Decentralized Control for Arbitrarily Interconnected Systems over Lossy Network

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Abstract   This paper deals with the control of arbitrarily topological interconnected systems where information communicated between subsystems may be lost due to unreliable links. First, the stochastic variable that is responsible for the communication status of lossy network is regarded as a source of model uncertainty. The system is modeled in the framework of linear fractional transformation with a deterministic nominal system and a stochastic model uncertainty. Then, the robust control theory is employed for system analysis. The largest probability of communication failure, tolerated by the interconnected systems keeping mean square stable, can be obtained by solving a μ synthesis optimization problem. Decentralized state feedback controllers are designed to ensure that the whole system is mean square stable for a given communication failure rate, based on the technique of linear matrix inequalities. An illustrative example is presented finally to verify the effectiveness of the proposed model and method.

Key words   Interconnected systems, communication failure, decentralized control, μ synthesis, linear matrix inequality

DOI   10.3724/SP.J.1004.2010.01321

Recently there has been a great interest in the control of large scale interconnected systems, necessitated by a broad class of potential applications such as large arrays of micro-electromechanical sensors and actuators[1], multi-agent formation systems[2], power network distributed systems[3], etc. It is difficult or even infeasible to use the classical centralized control for such complex systems, because there are a lot of inputs and outputs, which imposes high burden of computation. In addition, the interconnections between subsystems are usually realized through network communication. For example, in the multi-robot system, robots broadcast their positions to teammates over wireless network. The use of network gives rise to new challenges such as intermittent data packet losses, which results in different information patterns at each subsystem and leads to more complicated synthesis of control law. The goal of this paper is to develop decentralized controllers that can handle the effects of both complex structure and communication failure on the stability of arbitrarily topological interconnected systems.

There are already some results on the control of interconnected systems. References [4−7] developed distributed controllers for a class of spatially invariant interconnected systems. In these works, the interconnections were just between neighboring subsystems and the communication was assumed to be perfect. The control of interconnected systems with communication constraints between subsystems is a rising problem and there are relatively few results in this field. The authors in [8] employed tools from dissipativity theory. They constructed a class of structured controllers for spatially interconnected systems over arbitrary

graph. These results were later extended to analysis and synthesis of interconnected systems where small communication delays existed between subsystems[9]. Langbort et al.[10] have discussed the control of interconnected systems over failing channels based on the theory of Markovian jump linear systems. However, the controllers were obtained on the condition of existence of an arbitrarily failing packet loss model. The result is more conservative than those of using the general Markovian failure model. Reference [11] has discussed the kind of Markovian communication failure between subsystems, but interconnected systems considered were spatially-invariant, only having information exchange between neighboring subsystems. Jiang et al.[12] designed structured state feedback controllers by Youla-Kucera parameterization method for systems interconnected over lossy networks. However, the solving of Youla-Kucera parameter will cost a large amount of computation since it is based on centralized control scheme, which has a great number of inputs and outputs.

In the field of networked control systems, there are abundant outcomes to deal with the communication failure problem. However, what has been considered is about the lossy links from sensors to controllers, or from controllers to actuators. The results were derived in the context of a single plant leading to a centralized control structure. But the methods they used give us a great inspiration. For example, [13] viewed the packet transmission situation as a binary switching sequence and employed the filtering scheme to handle the effect of missing measurements. Reference [14] used a discrete time linear system with Markovian jumping parameters to represent the networked control system with random communication failure. Reference [15] modeled the situation of intermittent communication as a source of model uncertainty and converted the problem into the classical robust control framework. This method is also the one we employ here.

In this paper, we aim to tackle the analysis and synthesis of interconnected systems over an arbitrary graph, which suffer the intermittent information transmission between subsystems because of the unreliable communication. An erasure channel model is introduced to describe the communication link. It is more general than the model used in [10]. Inspired by the method of [15], we treat the stochastically unreliable link as a source of model uncertainty. The whole interconnected system is represented as a linear fractional transformation and the results of robust control theory are adopted. Based on the technique of computationally tractable linear matrix inequalities, a method of designing decentralized state feedback controllers is proposed to ensure that the whole system is mean square stable under the effect of stochastic communication failure. The largest communication failure probability can also be obtained by solving this robust synthesis problem.

The paper is organized as follows: In Section 1, the model of interconnected systems over an arbitrary graph is presented, under the effect of random communication losses between any two subsystems. Analysis conditions are proposed in Section 2, which ensure the mean square stability of the whole system. Section 3 shows the method of designing decentralized controllers based on the linear matrix inequalities technique. An illustrative example is given in Section 4, and the conclusion is in Section 5.

