular model, the behavior of such systems is much more complicated than that of regular switched time-delay systems or switched singular systems. To date, there are only a few results reported on the SSTD systems. In [28], the robust stability and H_∞ control problems for discrete-time uncertain SSTD systems under arbitrary switching were discussed by using the switched Lyapunov function method. In [29], a switching signal was constructed to guarantee the asymptotic stability of continuous-time SSTD systems. However, the aforementioned results are focused on the basic problem 1)[28] and problem 3)[29] for SSTD systems. Problem 2) is to identify stabilizing switching signals on the premise that all the individual subsystems of the switched systems are stable. Basically, we will find that stability is ensured if the switching is sufficiently slow[3], and it is well-known that dwell time and average dwell time are two simple but effective tools to define slow switching signals. In [7], it was shown that if all the individual subsystems are exponentially stable and that the dwell time of the switching signal is not smaller than a certain lower bound, then the switched system is exponentially stable. This result was extended to both continuous-time switched linear time-delay systems[14] and discrete-time cases[15]. Unfortunately, so far, to the best of the authors′ knowledge, solving the basic problem 2) for SSTD systems via the dwell time or average dwell time scheme remains open and unsolved. This forms the motivation of this paper.

In this paper, we are concerned with the robust admissibility problem for a class of continuous-time uncertain switched singular systems with interval time-varying delay. More precisely, a class of slow switching signals specified by the average dwell time is identified to guarantee the exponential admissibility of the considered system. In terms of linear matrix inequalities (LMIs), a delay-range-dependent sufficient condition, which is dependent on the switching signal, is first derived for the nominal system to be regular, impulse free, and exponentially stable by using a properly constructed decay-rate-dependent Lyapunov function and the average dwell time approach. Then, this condition is extended to uncertain case. The effectiveness of the obtained results is finally demonstrated by two illustrative examples.

Notations. $P>0$ ($P\geq 0$) means that matrix P is positive definite (semi-positive definite). $\lambda_{\min}(P)$ ($\lambda_{\max}(P)$) denotes the minimum (maximum) eigenvalue of symmetric matrix P. $\mathcal{C}_{n,d}=\mathcal{C}([-d,0],\mathbf{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ to $\mathbf{R}^n$. Let $\boldsymbol{x}_t\in\mathcal{C}_{n,d}$ be defined by $\boldsymbol{x}_t=\boldsymbol{x}(t+\theta)$, $\theta\in[-d,0]$. $\|\cdot\|$ denotes the Euclidean norm of a vector and induced norm of a matrix and $\|\boldsymbol{x}_t\|_d=\sup_{-d\leq\theta\leq 0}\|\boldsymbol{x}(t+\theta)\|$. The superscript "T" represents matrix transposition, and the symmetric terms in a matrix are denoted by "$*$".

1 Problem formulation

Consider a class of SSTD systems of the form:

$$\begin{cases} E\dot{\boldsymbol{x}}(t)=(A_{\sigma(t)}+\triangle A_{\sigma(t)})\boldsymbol{x}(t)+(A_{d\sigma(t)}+\\ \qquad\triangle A_{d\sigma(t)})\boldsymbol{x}(t-d(t))\\ \boldsymbol{x}(t)=\boldsymbol{\phi}(t),\quad t\in[-d_1-d_2,0]\end{cases}\tag{1}$$

where $\boldsymbol{x}(t)\in\mathbf{R}^n$ is the state, $\boldsymbol{\phi}(t)\in\mathcal{C}_{n,d_1+d_2}$ is a compatible vector valued initial function. $\sigma(t):[0,+\infty)\rightarrow\mathcal{I}=\{1,2,\cdots,N\}$ with integer $N>1$ is the switching signal.

Manuscript received May 13, 2010; accepted September 10, 2010
Supported by National Natural Science Foundation of China (60904020, 60835001, 61004032)
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The matrix $E \in \mathbf{R}^{n\times n}$ may be singular and it is assumed that rank$E = r \le n$. $d(t)$ is a time-varying continuous function that satisfies:

$$d_1 \le d(t) \le d_1 + d_2, \quad \dot{d}(t) \le \mu \tag{2}$$

where $d_1 \ge 0$, $d_2 > 0$, and $0 \le \mu < 1$ are constants. For each possible value $\sigma(t) = i$, $i \in \mathcal{I}$, A_i and A_{di} are constant real matrices with appropriate dimensions, and $\triangle A_i$ and $\triangle A_{di}$ are unknown matrices representing parameter uncertainties, and are assumed to be of the form

$$[\ \triangle A_i \quad \triangle A_{di}\ ] = M_i F_i [\ N_{ai} \quad N_{di}\ ] \tag{3}$$

where M_i, N_{ai}, and N_{di} are known real constant matrices, and F_i is the uncertain matrix satisfying

$$F_i^{\mathrm{T}} F_i \le I, \quad i \in \mathcal{I} \tag{4}$$

Since rank$E = r \le n$, there exist non-singular matrices P, $Q \in \mathbf{R}^{n\times n}$ such that $PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. In this paper, without loss of generality, let

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

Corresponding to the switching signal $\sigma(t)$, we denote the switching sequence by $\mathcal{S} = \{(i_0, t_0), \cdots, (i_k, t_k) | i_k \in \mathcal{I}, k = 0, 1, \cdots\}$ with $t_0 = 0$, which means that the i_k-th subsystem is activated when $t \in [t_k, t_{k+1})$.

Definition 1[30, 14]**.** For the switching signal $\sigma(t)$ and any delay $d(t)$ satisfying (2), the nominal part of system (1)

$$\begin{cases} E\dot{\boldsymbol{x}}(t) = A_{\sigma(t)}\boldsymbol{x}(t) + A_{d\sigma(t)}\boldsymbol{x}(t - d(t)) \\ \boldsymbol{x}_{t_0}(\theta) = \boldsymbol{x}(t_0 + \theta), \quad \theta \in [-d_1 - d_2, 0] \end{cases} \tag{6}$$

is said to be

1) regular if $\det(sE - A_i)$ is not identically zero for each $\sigma(t) = i$, $i \in \mathcal{I}$;

2) impulse free if $\deg(\det(sE - A_i)) = $ rank$E$ for each $\sigma(t) = i$, $i \in \mathcal{I}$;

3) exponentially stable under the switching signal $\sigma(t)$ if the solution $\boldsymbol{x}(t)$ of system (6) satisfies $\|\boldsymbol{x}(t)\| \le \iota \mathrm{e}^{-\lambda(t-t_0)} \|\boldsymbol{x}_{t_0}\|_{d_1+d_2}$, $\forall t \ge t_0$, where $\lambda > 0$ and $\iota > 0$ are called the decay rate and decay coefficient, respectively;

4) exponentially admissible if it is regular, impulse free, and exponentially stable under the switching signal $\sigma(t)$.

