Discretization of Continuous-time Systems with Input Delays

ZENG Li[1] HU Guang-Da[1]

Abstract In this paper, the Runge-Kutta (RK) method, which involves the polynomial interpolation is adopted to discretize continuous-time systems with input delay. The proposed scheme is an efficient and higher-order approach compared with conventional discretizing methods. The accuracy of the proposed conversion scheme is closely related to the order of RK as well as that of the polynomial interpolation. Both the approximate order and the maximal attainable order of the discretization are discussed. In addition, the input-to-state stability of the scheme is analyzed. In order to guarantee the stability of the corresponding discrete system, the sampling time can be chosen by investigating the absolute stability region of the RK method. Especially, when the RK method is A-stable, the sampling time can be selected without being constrained by stability considerations. A numerical experiment is provided to demonstrate the superior performance of the method.

Key words Runge-Kutta (RK) method, delay system, discretization, interpolation, A-stable, input-to-state stability (ISS)

DOI 10.3724/SP.J.1004.2010.01426

This paper deals with discretizing continuous-time systems with input delay by using Runge-Kutta (RK) methods. There are many methods for converting a continuous-time system to a discrete equivalence. The commonly used methods include Tustin approximation with or without frequency pre-warping, the impulse-invariance method, zero-pole mapping method, and the triangle hold equivalent, and so on. Some other approaches such as the higher-order s-to-z mapping method and weighed-sample method were analyzed in [1−4]. These conventional methods[5] are straightforward and applicable to delay-free cases. When a system contains delay, the conversion will be quite complex using classical methods.

Recently, some novel approaches were presented in [6−11] to tackle time-delayed systems. Both the matrix exponential and the Taylor series discretization methods in [7] considered that the input signal was piecewise constant over the sampling interval by the zero-hold assumption. Apart from the influences caused by the choice of parameters and the truncation order in the above two methods, the accuracy was lost due to the inaccurate piecewise constant input. Similarly, [8−11] adopted the zero-order and second-order holders to keep the input constant during the sampling intervals, respectively. The frequency domain recursive least square (RLS) based method in [6] was to obtain the discrete equivalence whose frequency response fitted that of the continuous one. Its result was fairly desirable. But the stability of discrete equivalence cannot be guaranteed and the RLS may give different local optimal convergent solutions with different initial points. Though RLS in [6] used the bilinear equivalence as the initial discrete model for its maintenance of stability during continuous and discrete conversion, this skill could not solve the stability problem essentially. Obviously, a fairly satisfying result can cost heavy computational efforts.

We apply the generally known RK methods to continuous-time systems with input delays. High accuracy and stability are the merits of the RK method. It should be mentioned that if the RK method is A-stable, the stability region includes all the left half-plane. The sampling time can be selected without being constrained by stability considerations. As for the discretizing, the Lagrange polynomial interpolation formula is used to approximate the non-integer step values of delayed input signals. The accuracy of the scheme can be maintained if the order of interpolating equals or exceeds that of the RK method itself. In addition, we analyze the maximal attainable order of the proposed scheme for discretizing. Furthermore, if the original continuous-time system is stable, we hope its discrete equivalence is also stable. So, the input-to-state stability (ISS) of the scheme is discussed.

The organization of this paper is as follows. RK method with polynomial interpolation for discretizing is presented in the next section. The approximate order of the scheme is derived in Section 2. The ISS of the scheme is discussed in Section 3. In Section 4, a numerical experiment is provided.

1 Using RK methods with polynomial interpolation for discretizing

We are concerned with an SISO continuous-time system with input delay whose transfer function is

$$H(s)=\frac{m_{d-1}s^{d-1}+\cdots+m_1s+m_0}{s^d+l_{d-1}s^{d-1}+\cdots+l_1s+l_0}\mathrm{e}^{-\tau s} \tag{1}$$

where l_i and m_i, $i=0,1,\cdots,d-1$ are real coefficients and τ is the time delay. Express this transfer function in the controllable canonical form defined in [12] as follows:

$$\begin{cases}\dot{\boldsymbol{x}}(t)=L\boldsymbol{x}(t)+\boldsymbol{G}u(t-\tau)\\ y(t)=\boldsymbol{M}\boldsymbol{x}(t)\end{cases} \tag{2}$$

where

$$L=\begin{bmatrix}0&1&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\-l_0&-l_1&\cdots&-l_{d-1}\end{bmatrix},\quad \boldsymbol{G}=\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

$$\boldsymbol{M}=\begin{bmatrix}m_0&m_1&\cdots&m_{d-1}\end{bmatrix}$$

We always assume that the initial vector $\boldsymbol{x}(0)$ is known throughout this paper.