Notations. For a matrix Z belonging to the set of real symmetric matrices R_s^n, $Z > 0$ means that Z is positive definite; $Z < 0$ means Z is negative definite. Given matrices $Z_k, k = 1, 2, \cdots, n$, $\mathrm{diag}_{k=1}^n \{Z_k\}$ denotes a block-diagonal

matrix with Z_k along the diagonal. It is usually denoted as $\mathrm{diag}_k\{Z_k\}$ for brevity. Likewise for signals or vectors $x_k, k = 1, 2, \cdots, n$, the notation $cat_{k=1}^n Z_k$ denotes a signal or a vector $[x_1^\mathrm{T}, x_2^\mathrm{T}, \cdots, x_n^\mathrm{T}]^\mathrm{T}$ formed by concatenating x_k, where x_k^T is the transposition of x_k. It is also denoted as $cat_k Z_k$ for brevity. I_n represents an identity matrix with the dimension $n \times n$. The dimension of a vector x is denoted by $\dim(x)$. For a stochastic variable $\gamma(k), \mu = \mathrm{E}\{\gamma(k)\}$ represents its mathematical expectation, $\sigma^2 = \mathrm{var}\{\gamma(k)\}$ represents the variance, and $\sigma = \sqrt{\sigma^2}$ is the standard deviation.

1 The model of interconnected systems with random connection

Consider the interconnected systems with arbitrary topology of L subsystems $G_i, i = 1, 2, \cdots, L$, as shown in Fig. 1. Each node represents a linear time-invariant (LTI) finite dimensional subsystem. Every directed edge (G_i, G_j) indicates that there is information flowing from subsystem G_i to G_j. The edge (G_i, G_j) denotes the case that information feeds back in subsystem G_i.
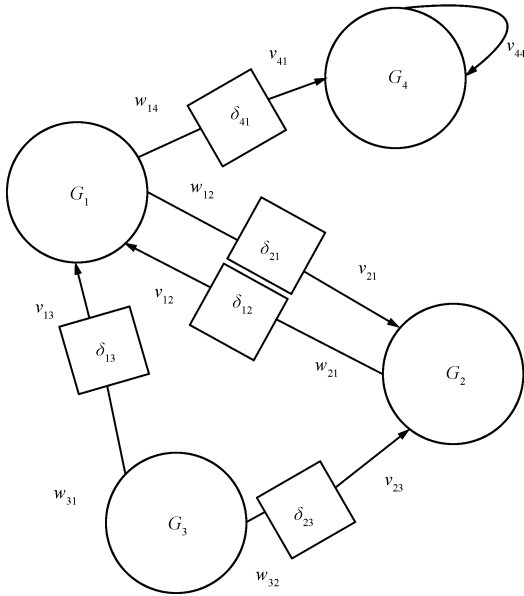
Fig. 1   Interconnected systems with random connection

For every pair of subsystems G_i and G_j, we use the following four signal notations[9]: 1) v_{ij}, the input of G_i coming from G_j; 2) w_{ij}, the output of G_i flowing towards G_j; similarly, 3) v_{ji} and 4) w_{ji}. Denote $n_{ij} = \dim(v_{ij})$. Assume that $\dim(w_{ji}) = \dim(v_{ij}) = n_{ij}$. $n_{ij} = 0$ indicates that there is no signal flowing from G_j to G_i.

1.1 The model of communication channel

Due to the characteristic of lossy communication network, information exchanged between subsystems may be lost randomly at each time-step. For simplicity, we assume that the channels either transmit information perfectly or lose it completely. Besides, the communication failure probabilities of different channels are independent and identical. The erasure channel model is introduced here to depict the lossy communication network channel used in this pape, which is a type of fading channel[15].

Definition 1. $\delta_{ij}(k)$ is a stochastic process taking values in $\{0, 1\}$, which is used to characterize the status of the communication between subsystem G_i and G_j at time k. An erasure channel, associated to $\delta_{ij}(k)$, is a multiplicative channel with a Bernoulli fading distribution, where $\Pr\{\delta_{ij}(k) = 0\} = p$, and $\Pr\{\delta_{ij}(k) = 1\} = 1 - p$. denotes the probability of communication failure.

Assume that for each $i, j = 1, 2, \cdots, L$, $\delta_{ij}(0)$, $\delta_{ij}(1)$, $\cdots, \delta_{ij}(k), \cdots$ are independent identically distributed random variables, and $\delta_{ij}(k)$ are independent from each other for different i, j. In other words, the state of a communication channel at a certain time is independent of the status of other channels at other time else.