Definition 2[1]**.** For the switching signal $\sigma(t)$ and any $T_2 > T_1 \ge 0$, let $N_\sigma(T_1, T_2)$ denotes the number of switching of $\sigma(t)$ over (T_1, T_2). If $N_\sigma(T_1, T_2) \le N_0 + (T_2 - T_1)/T_a$ holds for $T_a > 0$, $N_0 \ge 0$, then T_a is called average dwell time. As commonly used in the literature, we choose $N_0 = 0$.

The problem to be addressed in this paper can be formulated as follows: given the SSTD system (1), identify a class of switching signal $\sigma(t)$ such that the system is exponentially admissible under the switching signal $\sigma(t)$.

Lemma 1. For any constant matrix $Z \in \mathbf{R}^{n\times n}$, $Z = Z^{\mathrm{T}} > 0$, positive scalar α, and vector function $\dot{\boldsymbol{x}} : [-\tau, \infty) \to \mathbf{R}^n$ such that the following integration is well defined, then

$$\frac{\mathrm{e}^{\alpha\tau} - 1}{\alpha} \int_{t-d(t)}^{t} \mathrm{e}^{\alpha(s-t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(s) \mathrm{d}s \ge \left(\int_{t-d(t)}^{t} E\dot{\boldsymbol{x}}(s)\mathrm{d}s \right)^{\mathrm{T}} Z \left(\int_{t-d(t)}^{t} E\dot{\boldsymbol{x}}(s)\mathrm{d}s \right), \ t \ge 0$$

where $0 \le d(t) \le \tau$.

Proof. Using Schur complement, we have

$$\begin{bmatrix} \mathrm{e}^{\alpha(s-t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(s) & \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} \\ * & \mathrm{e}^{\alpha(t-s)} Z^{-1} \end{bmatrix} \ge 0$$

Integrating it from $t - d(t)$ to t, we get

$$\begin{bmatrix} \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(s)\mathrm{d}s & \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} \mathrm{d}s \\ * & \dfrac{(\mathrm{e}^{\alpha\tau} - 1) Z^{-1}}{\alpha} \end{bmatrix} \ge 0$$

Using Schur complement again, we find that Lemma 1 holds. □

Lemma 2[31]**.** Given matrices Ω, Γ, and Ξ with appropriate dimensions and with Ω symmetrical, then $\Omega + \Gamma F \Xi + \Xi^{\mathrm{T}} F^{\mathrm{T}} \Gamma^{\mathrm{T}} < 0$ holds for any F satisfying $F^{\mathrm{T}} F \le I$, if and only if there exists a scalar $\varepsilon > 0$ such that $\Omega + \varepsilon \Gamma \Gamma^{\mathrm{T}} + \varepsilon^{-1} \Xi^{\mathrm{T}} \Xi < 0$.

In the following, for representation simplicity, we let

$$\bar{d}_2 = d_1 + d_2$$

2 Main results

First, we apply the average dwell time approach and the piecewise Lyapunov function technique to investigate the exponential admissibility for the SSTD system (6), and give the following result.

Theorem 1. For prescribed scalars $\alpha > 0$, $d_1 \ge 0$, $d_2 > 0$, and $0 \le \mu < 1$, if for each $i \in \mathcal{I}$, there exist matrices $Q_{il} > 0$, $Z_{il} > 0$, $l = 1, 2$, and P_i of the following form

$$P_i = \begin{bmatrix} P_{i11} & 0 \\ P_{i21} & P_{i22} \end{bmatrix} \tag{7}$$

with $P_{i11} \in \mathbf{R}^r$, $P_{i11} > 0$, and P_{i22} being invertible, such that

$$\Phi_i = \begin{bmatrix} \Phi_{i11} & P_i^{\mathrm{T}} A_{di} & \Phi_{i13} & \Phi_{i14} & \Phi_{i15} \\ * & \Phi_{i22} & \Phi_{i23} & \Phi_{i24} & \Phi_{i25} \\ * & * & \Phi_{i33} & 0 & 0 \\ * & * & * & -Z_{i1} & 0 \\ * & * & * & * & -Z_{i2} \end{bmatrix} < 0 \tag{8}$$

where $c_1 = (\alpha d_1)/(\mathrm{e}^{\alpha d_1} - 1)$, $c_2 = (\alpha d_2)/(\mathrm{e}^{\alpha d_2} - 1)$, $\Phi_{i11} = P_i^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_i + \sum_{l=1}^{2} Q_{il} + \alpha E^{\mathrm{T}} P_i - c_1 E^{\mathrm{T}} Z_{i1} E$, $\Phi_{i13} = c_1 E^{\mathrm{T}} Z_{i1} E$, $\Phi_{i14} = d_1 A_i^{\mathrm{T}} Z_{i1}$, $\Phi_{i15} = d_2 \mathrm{e}^{\frac{1}{2}\alpha d_1} A_i^{\mathrm{T}} Z_{i2}$, $\Phi_{i22} = -(1-\mu)\mathrm{e}^{-\alpha \bar{d}_2} Q_{i2} - c_2 E^{\mathrm{T}} Z_{i2} E$, $\Phi_{i23} = c_2 E^{\mathrm{T}} Z_{i2} E$, $\Phi_{i24} = d_1 A_{di}^{\mathrm{T}} Z_{i1}$, $\Phi_{i25} = d_2 \mathrm{e}^{\frac{1}{2}\alpha d_1} A_{di}^{\mathrm{T}} Z_{i2}$, and $\Phi_{i33} = -\mathrm{e}^{-\alpha d_1} Q_{i1} - c_1 E^{\mathrm{T}} Z_{i1} E - c_2 E^{\mathrm{T}} Z_{i2} E$. Then, system (6) with $d(t)$ satisfying (2) is exponentially admissible for any switching sequence $\mathcal{S}$ with average dwell time $T_a \ge T_a^* = (\ln \beta)/\alpha$, where $\beta \ge 1$ satisfies

$$P_{i11} \le \beta P_{j11},\ Q_{il} \le \beta Q_{jl},\ Z_{il} \le \beta Z_{jl}, \quad l = 1, 2,\ \forall i, j \in \mathcal{I} \tag{9}$$

Moreover, an estimate on the exponential decay rate is $\lambda = \frac{1}{2}(\alpha - (\ln \beta)/T_a)$.