In this section, we are concerned with applying the RK method to the continuous-time system with input delay (2).

We already know that an s-stage RK method is completely specified by its-Butcher array in which

$$\boldsymbol{c}=[c_1,c_2,\cdots,c_s]^{\mathrm{T}},\quad \boldsymbol{b}=[b_1,b_2,\cdots,b_s]^{\mathrm{T}},\quad A=[a_{ij}]$$

Manuscript received June 30, 2010; accepted July 20, 2010

1. Department of Electronics and Information Engineering, School of Information Engineering, University of Science and Technology Beijing, Beijing 100083, P. R. China

and the row-sum condition

$$c_i = \sum_{j=1}^{s} a_{ij}, \quad i = 1, 2, \cdots, s$$

holds. Applying an s-stage RK method to problem (2), we have

$$\begin{cases} \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + h \sum\limits_{i=1}^{s} b_i \boldsymbol{g}_i \\ \boldsymbol{g}_i = L(\boldsymbol{x}_n + h \sum\limits_{j=1}^{s} a_{ij} \boldsymbol{g}_j) + \boldsymbol{G} u(nh - \tau + c_i h), \\ \qquad i = 1, 2, \cdots, s \end{cases} \tag{3}$$

where $\boldsymbol{x}_n$ stands for the state vector of the discrete-time system (3) at $t_n = nh$, while $\boldsymbol{x}(n)$ stands for the exact solution $\boldsymbol{x}(t_n)$ of continuous-time system (2) at $t_n = nh$, and h denotes a constant step-size.

The discrete-time system (3) can be rewritten by the Kronecker product as follows:

$$\begin{bmatrix} I_{sd} - h(A \otimes L) & 0 \\ -\boldsymbol{b}^{\mathrm{T}} \otimes I_d & I_d \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_n \\ \boldsymbol{x}_{n+1} \end{bmatrix} - \begin{bmatrix} 0 & h(\boldsymbol{e}_s \otimes L) \\ 0 & I_d \end{bmatrix} \times$$

$$\begin{bmatrix} \boldsymbol{K}_{n-1} \\ \boldsymbol{x}_n \end{bmatrix} - \begin{bmatrix} hI_s \otimes \boldsymbol{G} \\ 0 \end{bmatrix} \boldsymbol{U}_n = 0 \tag{4}$$

where $\boldsymbol{K}_{n,i} = h\boldsymbol{g}_i$, $\boldsymbol{K}_n = [\boldsymbol{K}_{n,1}, \boldsymbol{K}_{n,2}, \cdots, \boldsymbol{K}_{n,s}]^{\mathrm{T}}$, $\boldsymbol{e}_s = [1, 1, \cdots, 1]^{\mathrm{T}}$, and $\boldsymbol{U}_n = [u(nh - \tau + c_1 h), u(nh - \tau + c_2 h), \cdots, u(nh - \tau + c_s h)]^{\mathrm{T}}$.

Remark 1. Let

$$V = \begin{bmatrix} I_{sd} - h(A \otimes L) & 0 \\ -\boldsymbol{b}^{\mathrm{T}} \otimes I_d & I_d \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & h(\boldsymbol{e}_s \otimes L) \\ 0 & I_d \end{bmatrix}$$

$$T = \begin{bmatrix} hI_s \otimes \boldsymbol{G} \\ 0 \end{bmatrix}, \quad \boldsymbol{Z}_n = \begin{bmatrix} \boldsymbol{K}_{n-1} \\ \boldsymbol{x}_n \end{bmatrix}$$

When the RK method is explicit or A-stable, V^{-1} exists. Equation (4) can be rewritten as follows:

$$\boldsymbol{Z}_{n+1} = V^{-1} W \boldsymbol{Z}_n + V^{-1} T \boldsymbol{U}_n \tag{5}$$

With the help of the rule of solving the inverse of a partitioned matrix, we derive from (5) that

$$\boldsymbol{Z}_{n+1} = \begin{bmatrix} 0 & h[I_{sd} - h(A \otimes L)]^{-1}(\boldsymbol{e}_s \otimes L) \\ 0 & I_d + h(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)[I_{sd} - h(A \otimes L)]^{-1}(\boldsymbol{e}_s \otimes L) \end{bmatrix}$$

$$\boldsymbol{Z}_n + \begin{bmatrix} h[I_{sd} - h(A \otimes L)]^{-1}(I_s \otimes \boldsymbol{G}) \\ h(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)[I_{sd} - h(A \otimes L)]^{-1}(I_s \otimes \boldsymbol{G}) \end{bmatrix} \boldsymbol{U}_n \tag{6}$$