According to the distribution of the stochastic variable $\delta_{ij}(k)$, we have the mean $\mu_{ij} = \mathrm{E}\{\delta_{ij}(k)\} = 1 - p$ and the variance $\sigma_{ij}^2 = \mathrm{var}\{\delta_{ij}(k)\} = \mathrm{E}\{(\delta_{ij}(k) - \mu_{ij})^2\} = p(1-p)$. Define $\Delta_{ij}(k) = \delta_{ij}(k)/\mu_{ij} - 1$. Then, $\mu_\Delta = \mathrm{E}\{\Delta_{ij}(k)\} = 0$, $\sigma_\Delta^2 = \mathrm{var}\{\Delta_{ij}(k)\} = p/(1-p)$. So, $\Delta_{ij}(k)$ is a zero-mean stochastic variable, and

$$\delta_{ij}(k) = \mu_{ij}(1 + \Delta_{ij}(k)) \qquad (1)$$

can be used to depict a fading channel[15].

Referring to the definitions of v_{ij} and w_{ij}, the interconnected relationship between two subsystems can be represented by

$$v_{ij}(k) = \delta_{ij}(k)I_{n_{ij}}w_{ji}(k) = \mu_{ij}(1 + \Delta_{ij}(k))I_{n_{ij}}w_{ji}(k) \quad (2)$$

The communication channel is called to be a mean channel when $\Delta_{ij}(k) = 0$, which is the nominal instance. Thus, the fading channel is composed of a mean channel and a zero-mean channel associated with the stochastic perturbation $\Delta_{ij}(k)$, as shown in Fig. 2. It is similar to the description of uncertainty in robust control problem.
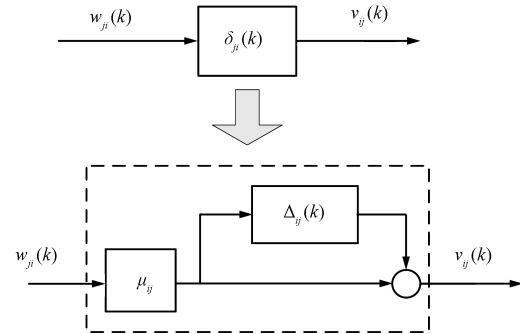
Fig. 2   Transformation of channel model

1.2 The model of interconnected systems

Denote the interconnected inputs to subsystem G_i as $v_i(k) = cat_{j=1}^L(v_{ij}(k))$. The interconnected outputs from G_i are $w_i(k) = cat_{j=1}^L(w_{ij}(k))$. Each subsystem can be described by the following state-space equations:

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i v_i(k) \\ w_i(k) &= C_i x_i(k) + D_i v_i(k) \end{aligned} \qquad (3)$$

where $B_i = [B_{i1}\ B_{i2}\ \cdots\ B_{iL}]$, $C_i = \begin{bmatrix} C_{1i} \\ C_{2i} \\ \vdots \\ C_{Li} \end{bmatrix}$, and $D_i = \begin{bmatrix} D_{i11} & \cdots & D_{i1L} \\ \vdots & \ddots & \vdots \\ D_{iL1} & \cdots & D_{iLL} \end{bmatrix}$ are the relevant coefficient matrices with appropriate dimensions respectively. $x_i \in \mathbf{R}^{m_i}$

are the state variables of the subsystem G_i. Suppose that the initial states $x_i(0)$ have bounded variances and are independent of $\Delta_{ij}(k), k \geq 0$.

The subsystem can also be expressed in the input-output form as $w_i(k) = G_i v_i(k)$.

Aggregate the interconnected signals as $v(k) = cat_{i=1}^L v_i(k)$, $w(k) = cat_{i=1}^L w_i(k)$, and denote $x(k) = cat_{i=1}^L x_i(k)$. The dynamics of the overall system with L subsystems can be formulated as

$$x(k+1) = Ax(k) + Bv(k)$$
$$w(k) = Cx(k) + Dv(k) \tag{4}$$

where $A = \mathrm{diag}_{i=1}^L \{A_i\}$. Similar expressions hold for B, C, and D. The overall system is denoted as G. There is $w(k) = Gv(k)$ with $G = \mathrm{diag}_{i=1}^L \{G_i\}$.

In order to represent the whole interconnected systems as the linear fractional transformation form for the convenience of system analysis, we define the following swapped interconnection signals

$$\bar{w}(k) = cat_{i=1}^L (cat_{j=1}^L w_{ji}(k)) \tag{5}$$

The permutation matrix P is defined correspondingly as

$$\bar{w}(k) = Pw(k) \tag{6}$$

Then,

$$v(k) = \Gamma(k)\bar{w}(k) \tag{7}$$

where

$$\Gamma(k) = \mathrm{diag}_{i=1}^L \{\mathrm{diag}_{j=1}^L \{\delta_{ij}(k) I_{n_{ij}}\}\} \tag{8}$$