Proof. The proof is divided into three parts: 1) to show the regularity and non-impulsiveness; 2) to show the exponential stability of the differential subsystem; 3) to show the exponential stability of the algebraic subsystem.

Part 1): regularity and non-impulsiveness. According to (5), for each $i\in\mathcal{I}$, denote

$$A_i=\left[\begin{array}{cc}A_{i11} & A_{i12}\\ A_{i21} & A_{i22}\end{array}\right],\quad Z_{i1}=\left[\begin{array}{cc}Z_{i111} & Z_{i112}\\ Z_{i112}^{\rm T} & Z_{i122}\end{array}\right] \tag{10}$$

where $A_{i11}\in\mathbf{R}^r$ and $Z_{i111}\in\mathbf{R}^r$. From (8), it is easy to see that $\Phi_{i11}<0$, $i\in\mathcal{I}$. Noting $Q_{il}>0$ and $Z_{il}>0$, $l=1,2$, we get $P_i^{\rm T}A_i+A_i^{\rm T}P_i+\alpha E^{\rm T}P_i-c_1E^{\rm T}Z_{i1}E<0$. Substituting P_i, A_i, Z_{i1}, and E given as (7), (10), and (5) into this inequality and using Schur complement, we have $A_{i22}^{\rm T}P_{i22}+P_{i22}^{\rm T}A_{i22}<0$, which implies that A_{i22}, $i\in\mathcal{I}$, is non-singular. Then by [21] and Definition 1, system (6) is regular and impulse free.

Part 2): exponential stability of the differential subsystem. Define the piecewise Lyapunov functional candidate for system (6) as follows:

$$\begin{aligned}V(\boldsymbol{x}_t)=&V_{\sigma(t)}(\boldsymbol{x}_t)=\\ &\boldsymbol{x}^{\rm T}(t)E^{\rm T}P_{\sigma(t)}\boldsymbol{x}(t)+\\ &\int_{t-d_1}^{t}{\rm e}^{\alpha(s-t)}\boldsymbol{x}^{\rm T}(s)Q_{\sigma(t)1}\boldsymbol{x}(s){\rm d}s+\\ &\int_{t-d(t)}^{t}{\rm e}^{\alpha(s-t)}\boldsymbol{x}^{\rm T}(s)Q_{\sigma(t)2}\boldsymbol{x}(s){\rm d}s+\\ &d_1\int_{-d_1}^{0}\int_{t+\theta}^{t}{\rm e}^{\alpha(s-t)}(E\dot{\boldsymbol{x}}(s))^{\rm T}Z_{\sigma(t)1}(E\dot{\boldsymbol{x}}(s)){\rm d}s{\rm d}\theta+\\ &d_2\int_{-\bar{d}_2}^{-d_1}\int_{t+\theta}^{t}{\rm e}^{\alpha(s-t+d_1)}(E\dot{\boldsymbol{x}}(s))^{\rm T}Z_{\sigma(t)2}(E\dot{\boldsymbol{x}}(s)){\rm d}s{\rm d}\theta\end{aligned} \tag{11}$$

Then, along the solution of system (6) for a fixed $\sigma(t)=i$, $i\in\mathcal{I}$, we have

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}_t)+\alpha V_i(\boldsymbol{x}_t)\le&\\ &2\boldsymbol{x}^{\rm T}(t)P_i^{\rm T}E\dot{\boldsymbol{x}}(t)+\boldsymbol{x}^{\rm T}(t)Q_{i1}\boldsymbol{x}(t)-\\ &{\rm e}^{-\alpha d_1}\boldsymbol{x}^{\rm T}(t-d_1)Q_{i1}\boldsymbol{x}(t-d_1)+\\ &\boldsymbol{x}^{\rm T}(t)Q_{i2}\boldsymbol{x}(t)+\alpha\boldsymbol{x}^{\rm T}(t)E^{\rm T}P_i\boldsymbol{x}(t)-\\ &(1-\mu){\rm e}^{-\alpha\bar{d}_2}\boldsymbol{x}^{\rm T}(t-d(t))Q_{i2}\boldsymbol{x}(t-d(t))+\\ &(E\dot{\boldsymbol{x}}(t))^{\rm T}(d_1^2Z_{i1}+d_2^2{\rm e}^{\alpha d_1}Z_{i2})(E\dot{\boldsymbol{x}}(t))-\\ &d_1\int_{t-d_1}^{t}{\rm e}^{\alpha(s-t)}(E\dot{\boldsymbol{x}}(s))^{\rm T}Z_{i1}(E\dot{\boldsymbol{x}}(s)){\rm d}s-\\ &d_2\int_{t-d(t)}^{t-d_1}{\rm e}^{\alpha(s-t+d_1)}(E\dot{\boldsymbol{x}}(s))^{\rm T}Z_{i2}(E\dot{\boldsymbol{x}}(s)){\rm d}s\end{aligned}$$