Let

$$S_1 = h[I_{sd} - h(A \otimes L)]^{-1}(\boldsymbol{e}_s \otimes L)$$
$$S_2 = I_d + h(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)[I_{sd} - h(A \otimes L)]^{-1}(\boldsymbol{e}_s \otimes L)$$
$$N_1 = h[I_{sd} - h(A \otimes L)]^{-1}(I_s \otimes \boldsymbol{G})$$
$$N_2 = h(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)[I_{sd} - h(A \otimes L)]^{-1}(I_s \otimes \boldsymbol{G})$$

We obtain the $\boldsymbol{x}_{n+1}$ part in (6) as follows:

$$\boldsymbol{x}_{n+1} = S_2 \boldsymbol{x}_n + N_2 \boldsymbol{U}_n \tag{7}$$

In the discrete-time system (3), components of input vectors $u(nh - \tau + c_i h)$, $i = 1, 2, \cdots, s$ include the noninteger step values. But a discrete system can only sample integral step values of the input signal $u(nh)$. It is necessary to use interpolation technique to approximate the term $u(nh - \tau + c_i h)$ for discretization with high-performance.

The Lagrange polynomial interpolation is used to approximate the noninteger step values of the input signal $u(nh - \tau + c_i h)$ as follows:

$$\tau = Dh + \gamma, \quad D = \{0, 1, 2, \cdots\}, \quad 0 \le \gamma < h$$

$$u(nh - \tau + \delta h) = \sum_{j=-r}^{0} L_j(\delta) u_{n-D+j} + o(h^{r+1}) \tag{8}$$

where $\delta \in [0, 1]$, $n = 0, 1, \cdots$, and $o(h^{r+1})$ is the interpolation residual, and

$$L_j(\delta) = \prod_{k=-r, j \ne k}^{0} \frac{nh - \tau + \delta h - (nh - Dh + kh)}{(nh - Dh + jh) - (nh - Dh + kh)} = \prod_{k=-r, j \ne k}^{0} \frac{\delta h - \tau + Dh - kh}{jh - kh}$$

Let

$$\bar{\boldsymbol{U}}_{n-D} = [u_{n-D-r}, u_{n-D-r+1}, \cdots, u_{n-D}]^{\mathrm{T}}$$

We have

$$\boldsymbol{U}_n = Q \bar{\boldsymbol{U}}_{n-D} + \boldsymbol{o}(h^{r+1}) \tag{9}$$

where

$$Q = \begin{bmatrix} L_{-r}(c_1) & L_{-r+1}(c_1) & \cdots & L_0(c_1) \\ L_{-r}(c_2) & L_{-r+1}(c_2) & \cdots & L_0(c_2) \\ \vdots & \vdots & \ddots & \vdots \\ L_{-r}(c_s) & L_{-r+1}(c_s) & \cdots & L_0(c_s) \end{bmatrix}$$

Then, we use $Q\bar{\boldsymbol{U}}_{n-D}$ to approximate $\boldsymbol{U}_n$ in (7) as

$$\boldsymbol{x}_{n+1} = S_2 \boldsymbol{x}_n + N_2 Q \bar{\boldsymbol{U}}_{n-D} \tag{10}$$

Remark 2. If the RK method is of order q, $r \ge q - 1$ is taken to guarantee the order of the scheme being q. This is the conclusion described in Theorem 1.

2 Approximate order of RK method with polynomial interpolation

In this section, we discuss the accuracy of the discretization with the RK method. As the Lagrange polynomial interpolation formula is used to approximate the input signal $u(nh - \tau + c_i h)$, the accuracy of the scheme is closely related to that of the RK as well as interpolation.

Definition 1[13]**.** The local truncation error $\boldsymbol{T}_{n+1}$ of (3) at $t_{n+1} = (n + 1)h$ is defined to be the residual when $\boldsymbol{x}_{n+j}$ is replaced by $\boldsymbol{x}(t_{n+j})$, $j = 0, 1$, i.e., $\boldsymbol{T}_{n+1} = \boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n) - h \sum_{i=1}^{s} b_i \boldsymbol{g}_i$, where $\boldsymbol{x}(t)$ is the exact solution of problem (2).