Combining the expressions (1) and (8) yields

$$\Gamma(k) = \Psi(I + \Delta(k)) \tag{9}$$

where

$$\Psi = \mathrm{diag}_{i=1}^L \mathrm{diag}_{j=1}^L \{\mu_{ij} I_{n_{ij}}\}$$
$$\Delta(k) = \mathrm{diag}_{i=1}^L \mathrm{diag}_{j=1}^L \{\Delta_{ij}(k) I_{n_{ij}}\} \tag{10}$$
$$I = \mathrm{diag}_{i=1}^L \{\mathrm{diag}_{j=1}^L \{I_{n_{ij}}\}\}$$

Denote $\tilde{v}(k) = \Delta(k)\tilde{w}(k)$, where $\tilde{v}(k) = cat_{i=1}^L (\tilde{v}_{i\bullet}(k))$ $= cat_{i=1}^L (cat_{j=1}^L \tilde{v}_{ij}(k))$, $\tilde{w}(k) = cat_{i=1}^L (\tilde{w}_{\bullet i}(k))$ $= cat_{i=1}^L (cat_{j=1}^L \tilde{w}_{ji}(k))$. Let the linear operator M represents the mapping from $\tilde{v}(k)$ to $\tilde{w}(k)$, i.e.

$$\begin{bmatrix} \tilde{w}_{\bullet 1}(k) \\ \vdots \\ \tilde{w}_{\bullet L}(k) \end{bmatrix} = \begin{bmatrix} M_{11} & \cdots & M_{1L} \\ \vdots & \ddots & \vdots \\ M_{L1} & \cdots & M_{LL} \end{bmatrix} \begin{bmatrix} \tilde{v}_{1\bullet}(k) \\ \vdots \\ \tilde{v}_{L\bullet}(k) \end{bmatrix} \tag{11}$$

Then the whole interconnected system can be represented as a linear fractional transformation between $\Delta(k)$ and M, denoted as $H = F(\Delta, M)$, which is shown in Fig. 3.
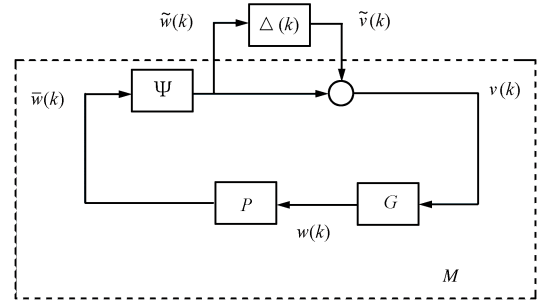

Fig. 3   The linear fractional transformation representation of interconnected systems

2   Analysis of interconnected systems

In this paper, we discuss two main features of interconnected systems: well-posedness and mean square stability.

2.1   Well-posedness

The interconnected systems are well-posed if the signals satisfying the interconnections exist, i.e., the systems are physically realizable. Reference [10] has given the mathematical definition on well-posedness of interconnected systems.

Definition 2[10]**.** The interconnected systems (4), which are composed of subsystems (3) and interconnections (2), are well-posed if for $\forall \delta_{ij} \in \{0, 1\}$, $1 \leq i, j \leq L$, the vectors $\{v_{ij}\}$ $\{w_{ij}\}$ satisfying $v_{ij} = \delta_{ij} w_{ji}$ and $\begin{bmatrix} w_i \\ v_i \end{bmatrix} \in$ $\mathbf{I}_m \begin{bmatrix} D_i \\ I \end{bmatrix}$ for all i, j, are zero, where $\mathbf{I}_m \begin{bmatrix} D_i \\ I \end{bmatrix}$ denotes the space generated by the vector $\begin{bmatrix} D_i \\ I \end{bmatrix}$.

In order to guarantee the well-posedness of interconnected systems, it is assumed that $D_i = 0$. This assumption may lead to a certain degree of conservatism, but it does not lose generality. It means that there is no feed-through loop between subsystems and no algebraic loop in the large-scale system. This can always be satisfied in practice.

2.2   Mean square stability

Definition 3[10]**.** The well-posed interconnected systems are mean square stable if $\lim_{k \to \infty} \mathrm{E}\{x(k)x(k)^{\mathrm{T}}\} = 0$

Definition 4[15]**.** The mean square norm of M is defined as

$$\|M\|_{MS} = \max_{i=1,\cdots,L} \sqrt{\sum_{j=1}^L \|M_{ij}\|_2^2} \tag{12}$$

It is noticeable that the square of the mean square norm represents the maximum output energy of all output channels.

For interconnected systems $H = F(\Delta, M)$, with the assumptions on $\delta_{ij}(k), x_i(0)$ and D_i mentioned previously, we have the following analysis result, which is adopted from [15].

Theorem 1. The following statements are equivalent:
1) The interconnected systems $H = F(\Delta, M)$ are mean square stable;
2) There exits a dialogic matrix Θ such that

$$\sigma^2 \inf_{\Theta > 0, \mathrm{diag}} \left\| \Theta^{-1} M \Theta \right\|_{MS}^2 < 1 \tag{13}$$

This theorem gives a condition for testing the mean square stability of interconnected systems from the per-

spective of the whole system. However, it is difficult to compute. In order to overcome this disadvantage, a sufficient condition for the mean square stability of interconnected systems is presented based on the dynamics of a basic building block. Although the condition is a little conservative, it is convenient for system analysis and synthesis.