By replacing $E\dot{\boldsymbol{x}}(t)$ with $A_i\boldsymbol{x}(t)+A_{di}\boldsymbol{x}(t-d(t))$ and using Lemma 1 and Schur complement, LMI (8) yields

$$\dot{V}_i(\boldsymbol{x}_t)+\alpha V_i(\boldsymbol{x}_t)<0 \tag{12}$$

As mentioned earlier, the i_k-th subsystem is activated when $t\in[t_k,t_{k+1})$. Integrating (12) from t_k to t_{k+1} gives

$$V(\boldsymbol{x}_t)=V_{\sigma(t)}(\boldsymbol{x}_t)\le{\rm e}^{-\alpha(t-t_k)}V_{\sigma(t_k)}(\boldsymbol{x}_{t_k}),\ t\in[t_k,t_{k+1}) \tag{13}$$

Let $\boldsymbol{x}(t)=\left[\begin{array}{c}\boldsymbol{x}_1(t)\\ \boldsymbol{x}_2(t)\end{array}\right]$, where $\boldsymbol{x}_1(t)\in\mathbf{R}^r$ and $\boldsymbol{x}_2(t)\in\mathbf{R}^{n-r}$. From (5) and (7), it can be seen that for each $\sigma(t)=i$, $i\in\mathcal{I}$, $\boldsymbol{x}^{\rm T}(t)E^{\rm T}P_i\boldsymbol{x}(t)=\boldsymbol{x}_1^{\rm T}(t)P_{i11}\boldsymbol{x}_1(t)$. In view of this, and using (9) and (11), at switching instant t_i, we have

$$V_{\sigma(t_i)}(\boldsymbol{x}_{t_i})\le\beta V_{\sigma(t_i^-)}(\boldsymbol{x}_{t_i^-}),\ i=1,2,\cdots \tag{14}$$

where t_i^- denotes the left limitation of t_i. Then, it follows from (13), (14), and the relation $k=N_\sigma(t_0,t)\le(t-t_0)/T_a$ that

$$\begin{aligned}V_{\sigma(t)}(\boldsymbol{x}_t)\le&{\rm e}^{-\alpha(t-t_k)}\beta V_{\sigma(t_i^-)}(\boldsymbol{x}_{t_i^-})\le\cdots\le\\ &{\rm e}^{-\alpha(t-t_0)}\beta^kV_{\sigma(t_0)}(t_0)\le\\ &{\rm e}^{-(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}V_{\sigma(t_0)}(\boldsymbol{x}_{t_0})\end{aligned} \tag{15}$$

According to (11) and (15), we obtain

$$\lambda_1\|\boldsymbol{x}_1(t)\|^2\le V_{\sigma(t)}(t),\quad V_{\sigma(t_0)}(\boldsymbol{x}_{t_0})\le\lambda_2\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2}^2 \tag{16}$$

where $\lambda_1=\min_{\forall i\in\mathcal{I}}\lambda_{\min}(P_{i11})$, and $\lambda_2=\max_{\forall i\in\mathcal{I}}\lambda_{\max}(P_{i11})+\frac{1}{\alpha}(1-{\rm e}^{-\alpha d_1})\max_{\forall i\in\mathcal{I}}\lambda_{\max}(Q_{i1})+\frac{1}{\alpha}(1-{\rm e}^{-\alpha\bar{d}_2})\max_{\forall i\in\mathcal{I}}\lambda_{\max}(Q_{i2})+\frac{d_1}{\alpha^2}(\alpha d_1-1+{\rm e}^{-\alpha d_1})\max_{\forall i\in\mathcal{I}}(2\lambda_{\max}(Z_{i1})(\|A_i\|+\|A_{di}\|))+\frac{1}{\alpha^2}(-d_2+\alpha d_2^2{\rm e}^{\alpha d_1}+d_2{\rm e}^{-\alpha d_2})\max_{\forall i\in\mathcal{I}}(2\lambda_{\max}(Z_{i2})(\|A_i\|+\|A_{di}\|))$. Then, combining (15) with (16) yields

$$\|\boldsymbol{x}_1(t)\|\le\sqrt{\frac{\lambda_2}{\lambda_1}}{\rm e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{17}$$

Part 3): exponential stability of the algebraic subsystem. Set $G_i=\left[\begin{array}{cc}I_r & -A_{i12}A_{i22}^{-1}\\ 0 & A_{i22}^{-1}\end{array}\right]$ and $H=\left[\begin{array}{cc}I_r & 0\\ 0 & I_{n-r}\end{array}\right]$. It is easy to get

$$\begin{aligned}&\hat{E}=G_iEH=\left[\begin{array}{cc}I_r & 0\\ 0 & 0\end{array}\right]\\ &\hat{A}_i=G_iA_iH=\left[\begin{array}{cc}\hat{A}_{i11} & 0\\ \hat{A}_{i21} & I_{n-r}\end{array}\right]\\ &\hat{P}_i=G_i^{-\rm T}P_iH=\left[\begin{array}{cc}\hat{P}_{i11} & 0\\ \hat{P}_{i21} & \hat{P}_{i22}\end{array}\right]\end{aligned} \tag{18}$$

where $\hat{A}_{i11}=A_{i11}-A_{i12}A_{i22}^{-1}A_{i21}$, $\hat{A}_{i21}=A_{i22}^{-1}A_{i21}$, $\hat{P}_{i11}=P_{i11}$, $\hat{P}_{i21}=A_{i12}^{\rm T}P_{i11}+A_{i22}^{\rm T}P_{i21}$, and $\hat{P}_{i22}=A_{i22}^{\rm T}P_{i22}$. According to (18), denote

$$\begin{aligned}&\hat{A}_{di}=G_iA_{di}H=\left[\begin{array}{cc}\hat{A}_{di11} & \hat{A}_{di12}\\ \hat{A}_{di21} & \hat{A}_{di22}\end{array}\right]\\ &\hat{Q}_{il}=H^{\rm T}Q_{il}H=\left[\begin{array}{cc}\hat{Q}_{il11} & \hat{Q}_{il12}\\ \hat{Q}_{il21} & \hat{Q}_{il22}\end{array}\right]\\ &\hat{Z}_{il}=G_i^{-\rm T}Z_{il}G_i^{-1}=\left[\begin{array}{cc}\hat{Z}_{il11} & \hat{Z}_{il12}\\ \hat{Z}_{il21} & \hat{Z}_{il22}\end{array}\right],\ l=1,2\end{aligned} \tag{19}$$