Definition 2[13]**.** If q is the largest integer such that $\boldsymbol{T}_{n+1} = \boldsymbol{o}(h^{q+1})$, then we say that the method has order q.

Remark 3. If we denote by $\boldsymbol{x}_{n+1}$ the value at t_{n+1} generated by the RK method when the localizing assumption that $\boldsymbol{x}(t_n) = \boldsymbol{x}_n$ is made, from the definition of local truncation error, then we have that $\boldsymbol{T}_{n+1} = \boldsymbol{x}(t_{n+1}) - \boldsymbol{x}_{n+1}$.

Remark 4. If the RK method is of order q, and the localizing assumption $\boldsymbol{x}(t_n) = \boldsymbol{x}_n$ is made, then we have $\boldsymbol{x}(t_{n+1}) = \boldsymbol{x}_{n+1} + \boldsymbol{o}(h^{q+1})$.

Theorem 1. When an RK method of order q is applied to problem (2), the order of the interpolation for input signal should be equal to or higher than that of the RK method in order to guarantee the order of the scheme is q, that is, the number of the interpolating points should be at least q or larger, namely, $r \geq q - 1$.

Proof. From (10), it follows that

$$\boldsymbol{x}_{n+1} = S_2 \boldsymbol{x}_n + N_2 Q \bar{\boldsymbol{U}}_{n-D} \tag{11}$$

If the RK method is of order q, according to Remark 4, we have

$$\boldsymbol{x}(t_{n+1}) = S_2 \boldsymbol{x}(t_n) + N_2 \boldsymbol{U}_n + \boldsymbol{o}(h^{q+1}) \tag{12}$$

Assume that the local linearizing assumption is satisfied, that is, $\boldsymbol{x}(t_n) = \boldsymbol{x}_n$, and let $\boldsymbol{T}_{n+1} = \boldsymbol{x}(t_{n+1}) - \boldsymbol{x}_{n+1}$. Comparing (11) with (12) and considering (9), we have

$$\begin{aligned}\boldsymbol{T}_{n+1} = N_2(\boldsymbol{U}_n - Q\bar{\boldsymbol{U}}_{n-D}) + \boldsymbol{o}(h^{q+1}) &= \\ \boldsymbol{o}(h^{r+2}) + \boldsymbol{o}(h^{q+1})&\end{aligned} \tag{13}$$

According to Definition 2, we can see that the condition $r \geq q - 1$ should be satisfied to guarantee the order of the scheme is q. □

Remark 5[13]**.** When an s-stage Gauss method is applied to the scalar test equation $\dot{y} = \lambda y, \lambda \in \mathbf{C}$, it yields the difference equation $y_{n+1} = R(\hat{h}) y_n$, where $\hat{h} = h\lambda$ and $R(\hat{h}) = \exp(\hat{h}) + o(h^{2s+1})$. According to the Padé approximation theory, this means $R(\hat{h})$ is the (s, s) Padé approximation of order $2s$ to the exponential $\exp(\hat{h})$. Furthermore, it indicates that the maximal attainable order of an s-stage RK method is $2s$.

Theorem 2. The maximal attainable order of the RK method for converting the continuous-time system with input delay (2) to its discrete equivalence is $2s$ if the order of the interpolation is equal to or higher than $2s$, that is, the number of the points for interpolating should be at least $2s$ or larger, namely, $r \geq 2s - 1$.

Proof. We rewrite (11), (12), and (13) in Theorem 1 as follows:

$$\begin{aligned}&\boldsymbol{x}_{n+1} = S_2 \boldsymbol{x}_n + N_2 Q \bar{\boldsymbol{U}}_{n-D} \\ &\boldsymbol{x}(t_{n+1}) = S_2 \boldsymbol{x}(t_n) + N_2 \boldsymbol{U}_n + \boldsymbol{o}(h^{q+1}) \\ &\boldsymbol{T}_{n+1} = N_2(\boldsymbol{U}_n - Q\bar{\boldsymbol{U}}_{n-D}) + \boldsymbol{o}(h^{q+1}) = \\ &\qquad \boldsymbol{o}(h^{r+2}) + \boldsymbol{o}(h^{q+1})\end{aligned} \tag{14}$$

From Remark 5, we know that the maximal attainable order of an s-stage RK method is the Gauss method which is of order $2s$. When an s-stage Guass method is applied to (2), the local truncation error at t_{n+1} is

$$\boldsymbol{T}_{n+1} = \boldsymbol{o}(h^{r+2}) + \boldsymbol{o}(h^{q+1}) = \boldsymbol{o}(h^{r+2}) + \boldsymbol{o}(h^{2s+1})$$

It follows that the maximal attainable order of the scheme is $2s$ based on the condition $r \geq 2s - 1$. □

The correctness of this conclusion can be verified by the example in Section 4.