Theorem 2. The interconnected systems $H = F(\Delta, M)$, with the assumptions on $\delta_{ij}(k), x_i(0)$, and D_i mentioned previously, are mean square stable if

$$\eta^2 \inf_{\theta>0,\text{diag}} \left\| \theta^{-1} G_i \theta \right\|_{MS}^2 < 1 \qquad (14)$$

for each $i = 1, 2, \cdots, L, \eta^2 = \mu_{ij}^2 (1 + \sigma)^2$.

Proof. Denote $M_1 = \Psi P G$ in Fig. 3. Then, $M = (I - M_1)^{-1} M_1$. Note that the scaling matrix Θ is positively diagonal, so

$$\left\| \Theta^{-1} M \Theta \right\|_{MS}^2 = \left\| \Theta^{-1} (I - M_1)^{-1} M_1 \Theta \right\|_{MS}^2 =$$

$$\left\| \Theta^{-1} (I - M_1)^{-1} \Theta \Theta^{-1} M_1 \Theta \right\|_{MS}^2 \leq$$

$$\left\| \Theta^{-1} (I - M_1)^{-1} \Theta \right\|_{MS}^2 \left\| \Theta^{-1} M_1 \Theta \right\|_{MS}^2 \overset{(a)}{\leq}$$

$$\frac{\left\| \Theta^{-1} M_1 \Theta \right\|_{MS}^2}{\left(1 - \left\| \Theta^{-1} M_1 \Theta \right\|_{MS} \right)^2} \qquad (15)$$

Inequality (a) holds true because

$$\left\| \Theta^{-1} (I - M_1)^{-1} \Theta \right\|_{MS} = \left\| (I - \Theta^{-1} M_1 \Theta)^{-1} \right\|_{MS} \leq$$

$$\frac{1}{1 - \left\| \Theta^{-1} M_1 \Theta \right\|_{MS}} \qquad (16)$$

On the other hand, since the permutation matrix P is orthogonal, and all the link channels have the same communication failure probability, i.e. μ_{ij} are all the same for $\forall i, j$,

$$\left\| \Theta^{-1} M_1 \Theta \right\|_{MS} = \left\| \Theta^{-1} \Psi G \Theta \right\|_{MS} \leq \mu_{ij} \left\| \Theta^{-1} G \Theta \right\|_{MS} \qquad (17)$$

Note that $G = \text{diag}_i \{ G_i \}$ and $\left\| \Theta^{-1} G \Theta \right\|_{MS} = \max_i \left\| \Theta^{-1} G_i \theta \right\|_{MS}$. If (14) holds true, i.e., $\mu_{ij} \inf_{\theta>0,\text{diag}} \left\| \theta^{-1} G_i \theta \right\|_{MS} < 1/(1+\sigma)$ for each i, we can get

$$\inf_{\Theta>0,\text{diag}} \left\| \Theta^{-1} M_1 \Theta \right\|_{MS} < \frac{1}{1+\sigma} \qquad (18)$$

Then

$$\inf_{\Theta>0,\text{diag}} \frac{\left\| \Theta^{-1} M_1 \Theta \right\|_{MS}^2}{\left(1 - \left\| \Theta^{-1} M_1 \Theta \right\|_{MS} \right)^2} < \frac{1}{\sigma^2} \qquad (19)$$

By (15) and Theorem 1, the result follows immediately.□

Lemma 1[15]**.** Given G_i which is stable for each i, and a diagonal matrix $\theta > 0$, then

$$\left\| \theta^{-1} G_i \theta \right\|_{MS}^2 = \inf_{Q>0,\gamma} \gamma$$

$$Q > A_i Q A_i^{\mathrm{T}} + B_i \theta^2 B_i^{\mathrm{T}}$$

$$\gamma \theta_{jj}^2 > C_{ji} Q C_{ji}^{\mathrm{T}} + D_{ij\bullet} \theta^2 D_{ij\bullet}^{\mathrm{T}}$$

$$j = 1, 2, \cdots, L$$

$$D_{ij\bullet} = \begin{bmatrix} D_{ij1} & \cdots & D_{ijL} \end{bmatrix} = \qquad (20)$$

$$\inf_{R>0,S>0,\gamma} \gamma$$

$$\begin{bmatrix} R & RA_i & RB_i\theta \\ A_i^{\mathrm{T}}R & R & 0 \\ \theta B_i^{\mathrm{T}}R & 0 & I \end{bmatrix} > 0$$

$$\begin{bmatrix} \theta S \theta & C_i & D_i\theta \\ C_i^{\mathrm{T}} & R & 0 \\ \theta D_i^{\mathrm{T}} & 0 & I \end{bmatrix} > 0$$

$$S_{jj} < \gamma, \ j = 1, 2, \cdots, L \qquad (21)$$

Proof. The result (20) is similar to the one in linear matrix inequality optimization for the computation of $\left\| \theta^{-1} G_i \theta \right\|_2^{2}$[16]. The only difference is that $\|G_i\|_2^2 = \text{tr}\{S\} = \sum_{j=1}^{L} S_{jj}$, while $\|G_i\|_{MS}^2 = \max_j S_{jj}$. The result (21) is drawn from the Schur complement lemma. □