Let $\boldsymbol{\xi}(t)=\left[\begin{array}{c}\boldsymbol{\xi}_1(t)\\ \boldsymbol{\xi}_2(t)\end{array}\right]=H^{-1}\boldsymbol{x}(t)=\boldsymbol{x}(t)$, where $\boldsymbol{\xi}_1(t)\in\mathbf{R}^r$ and $\boldsymbol{\xi}_2(t)\in\mathbf{R}^{n-r}$. Then, for any $\sigma(t)=i$, $i\in\mathcal{I}$, system (6) is restricted system equivalent (r.s.e.) to

$$\begin{aligned}\dot{\boldsymbol{\xi}}_1(t)=&\hat{A}_{i11}\boldsymbol{\xi}_1(t)+\\ &\hat{A}_{di11}\boldsymbol{\xi}_1(t-d(t))+\hat{A}_{di12}\boldsymbol{\xi}_2(t-d(t))\end{aligned} \tag{20}$$

$$\begin{aligned}-\boldsymbol{\xi}_2(t)=&\hat{A}_{i21}\boldsymbol{\xi}_1(t)+\\ &\hat{A}_{di21}\boldsymbol{\xi}_1(t-d(t))+\hat{A}_{di22}\boldsymbol{\xi}_2(t-d(t))\end{aligned} \tag{21}$$

From (8), we have $\left[\begin{array}{cc}\Phi_{i11} & P_i^{\rm T}A_{di}\\ * & \Phi_{i22}\end{array}\right]<0$. Pre- and post-multiplying this inequality by ${\rm diag}\{H^{\rm T},H^{\rm T}\}$ and diag$\{H, H\}$, respectively, noting the expressions in (18) and (19), and using Schur complement, we have

$$\begin{bmatrix} \hat{P}_{i22}^{\rm T} + \hat{P}_{i22} + \sum\limits_{l=1}^{2} \hat{Q}_{il22} & \hat{P}_{i22}^{\rm T}\hat{A}_{di22} \\ * & -(1-\mu){\rm e}^{-\alpha\bar{d}_2}\hat{Q}_{i222} \end{bmatrix} < 0$$

Pre- and post-multiplying this inequality by $[-\hat{A}_{di22}^{\rm T} \quad I]$ and its transpose, respectively, and noting $\hat{Q}_{i122} > 0$ and $0 \le \mu < 1$, we obtain $\hat{A}_{di22}^{\rm T}\hat{Q}_{i222}\hat{A}_{di22} - {\rm e}^{-\alpha\bar{d}_2}\hat{Q}_{i222} < 0$. Then, according to Lemma 7 in [32], we can deduce that there exist constants $\hbar_i > 1$ and $\eta_i > 0$ such that

$$\|({\rm e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{di22})^l\| \le \hbar_i {\rm e}^{-\eta_i l}, \quad l = 0, 1, \cdots \tag{22}$$

Define

$$t^0 = t, \quad t^j = t^{j-1} - d(t^{j-1}), \ j = 1, 2, \cdots \tag{23}$$

$$\|\hat{A}_{21}\| = \max_{\forall i \in \mathcal{I}} \|\hat{A}_{i21}\|, \quad \|\hat{A}_{d21}\| = \max_{\forall i \in \mathcal{I}} \|\hat{A}_{di21}\|,$$
$$\|\hat{A}_{d22}\| = \max_{\forall i \in \mathcal{I}} \|\hat{A}_{di22}\|, \quad \forall i \in \mathcal{I} \tag{24}$$

As mentioned above, for $t \in [t_k, t_{k+1})$, the i_k-th subsystem is activated. Then, from (21) and (23), we have

$$\boldsymbol{\xi}_2(t) = -\hat{A}_{i_k21}\boldsymbol{\xi}_1(t^0) - \hat{A}_{di_k21}\boldsymbol{\xi}_1(t^1) - \hat{A}_{di_k22}\boldsymbol{\xi}_2(t^1) \tag{25}$$

Similarly, it can be obtained that $\boldsymbol{\xi}_2(t^1) = -\hat{A}_{i_k21}\boldsymbol{\xi}_1(t^1) - \hat{A}_{di_k21}\boldsymbol{\xi}_1(t^2) - \hat{A}_{di_k22}\boldsymbol{\xi}_2(t^2)$. Substituting this into (25), we get $\boldsymbol{\xi}_2(t) = (-\hat{A}_{di_k22})^2\boldsymbol{\xi}_2(t^2) - \sum_{j=0}^{1}(-\hat{A}_{di_k22})^j(\hat{A}_{i_k21}\boldsymbol{\xi}_1(t^j) + \hat{A}_{di_k21}\boldsymbol{\xi}_1(t^{j+1}))$. Continuing in the same manner and noting that $t^j < t^{j-1}$, then there exists a finite positive integer T_{i_k} such that

$$\boldsymbol{\xi}_2(t) = (-\hat{A}_{di_k22})^{T_{i_k}}\boldsymbol{\xi}_2(t^{T_{i_k}}) - \sum_{j_{i_k}=0}^{T_{i_k}-1}(-\hat{A}_{di_k22})^{j_{i_k}} \times$$
$$(\hat{A}_{i_k21}\boldsymbol{\xi}_1(t^{j_{i_k}}) + \hat{A}_{di_k21}\boldsymbol{\xi}_1(t^{j_{i_k}+1})) \tag{26}$$

where $t^{T_{i_k}} \in (t_{k-1}, t_k]$ and $t^{T_{i_k}} \to t_k$. When $t \in [t_{k-1}, t_k)$, the i_{k-1}-th subsystem is activated. Then, following a similar procedure as the above, there exists a finite positive integer $T_{i_{k-1}}$ such that

$$\boldsymbol{\xi}_2(t^{T_{i_k}}) = (-\hat{A}_{di_{k-1}22})^{T_{i_{k-1}}}\boldsymbol{\xi}_2(t^{T_{i_k}+T_{i_{k-1}}}) -$$
$$\sum_{j_{i_{k-1}}=T_{i_k}}^{T_{i_k}+T_{i_{k-1}}-1}(-\hat{A}_{di_{k-1}22})^{j_{i_{k-1}}-T_{i_k}} \times$$
$$(\hat{A}_{i_{k-1}21}\boldsymbol{\xi}_1(t^{j_{i_{k-1}}}) + \hat{A}_{di_{k-1}21}\boldsymbol{\xi}_1(t^{j_{i_{k-1}}+1}))$$