3 ISS of the scheme

In this section, we discuss the ISS of the scheme. Assuming that the original continuous-time system is ISS, we hope that the resulting discrete equivalence derived from RK methods is also ISS. Theorems 3 and 4 give the relationships between the ISS and the absolute stability of the RK method.

Lemma 1[12]**.** For the linear system (2), suppose that all eigenvalues of L have negative parts. Then, system (2) is ISS.

Lemma 2[12]**.** For the linear system $\boldsymbol{x}(k+1) = L\boldsymbol{x}(k) + \boldsymbol{G}u(k)$, where $L \in \mathbf{R}^{d\times d}$ and $\boldsymbol{G} \in \mathbf{R}^{d\times 1}$ are constant matrices, suppose that all eigenvalues of L have magnitudes less than unity. Then, this system is ISS.

Definition 3. An RK method with the polynomial interpolation for linear system (2) is called ISS if the difference system (3) is ISS in terms of the input term $u(nh - \tau + c_i h)$, $i = 1, 2, \cdots, s$.

Definition 4. The stability function of the RK method is given by

$$r(\hat{h}) = 1 + \hat{h}\boldsymbol{b}^{\mathrm{T}}(I_s - \hat{h}A)^{-1}\boldsymbol{e}_s = \frac{\det[I_s - \hat{h}(A - \boldsymbol{e}_s \boldsymbol{b}^{\mathrm{T}})]}{\det(I_s - \hat{h}A)}$$

based on the test equation $\dot{y}(t) = \lambda y(t)$, where $\lambda \in \mathbf{C}$, $\hat{h} = h\lambda$ and $\mathrm{Re}(\lambda) < 0$. The region

$$\mathbf{R}_{RK} = \{\hat{h} \in \mathbf{C} : |r(\hat{h})| < 1\}$$

is called the region of absolute stability of the RK method. The method is said to be A-stable if $\mathbf{R}_{RK}$ includes all the left half-plane of $\hat{h}$.

Theorem 3. We consider that an RK method $(A, \boldsymbol{b}, \boldsymbol{c})$ with the polynomial interpolation is applied to the linear system (2). Assume that

1) $\mathrm{Re}(\lambda_i(L)) < 0$, for $i = 1, 2, \cdots, d$;

2) The RK method is explicit and $h\lambda_i(L) \in \mathbf{R}_{RK}$, for $i = 1, 2, \cdots, d$.

Then, the scheme for (2) is ISS. Here, $\lambda_i(L)$ stands for the i-th eigenvalue of the matrix L.

Proof. According to Lemma 1, the linear system (2) is ISS for $\mathrm{Re}(\lambda_i(L)) < 0$. When an RK method with polynomial interpolation is applied to (2), we obtain the non-homogeneous difference system (4). First, we can prove the corresponding homogeneous system of (4) is ISS. The proof is similar to that in [14−15]. Then by Lemma 2, the non-homogeneous system (4) is ISS. □

We know from the theory of numerical analysis that if the RK method is A-stable, the sampling time can be selected without being constrained by stability considerations. For the A-stable RK methods, we have the following result.

Theorem 4. We consider that an RK method $(A, \boldsymbol{b}, \boldsymbol{c})$ with the polynomial interpolation is applied to the linear system (2). Assume that

1) $\mathrm{Re}(\lambda_i(L)) < 0$, for $i = 1, 2, \cdots, d$;

2) The RK method is A-stable and $\mathrm{Re}(\lambda_i(A)) \geq 0$, for $i = 1, 2, \cdots, s$.

Then, the scheme for (2) is ISS.

Proof. The proof can be carried out similarly to that of Theorem 3. □

Remark 6. The above results show that the ISS of RK method for the linear system (2) can be described by the region of absolute stability of the RK method for the scalar test equation without the control term. Theorems 3 and 4 give the relationships between the ISS and the absolute stability of the RK method.

4 Numerical example

We will give an example to illustrate the scheme. All the computations are carried out by Matlab.