Thus, the stability condition (14) can be transformed into a group of linear matrix inequalities by Lemma 1, which is easier for computation.

Theorem 3. The interconnected systems H are mean square stable, if there exist a matrix $Q > 0$ and a vector $\alpha \in \mathbf{R}^L$ of positive elements satisfying the following linear matrix inequalities:

$$Q > A_i Q A_i^{\mathrm{T}} + \sum_{j=1}^{L} B_{ij} \alpha_j B_{ij}^{\mathrm{T}}$$

$$\alpha_j > \eta^2 C_{ji} Q C_{ji}^{\mathrm{T}} + \eta^2 \sum_{j=1}^{L} D_{ij\bullet} \alpha_j D_{ij\bullet}^{\mathrm{T}},$$

$$i = 1, 2, \cdots, L \qquad (22)$$

Furthermore, observe that η^2 is monotonic in p, so $\inf_{\theta>0,\text{diag}} \left\| \theta^{-1} G_i \theta \right\|_{MS}^2$ corresponds to the maximal η^2 and then the largest communication failure probability p which can be tolerated by the interconnected systems.

Finally, the definition of mean square structured norm[15] is introduced to describe the largest stability margin for interconnected systems.

Definition 5. The mean square structured norm of M, denoted by $\mu_{MS}(M, \Delta)$, is defined as follows:

$$\mu_{MS}(M, \Delta) = \qquad (23)$$

$$\frac{1}{\{\sup \eta^2 : \text{the interconnected systems are mean square stable}\}}$$

It is worthy of noting that $1/\mu_{MS}$ gives the largest mean square stability margin, which associates with the largest communication failure probability that can be tolerated by H.

Based on the Theorem 2 and Definition 5, Corollary 1 follows immediately.

Corollary 1.

$$\mu_{MS}(M, \Delta) = \inf_i \inf_{\theta>0,\text{diag}} \left\| \theta^{-1} G_i \theta \right\|_{MS}^2 \qquad (24)$$

3 Controller synthesis

In this section, we aim to design a group of decentralized state feedback controllers so that the closed loop interconnected systems are mean square stable under the effect of random communication failure between subsystems. The reason for designing decentralized state feedback controllers is that: 1) Decentralized controllers can not only reduce the computation cost, but also lessen the communication burden; 2) The state feedback controllers may ensure that the

closed loop systems do not have algebraic loop, which is required for the well-posedness of interconnected systems.

Augmented with the sensor output signals $y_i \in \mathbf{R}^q$ and the control inputs $u_i \in \mathbf{R}^p$, the subplant can be described as:

$$P_i : \begin{cases} x_i(k+1) = A_i x_i(k) + B_i v_i(k) + B_i^u u_i(k) \\ w_i(k) = C_i x_i(k) + D_i v_i(k) + D_i^u u_i(k) \\ y_i(k) = C_i^y x_i(k) + D_i^y v_i(k) \end{cases} \quad (25)$$

We want to design controllers

$$K_i : u_i = K_i x_i \quad (26)$$

so that the closed loop interconnected systems are mean square stable when the information exchanging between subsystems is through erasure channels. Likewise it is assumed that the channels are independent from each other, and have identical communication failure probability. As to the largest stability margin, we have

$$\mu_{MS}^* = \inf_{K-stab} \inf_i \inf_{\theta > 0, \text{diag}} \left\| \theta^{-1} G_{cl,i} \theta \right\|_{MS}^2 =$$
$$\inf_{\theta > 0, \text{diag}} \inf_i \inf_{K-stab} \left\| \theta^{-1} G_{cl,i} \theta \right\|_{MS}^2 \quad (27)$$

where $G_{cl,i}$ is the closed loop subsystem and $K - stab$ denotes the set of all the stabilizing state feedback controllers.

For a fixed diagonal matrix $\theta_0 > 0$, the minimal mean square norm synthesis is a convex problem. The controller gain can be obtained by solving a group of linear matrix inequalities. It will be discussed in the following context in detail. However, the search for the optimal controller is non-convex for non-deterministic θ. We use a heuristic algorithm, termed D-K iteration[17] to accomplish the optimal controller synthesis. The problem is split into two simple parts: 1) solving the controller K_i in H_∞ synthesis framework for a fixed scaling $\theta_0 > 0$; 2) finding a scaling θ that minimizes the mean square norm.