where $t^{T_{i_k}+T_{i_{k-1}}} \in (t_{k-2}, t_{k-1}]$ and $t^{T_{i_k}+T_{i_{k-1}}} \to t_{k-1}$. After k-times iterative manipulations, t belongs to $[t_0, t_1)$, and there exists a finite positive integer T_{i_0} such that

$$\boldsymbol{\xi}_2(t^{T_{i_k}+\cdots+T_{i_1}}) = (-\hat{A}_{di_022})^{T_{i_0}}\boldsymbol{\xi}_2(t^{T_{i_k}+\cdots+T_{i_0}}) -$$
$$\sum_{j_{i_0}=T_{i_k}+\cdots+T_{i_1}}^{T_{i_k}+\cdots+T_{i_0}-1}(-\hat{A}_{di_022})^{j_{i_0}-T_{i_k}-\cdots-T_{i_1}} \times$$
$$(\hat{A}_{i_021}\boldsymbol{\xi}_1(t^{j_{i_0}}) + \hat{A}_{di_021}\boldsymbol{\xi}_1(t^{j_{i_0}+1}))$$

where $t^{T_{i_k}+\cdots+T_{i_0}} \in (-\bar{d}_2, t_0]$ and $t^{T_{i_k}+\cdots+T_{i_0}} \to t_0$. By a simple induction, we have

$$\boldsymbol{\xi}_2(t) = \left[\prod_{j=0}^{k}(-\hat{A}_{di_j22})^{T_{i_j}}\right]\boldsymbol{\xi}_2(t^{T_{i_k}+\cdots+T_{i_1}+T_{i_0}}) -$$
$$\sum_{j_{i_k}=0}^{T_{i_k}-1}(-\hat{A}_{di_k22})^{j_{i_k}}\hat{A}_{i_k21}\boldsymbol{\xi}_1(t^{j_{i_k}}) -$$
$$\sum_{j_{i_k}=0}^{T_{i_k}-1}(-\hat{A}_{di_k22})^{j_{i_k}}\hat{A}_{di_k21}\boldsymbol{\xi}_1(t^{j_{i_k}+1}) -$$
$$\sum_{p=1}^{k}\left[\prod_{q=p}^{k}(-\hat{A}_{di_q22})^{T_{i_q}}\right] \times$$
$$\sum_{j_{i_{p-1}}=T_{i_k}+\cdots+T_{i_p}}^{T_{i_k}+\cdots+T_{i_{p-1}}-1}(\boldsymbol{\varphi}_1(t) + \boldsymbol{\varphi}_2(t)) \tag{27}$$

where

$$\boldsymbol{\varphi}_1(t) = (-\hat{A}_{di_{p-1}22})^{j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p}}\hat{A}_{i_{p-1}21}\boldsymbol{\xi}_1(t^{j_{i_{p-1}}})$$
$$\boldsymbol{\varphi}_2(t) = (-\hat{A}_{di_{p-1}22})^{j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p}}\hat{A}_{di_{p-1}21}\boldsymbol{\xi}_1(t^{j_{i_{p-1}}+1})$$

Therefore, from (24) and (27), and noting $t^{T_{i_k}+\cdots+T_{i_0}} \in (-\bar{d}_2, t_0]$, we obtain

$$\|\boldsymbol{\xi}_2(t)\| \le \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \tag{28}$$

where

$$\Delta_1 = \left[\prod_{j=0}^{k}\|(\hat{A}_{di_j22})^{T_{i_j}}\|\right]\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2}$$
$$\Delta_2 = \hat{A}_{21}\sum_{j_{i_k}=0}^{T_{i_k}-1}\|(\hat{A}_{di_k22})^{j_{i_k}}\|\|\boldsymbol{\xi}_1(t^{j_{i_k}})\|$$
$$\Delta_3 = \hat{A}_{d21}\sum_{j_{i_k}=0}^{T_{i_k}-1}\|(\hat{A}_{di_k22})^{j_{i_k}}\|\|\boldsymbol{\xi}_1(t^{j_{i_k}+1})\|$$
$$\Delta_4 = \hat{A}_{21}\sum_{p=1}^{k}\left\{\left[\prod_{q=p}^{k}\|(\hat{A}_{di_q22})^{T_{i_q}}\|\right]\varphi_1'\right\}$$
$$\Delta_5 = \hat{A}_{d21}\sum_{p=1}^{k}\left\{\left[\prod_{q=p}^{k}\|(\hat{A}_{di_q22})^{T_{i_q}}\|\right]\varphi_2'\right\}$$

with

$$\varphi_1' = \sum_{j_{i_{p-1}}=T_{i_k}+\cdots+T_{i_p}}^{T_{i_k}+\cdots+T_{i_{p-1}}-1}\|(\hat{A}_{di_{p-1}22})^{j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p}}\| \times$$
$$\|\boldsymbol{\xi}_1(t^{j_{i_{p-1}}})\|$$

$$\varphi_2' = \sum_{j_{i_{p-1}}=T_{i_k}+\cdots+T_{i_p}}^{T_{i_k}+\cdots+T_{i_{p-1}}-1}\|(\hat{A}_{di_{p-1}22})^{j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p}}\| \times$$
$$\|\boldsymbol{\xi}_1(t^{j_{i_{p-1}}+1})\|$$

Note

$$t_0 \geq t^{T_{i_k}+\cdots+T_{i_0}} = t - \sum_{j=0}^{T_{i_k}+\cdots+T_{i_0}-1} d(t^j) \geq t - (T_{i_k} + \cdots + T_{i_0})\bar{d}_2$$