Example 1. Consider the following transfer function

$$H(s)=\frac{s-1}{s^2+4s+5}\mathrm{e}^{-0.35s}$$

which appeared in [6]. Its controllable canonical form is

$$\begin{cases} \dot{\boldsymbol{x}}(t)=\begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t-0.35) \\ y(t)=\begin{bmatrix} -1 & 1 \end{bmatrix}\boldsymbol{x}(t) \end{cases} \tag{15}$$

Assume that the initial state is $\boldsymbol{x}(0)=[1,1]^{\mathrm{T}}$ and the input $u(t)$ to this system is $\sin(t)/4$.

We consider the influence of interpolating input signal on the order of the scheme. Applying the two-stage Gauss method with different interpolations to (15), we get the global errors at $t=1.4$ for a range of step lengths. The results are given in Table 1.

What is persuasive is the column of entries named "Ratio", where we have calculated, for each h, the ratio of the error when the step length is h to the error when the step length is $h/2$. It is known that for a q-order method, the global error is $o(h^q)$. Thus, for the Gauss method of 4 order, we expect this ratio to tend to $2^4=16$ as $h\rightarrow 0$. The results in Table 1 indicate that we are achieving fourth-order behaviors for $r=3$ and $r=4$, but not for $r=2$. At the same time, the results verify the correctness of Theorem 2, namely, taking $r\geq q-1=2s-1=3$ can guarantee the order of the scheme.

Next, on the basis of the conditions for interpolating being satisfied, we discuss the accuracies of the system obtained by different RK methods, which are applied to (15). The sampling frequency ω_s varies from 10π to 160π rad/s. An explicit RK method ($s=2$ and $q=2$), Gauss methods ($s=2,3$ and $q=4,6$), and Radau method ($s=2$ and $q=3$) are used for comparison. The root-mean-square (RMS) errors are plotted in Fig. 1.

$$E_{RMS}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\|\boldsymbol{x}(i)-\boldsymbol{x}_i\|^2}$$

In the following, we discuss the frequency responses of different discrete-time systems that appeared in [6] and our paper. Plots of the magnitude and phase frequency responses are in Fig. 2 for the continuous-time system, and for the discrete-time ones obtained with 4th-order Gauss method with $h=0.05$, 4th-order Gauss method with $h=0.1$, RLS method with $h=0.1$ in [6], Tustin rule with $h=0.1$ in [6], and second-order explicit RK method with $h=0.05$. In Fig. 3, the deviation in the response of each discrete-time one from the continuous system is plotted. The time responses of discrete equivalence of (15) are plotted in Fig. 4. And the deviations in the response of each discrete one from the original continuous system in the time domain are plotted in Fig. 5. The results show that over the useful frequency band, the discrete equivalence closely duplicates its original continuous one in the frequency domain as well as in the time domain. And the performance improves as the order of RK method increases.

The results in the above example indicate the followings:

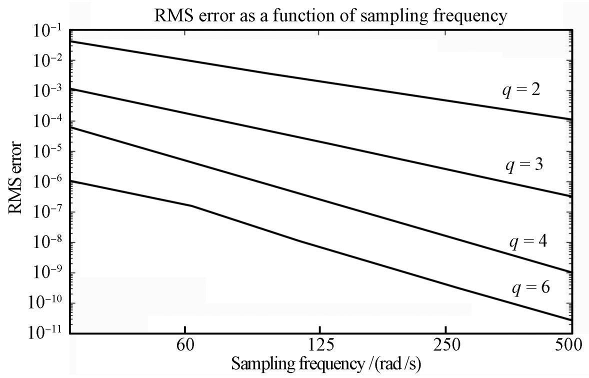

Fig. 1 RMS errors of Example 1

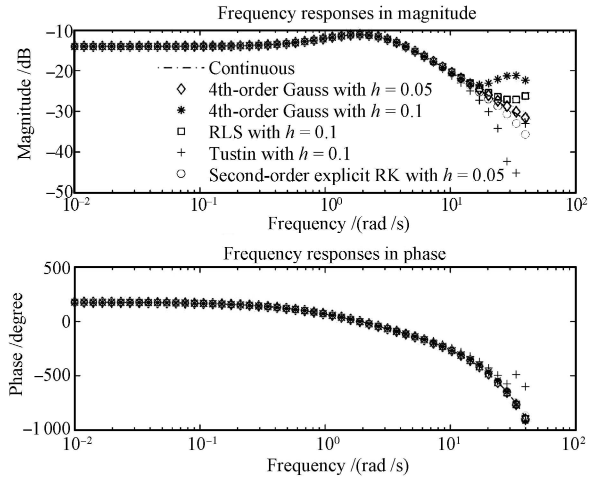

Fig. 2 Frequency responses of Example 1

Table 1 Global errors

	r=2		r=3		r=4	
h	Error	Ratio	Error	Ratio	Error	Ratio
0.2	9.8032×10^{-5}		8.7745×10^{-5}		8.7484×10^{-5}	
		22.26		16.59		16.12
0.1	4.4047×10^{-6}		5.2895×10^{-6}		5.4281×10^{-6}	
		8.18		15.51		16.08
0.05	5.3855×10^{-7}		3.4106×10^{-7}		3.3758×10^{-7}	
		11.59		16.02		16.01
0.025	4.6484×10^{-8}		2.1289×10^{-8}		2.1091×10^{-8}	
		10.31		16.01		16.00
0.0125	4.5090×10^{-9}		1.3299×10^{-9}		1.3182×10^{-9}	

Remark 7. The accuracy of the scheme is determined by the order of the interpolation as well as that of the RK. If the condition $r \geq q - 1$ is satisfied, the order of the RK itself decides the accuracy of the scheme.

Remark 8. The deviations in both time and frequency domains decrease as the order for discretizing increases. And the equivalent discrete one well duplicates its original continuous-time system when the discretizing accuracy is high. So, the RK method is efficient and preferable to discretizing compared with conventional methods.

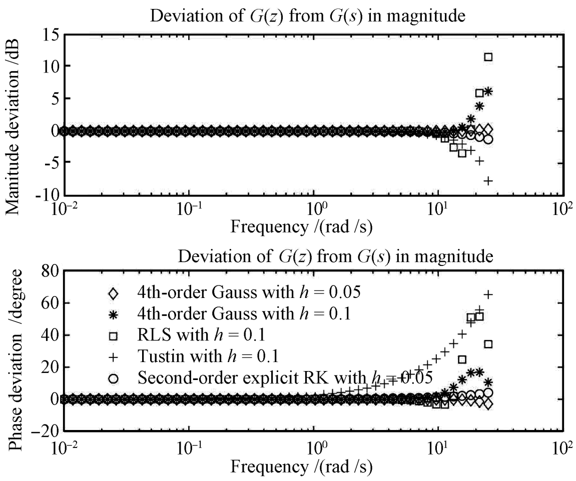

Fig. 3 Deviations of frequency responses of Example 1

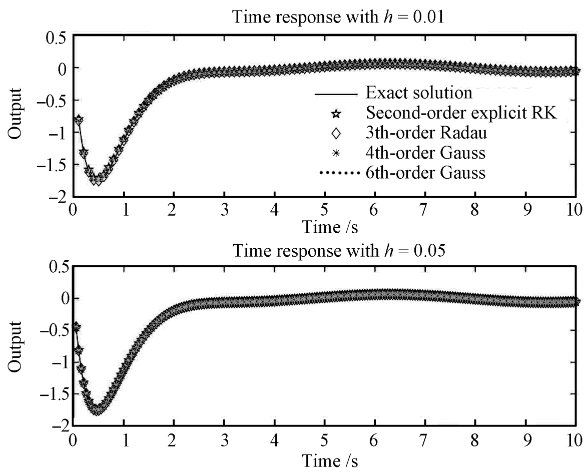

Fig. 4 Time responses of Example 1

Remark 9. The implementation of the proposed method is easier compared to that of the RLS method in [6]. RLS adopts parameters derived from the bilinear discretizing as the initial values for recursion. It takes many steps and requires heavy computational efforts to attain a fairly desirable convergent result. And, the stability of the resultant discrete filter may not be guaranteed. However, a good discrete equivalence can be obtained by computing just once for RK methods. The computational efforts largely decrease as compared to that of RLS method. As for the stability of discrete one, Theorems 3 and 4 give clear explanations. All these show the superiority of the RK method.