Theorem 4. For a subsystem $G_{cl,i}$ and a fixed diagonal matrix $\theta_0 > 0$, the optimization problem of $\inf_{K-stab} \left\| \theta_0^{-1} G_{cl,i} \theta_0 \right\|_{MS}^2$ is equivalent to the following linear matrix optimization:

$$\inf_{R>0, S>0, X, \gamma} \gamma$$

s.t.

$$\begin{bmatrix} R & RA_i + X & RB_i \theta_0 \\ A_i^T R + X^T & R & 0 \\ \theta_0 B_i^T R & 0 & I \end{bmatrix} > 0$$

$$\begin{bmatrix} \theta S \theta & C_i & D_i \theta \\ C_i^T & R & 0 \\ \theta D_i^T & 0 & I \end{bmatrix} > 0$$

$$S_{jj} < \gamma, \ j = 1, 2, \cdots, L \quad (28)$$

Moreover, if these linear matrix inequalities are feasible, $K_i = (RB_i^u)^- X$ is the needed state feedback gain.

Proof. This result follows immediately from Theorem 3 and Lemma 1, with the system matrices A_i, B_i, C_i, D_i being that of the closed loop system instead. Then, the approach of variable substitution[16] is used to convert them into the form of linear matrix inequalities. □

4   A power network example

In this section, the tools presented above are applied to an example of power network where the transmission between subsystems may fail stochastically. The goal is to control each power plant individually so that the whole network is mean square stable in face of transmission failure. This model is adopted from [10].

The power network consists of a group of load-driving generators interconnected by unreliable transmission lines. The dynamics of the i-th generator is governed by

$$x_i(k+1) = T_i x_i(k) + L_i I_i(k) + B_i^u u_i(k)$$
$$V_i(k) = F_i x_i(k) \quad (29)$$

where $x_i(k) \in \mathbf{R}^2$ are the state variables of the i-th generator, corresponding to the deviations from the reference profiles of rotor's angular velocities and angles. $I_i(k) \in \mathbf{R}^2$ are the current deviations from the reference operating points. $V_i(k) \in \mathbf{R}^2$ are the voltage deviations. $u_i(k) \in \mathbf{R}$ are the control torques.

As shown in Fig. 4, each generator is connected to a load resistance with admittance matrix $Y_{ii} \in \mathbf{R}^{2 \times 2}$ and to other generators through a transmission line with stochastically varying admittance $\delta_{ij}(k) Y_{ii} \in \mathbf{R}^{2 \times 2}$, $j = 1, 2, \cdots, L$. The stochastic variable $\delta_{ij}(k)$ is a Markovian random process, which has the distribution characteristic as depicted in Section 1. Assume that the line fails and is independent of other lines. By the Kirchhoff Laws, we have

$$I_i(k) = Y_{ii} V_i(k) + \sum_{j \neq i} \delta_{ij}(k) Y_{ij}(V_i(k) - V_j(k)) \quad (30)$$

Then

$$x_i(k+1) = (T_i + L_i Y_{ii} F_i) x_i(k) + \quad (31)$$
$$L_i \sum_{j \neq i} \delta_{ij}(k) Y_{ij}(F_i x_i(k) - F_j x_j(k)) + B_i^u u_i(k)$$
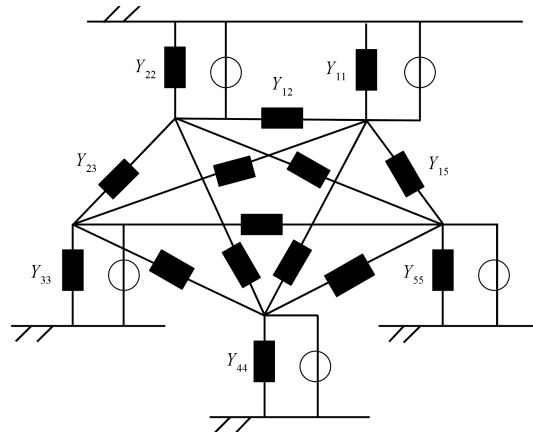
for $i = 1, 2, \cdots, L$.