Using (22) and the relation $T_a \geq T_a^* = (\ln\beta)/\alpha$, the first term in (28) can be estimated as

$$\Delta_1 = \left[\prod_{j=0}^{k} \|(\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{di_j22})^{T_{i_j}}\|\mathrm{e}^{-\frac{1}{2}\alpha(T_{i_k}+\cdots+T_{i_0})\bar{d}_2}\right] \times \|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \left[\prod_{j=0}^{k} \hbar_{i_j}\mathrm{e}^{-\eta_{i_j}T_{i_j}}\right]\mathrm{e}^{-\frac{1}{2}\alpha(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \left[\prod_{j=0}^{k} \hbar_{i_j}\mathrm{e}^{-\eta_{i_j}T_{i_j}}\right]\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} = \chi_1\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{29}$$

By (17) and (22) $\sim$ (24), we get

$$\|(\hat{A}_{di_k22})^{j_{i_k}}\|\|\boldsymbol{\xi}_1(t^{j_{i_k}})\| \leq \|(\hat{A}_{di_k22})^{j_{i_k}}\|\sqrt{\frac{\lambda_2}{\lambda_1}}\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t^{j_{i_k}}-t_0)} \times \|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}(\hat{A}_{di_k22})^{j_{i_k}}\| \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t^{j_{i_k}-1}-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \cdots \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|(\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{di_k22})^{j_{i_k}}\| \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t^0-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\hbar_{i_k}\mathrm{e}^{-\eta_{i_k}j_{i_k}}\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{30}$$

Then, the second term in (28) can be estimated as

$$\Delta_2 \leq \hat{A}_{21}\sqrt{\frac{\lambda_2}{\lambda_1}}\left[\sum_{j_{i_k}=0}^{T_{i_k}-1}\hbar_{i_k}\mathrm{e}^{-\eta_{i_k}j_{i_k}}\right] \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \hbar_{i_k}\hat{A}_{21}\sqrt{\frac{\lambda_2}{\lambda_1}}\frac{\mathrm{e}^{\eta_{i_k}}}{\mathrm{e}^{\eta_{i_k}}-1} \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} = \chi_2\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{31}$$

Similarly, the third term in (28) can be bounded by

$$\Delta_3 \leq \hbar_{i_k}\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{d21}\sqrt{\frac{\lambda_2}{\lambda_1}}\frac{\mathrm{e}^{\eta_{i_k}}}{\mathrm{e}^{\eta_{i_k}}-1} \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} = \chi_3\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{32}$$

On the other hand, following a similar deduction as that in (30), we obtain

$$\varphi_1' \leq \sqrt{\frac{\lambda_2}{\lambda_1}}(\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2})^{T_{i_k}+\cdots+T_{i_p}}(\hbar_{i_{p-1}} \times \mathrm{e}^{-\eta_{i_{p-1}}(j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p})}) \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2}$$

Then, considering this and (22), the fourth term in (28) can be estimated as

$$\Delta_4 \leq \hat{A}_{21}\sqrt{\frac{\lambda_2}{\lambda_1}}\sum_{p=1}^{k}\left[\prod_{q=p}^{k}\|(\mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{di_q22})^{T_{i_q}}\|\right] \times \sum_{j_{i_{p-1}}=T_{i_k}+\cdots+T_{i_p}}^{T_{i_k}+\cdots+T_{i_{p-1}}-1}\hbar_{i_{p-1}}\mathrm{e}^{-\eta_{i_{p-1}}(j_{i_{p-1}}-T_{i_k}-\cdots-T_{i_p})} \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \leq \hat{A}_{21}\sqrt{\frac{\lambda_2}{\lambda_1}}\sum_{p=1}^{k}\hbar_{i_{p-1}}\left[\prod_{q=p}^{k}\hbar_{i_q}\mathrm{e}^{-\eta_{i_q}T_{i_q}}\right] \times \frac{\mathrm{e}^{\eta_{i_{p-1}}}}{\mathrm{e}^{\eta_{i_{p-1}}}-1}\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} = \chi_4\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{33}$$

Similarly, the fifth term in (28) can be bounded by

$$\Delta_5 \leq \mathrm{e}^{\frac{1}{2}\alpha\bar{d}_2}\hat{A}_{d21}\sqrt{\frac{\lambda_2}{\lambda_1}} \times \sum_{p=1}^{k}\left\{\hbar_{i_{p-1}}\left[\prod_{q=p}^{k}\hbar_{i_q}\mathrm{e}^{-\eta_{i_q}T_{i_q}}\right]\frac{\mathrm{e}^{\eta_{i_{p-1}}}}{\mathrm{e}^{\eta_{i_{p-1}}}-1}\right\} \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} = \chi_5\mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{34}$$

Therefore, using (29), (31) $\sim$ (34), $\|\boldsymbol{\xi}_2(t)\|$ can be estimated as

$$\|\boldsymbol{\xi}_2(t)\| \leq (\chi_1+\chi_2+\chi_3+\chi_4+\chi_5) \times \mathrm{e}^{-\frac{1}{2}(\alpha-\frac{\ln\beta}{T_a})(t-t_0)}\|\boldsymbol{x}_{t_0}\|_{\bar{d}_2} \tag{35}$$

Combining (17) and (35) yields that system (6) is exponentially stable for any switching sequence $\mathcal{S}$ with average dwell time $T_a \geq T_a^* = (\ln\beta)/\alpha$. □

Remark 1. In terms of LMIs, Theorem 1 presents a delay-range-dependent exponential admissibility condition for the switched singular systems with interval time-varying delay. It is noted that this condition is obtained by using the integral inequality (Lemma 1); no free-weighting matrices are introduced in the derivation of Theorem 1. Therefore, the condition proposed here involves much less decision variables than those obtained by using the free-weighting matrices method[14, 17, 19−20] if the same Lyapunov function is chosen. It is also noted that the Lyapunov function (11) not only makes use of the information on the time-delay upper bound $\bar{d}_2$, but also uses the information on the time-delay lower bound d_1 and the time-delay interval d_2. Therefore, the stability condition in Theorem 1 is expected to be less conservative.