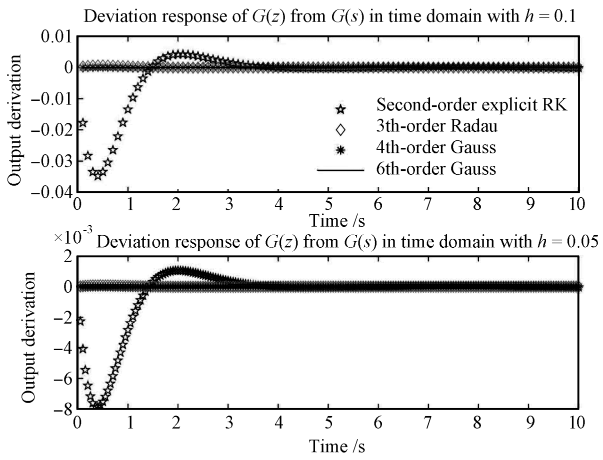

Fig. 5 Deviations of time responses of Example 1

5 Conclusion

In this paper, we study the RK method for discretizing a continuous-time system with input delay. The technique of Lagrange polynomial interpolation is employed for approximation to tackle the noninteger step values of delayed input. This action needs us to make further analysis of the influence caused by the interpolation to the accuracy of discretizing. The approximate order and the maximal attainable order of the scheme are analyzed, respectively. Furthermore, we discuss the ISS of the scheme. The ISS of the discrete system is closely related to the absolute stability of the RK method. The numerical example verifies the efficiency and superiority of the proposed method.

References

1 Schneider A M, Kaneshige J T, Groutage F D. High-order s-to-z mapping functions and their application in digitizing continuous-time filters. *Proceedings of the IEEE*, 1991, **79**(11): 1661−1674

2 Schneider A M, Anuskiewicz J A, Barghouti I S. Accuracy and stability of discrete-time filters generated by higher-order s-to-z mapping functions. *IEEE Transactions on Automatic Control*, 1994, **39**(2): 435−441

3 Wan C X, Schneider A M. Further improvements in digitizing continuous-time filters. *IEEE Transactions on Signal Processing*, 1997, **45**(3): 533−542

4 Wan C X, Schneider A M. Extensions of the weighted-sample method for digitizing continuous-time filters. *IEEE Transactions on Signal Processing*, 2001, **49**(8): 1627−1637

5 Liu P, Wang Q, Gu Y J. Study on comparison of discretization methods. In: Proceedings of the 2009 International Conference on Artificial Intelligence and Computational Intelligence. Shanghai, China: IEEE, 2009. 380−384

6 Wan Q, Bi Q, Yang X. High-performance conversion between continuous- and discrete-time systems. *Signal Processing*, 2001, **81**(9): 1865−1877

7 Zhang Z, An D U, Kim H, Chong K T. Comparative study of matrix exponential and Taylor series discretization methods for nonlinear ODEs. *Simulation Modelling Practice and Theory*, 2009, **17**(2): 471−484

8 Zhang Z, Chong K T. Second order hold based discretization method of input time-delay systems. In: Proceedings of the 2007 International Symposium on Information Technology Convergence. Jeonju, Korea: IEEE, 2007. 348−352

9 Unzueta M V R, Piqueira J R C. Discretization of a continuous time delay system via interpolation. In: Proceedings of the 40th Southeastern Symposium on System Theory. New Orleans, USA: IEEE, 2008. 108−112

10 Jugo J. Discretization of continuous time-delay systems. In: Proceedings of the 15th Triennial World Congress of the International Federation of Automatic Control. Barcelona, Spain: Elsevier, 2002. 1−6

11 Zhang Z, Kostyukova O, Zhang Y L, Chong K T. Hybrid discretization method for time-delay nonlinear systems. *Journal of Mechanical Science and Technology*, 2010, **24**(3): 731−741

12 Rugh W J. *Linear System Theory*. New Jersey: Prentice-Hall, 1996

13 Lambert J D. *Numerical Methods for Ordinary Differential Systems*. New York: John Wiley and Sons, 1991

14 Hu G D, Mitsui T. Stability analysis of numerical methods for systems of neutral delay-differential equations. *BIT Numerical Mathematics*, 1995, **35**(4): 504−515

15 Koto T. A stability property of A-stable natural Runge-Kutta methods for systems of delay differential equations. *BIT Numerical Mathematics*, 1994, **34**(2): 262−267

ZENG Li Received her bachelor degree from University of Science and Technology Beijing (USTB) in 2007. She is currently a Ph. D. candidate at USTB. Her research interest covers nonlinear control, time-delay system control, and numerical methods. Corresponding author of this paper. E-mail: zengliustb@gmail.com

HU Guang-Da Received his bachelor and master degrees from Harbin Institute of Technology, China in 1984 and 1989, and his Ph. D. degree from Nagova University, Japan in 1996, respectively. He was a visiting fellow at Manchester University, Britain from 1998 to 1999, and a postdoctoral fellow and a visiting professor at Newfoundland University, Canada from 1999 to 2000 and from 2000 to 2001, respectively. Currently, he is a professor at University of Science and Technology Beijing, China. His research interest covers time-delay control, nonlinear control, network control, and numerical methods.
E-mail: ghu@hit.edu.cn, ghuca@yahoo.ca