Remark 2. Equation (27) plays an important role in analyzing the exponential stability of the algebraic subsystem, which can be seen as a generalization of the iterative

Table 1 Comparison of allowable upper bound $\bar{d}_2$ for different d_1 in Example 1

Methods	$\bar{d}_2$			Number of variables
Lemma 1[19]	1.130 ($d_1 = 0.1$)	1.099 ($d_1 = 0.3$)	1.084 ($d_1 = 0.7$)	84
Theorem 1[20]	1.130 ($d_1 = 0.1$)	1.099 ($d_1 = 0.3$)	1.084 ($d_1 = 0.7$)	84
Theorem 1	1.134 ($d_1 = 0.1$)	1.133 ($d_1 = 0.3$)	1.133 ($d_1 = 0.7$)	30

equation in [30] for non-switched singular time-delay system to SSTD system.

Remark 3. If $\beta = 1$ in $T_a \geq T_a^* = (\ln\beta)/\alpha$, which leads to $P_{i11} \equiv P_{j11}$, $Q_{il} \equiv Q_{jl}$, $Z_{il} \equiv Z_{jl}$, $l = 1, 2$, $\forall i, j \in \mathcal{I}$, and $T_a^* = 0$, then system (6) possesses a common Lyapunov function and the switching signals can be arbitrary.

Now, extending Theorem 1 to uncertain system (1) yields the following theorem.

Theorem 2. For prescribed scalars $\alpha > 0$, $d_1 \geq 0$, $d_2 > 0$, and $0 \leq \mu < 1$, if for each $i \in \mathcal{I}$, there exist matrices P_i of (7), $Q_{il} > 0$, $Z_{il} > 0$, $l = 1, 2$, and scalar $\varepsilon_i > 0$, such that

$$\begin{bmatrix} \Phi_i & \Gamma_i & \varepsilon_i \Xi_i^{\mathrm{T}} \\ * & -\varepsilon_i I & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0 \tag{36}$$

where Φ_i follows the same definition as that in Theorem 1, $\Gamma_i = [\ M_i^{\mathrm{T}} P_i \quad 0 \quad 0 \quad d_1 M_i^{\mathrm{T}} Z_{i1} \quad d_2 \mathrm{e}^{\frac{1}{2}\alpha d_1} M_i^{\mathrm{T}} Z_{i2}\ ]^{\mathrm{T}}$ and $\Xi_i = [\ N_{ai} \quad N_{di} \quad 0 \quad 0 \quad 0\ ]$. Then, system (1) with $d(t)$ satisfying (2) is robustly exponentially admissible for any switching sequence $\mathcal{S}$ with average dwell time $T_a \geq T_a^* = (\ln\beta)/\alpha$, where $\beta \geq 1$ satisfies (9). Moreover, an estimate on the exponential decay rate is $\lambda = \frac{1}{2}(\alpha - (\ln\beta)/T_a)$.

Proof. By Theorem 1 and Lemma 2 and using the idea of generalized quadratic stability, Theorem 2 can be easily proved. So the proof is omitted. □

Remark 4. In this paper, the derivative matrix E is assumed to be switch-mode-independent. If E is also switch-mode-dependent, then E is changed to E_i, $i \in \mathcal{I}$, and the transformation matrices P and Q should become P_i and Q_i so that $P_i E_i Q_i = \begin{bmatrix} I_{r_i} & 0 \\ * & 0 \end{bmatrix}$. In this case, the state of the transformed system becomes $\tilde{\boldsymbol{x}}(t) = Q_i^{-1}(t) = [\tilde{\boldsymbol{x}}_{i1}^{\mathrm{T}}(t) \quad \tilde{\boldsymbol{x}}_{i2}^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $\tilde{\boldsymbol{x}}_{i1}^{\mathrm{T}}(t) \in \mathbf{R}^{r_i}$ and $\tilde{\boldsymbol{x}}_{i1}^{\mathrm{T}}(t) \in \mathbf{R}^{n-r_i}$, which means that there does not exist one common state space coordinate basis for different subsystems; thus it is rather complicated to discuss the transformed system. Hence, some assumptions for E_i (for example, E_i, $i \in \mathcal{I}$, have the same right zero subspace[22]) should be given so that Q_i remains the same; in this case, the method presented in this paper is also valid. Nonetheless, the general case with E being switch-mode-dependent is an interesting problem for future investigation via other methods.

3 Numerical examples

Example 1. Consider the switched system (6) with $E = I$, $N = 2$, and the following parameters, which are borrowed from[19]

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

For $\mu = 0.4$, $\alpha = 0.5$, and $\beta = 1.1$, employing the LMIs in [19−20] and those in Theorem 1 yields an allowable upper bound $\bar{d}_2$ of the delay $d(t)$ that guarantees the stability of system (6). Table 1 shows the values of $\bar{d}_2$ for various d_1 and the number of involved variables by using different methods. It is clear that Theorem 1 of this paper not only gives better results than the criteria in [19−20] but also reduces the computational complexity to some extent.

Example 2. Consider the switched system (1) with $N = 2$ and the following parameters:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.73 & 0 \\ 0 & -1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix}$$
$$M_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ N_{a1} = [\ 0.1 \quad 1\ ],\ N_{d1} = [\ 0.1 \quad 0.1\ ]$$
$$A_2 = \begin{bmatrix} 0.4 & 0 \\ -0.1 & -1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1 & 0.1 \\ 0 & 0.1 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix},\ N_{a2} = [\ 0.2 \quad 0.5\ ],\ N_{d2} = [\ 0.1 \quad 0.2\ ]$$

and $d_1 = 0.1$, $d_2 = 0.1$, $\mu = 0.3$, and $\alpha = 0.4$. It can be checked that the above two subsystems are both stable independently. Let $\beta = 1$; by simulation, it can be found that there is no feasible solution to this case, which means that there is no common Lyapunov function for all subsystems (see Remark 3). Now, we consider the average dwell time scheme. By analysis, it can be found that the allowable minimum of β is $\beta_{\min} = 1.19$ when $\alpha = 0.4$ is fixed; in this case $T_a^* = (\ln\beta_{\min})/\alpha = 0.4349$.

4 Conclusions

In this paper, the problem of robust exponential admissibility for a class of continuous-time uncertain switched singular systems with interval time-varying delay has been investigated. A class of switching signals has been identified for the switched singular time-delay systems to be robustly exponentially admissible under the average dwell time scheme. Numerical examples have been provided to demonstrate the effectiveness of the obtained results.

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