Fig. 4   A power network with five generators

Let the sensor output signals $y_i(k) = x_i(k)$. Thus, (31) can be rewritten in the form of (25), i.e.,

$$x_i(k+1) = A_i x_i(k) + B_i v_i(k) + B_i^u u_i(k) \quad (32)$$

with $A_i = T_i + L_i Y_{ii} F_i$ and $B_i = \underbrace{[L_i \cdots L_i]}_{2(L-1)}$. The interconnected output signals are

$$w_i(k) = cat_j w_{ij}(k) \quad (33)$$

where

$$w_{ij}(k) = -Y_{ji}F_i x_i(k), \quad i \neq j$$
$$w_{ii}(k) = \begin{bmatrix} Y_{i1} \cdots Y_{i(i-1)} \ Y_{i(i+1)} \cdots Y_{iL} \end{bmatrix}^{\mathrm{T}} F_i x_i(k) \quad (34)$$

The interconnections between subsystems are

$$v_{ij}(k) = \delta_{ij}(k)I_{n_{ij}} w_{ji}(k), \quad i \neq j$$
$$v_{ii}(k) = \mathrm{diag}_{1 \leq j \leq L, j \neq i}\{\delta_{ij}(k)I_{n_{ij}}\} w_{ii}(k) \quad (35)$$

and

$$v_i(k) = cat_j v_{ij}(k) \quad (36)$$

Remark 1. In this example, at each instant k, $v_{ii}(k)$ and $w_{ii}(k)$ are related through a block-diagonal matrix, instead of a diagonal one. This only increases the dimension of interconnection. Our tools for system analysis and control synthesis can be extended straightforwardly to this case. The characteristics of the generator are supposed to be $T_i = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, $L_i = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$, $B_i^u = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and $F_i = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}$. The load matrices are $Y_{ii} = \begin{bmatrix} 10 & -2 \\ -2 & 10 \end{bmatrix}$ and $Y_{ij} = \begin{bmatrix} 10 & -5 \\ 5 & 10 \end{bmatrix}$.

Assume that there are five generators interconnected over a complete graph, i.e., every generator interconnects with the other four generators, $n_{ij} = 2$ for all $i, j = 1, 2, \cdots, 5$. All the links independently fail with failure rate $p = 0.3$. Using the control synthesis method described in Theorem 4, we can get the suboptimal state feedback controller gains as $K_i = [5.6652 \ -244.6163]$ while $\theta_{ii} = 1$. With these decentralized controllers, $A_i + B_i^u K_i$ have eigenvalues $\{-0.6652, 0\}$ which locate inside the unit circle. The simulation result is shown in Fig. 5. It is found that the interconnected systems can keep mean square stable under the effect of stochastic link failure.

If the failure rate is changed to be $p = 0.8$, the simulation result is presented as Fig. 6. The systems are still mean square stable, but they need longer setting time.

The mean square structured norm is calculated to be $\mu_{MS}^* = 1 \times 10^{-4}$ in this example. That is to say, the system can keep mean square stable even with a failure rate $p \approx 1$. However, it may not be the case always. For instance, if we change F_i and Y_{ij} to $F_i = \begin{bmatrix} 0 & 0 \\ 0 & -100 \end{bmatrix}$ and $Y_{ij} = \begin{bmatrix} 10 & -50 \\ 50 & 10 \end{bmatrix}$, it follows that $\mu_{MS}^* = 2.0934$. The maximal failure probability for system keeping mean square stable is $p = 0.7461$.

5 Conclusion

The problems of stability analysis and controller synthesis of arbitrarily interconnected systems linked over lossy communication network are discussed in this paper. The intermittent communication failure was treated as a source of model uncertainty. The system was put into the framework of linear fractional transformation representation. Analysis conditions for system maintaining mean square stable were obtained based on the robust control theory. The conditions were then expressed in terms of subsystem and a group of decentralized controllers were designed based on the technique of linear matrix inequalities. A power network example was given finally to show the effectiveness of our approach. In the future, we will try our best to deal with the effects of packet losses and time delays on the

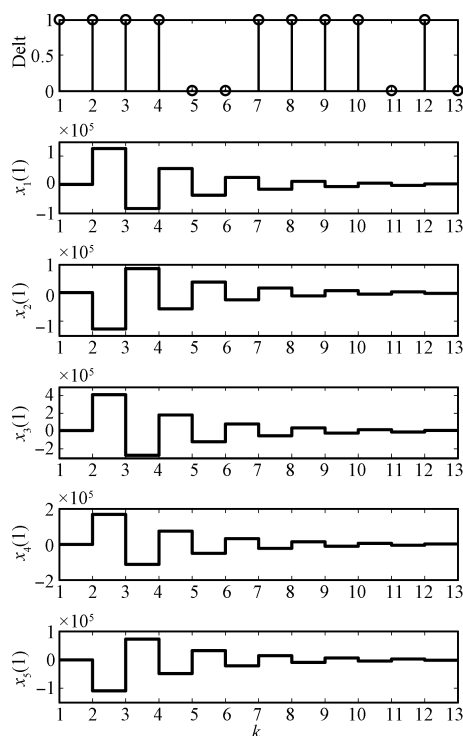stability of interconnected systems having network-based communication between subsystems.

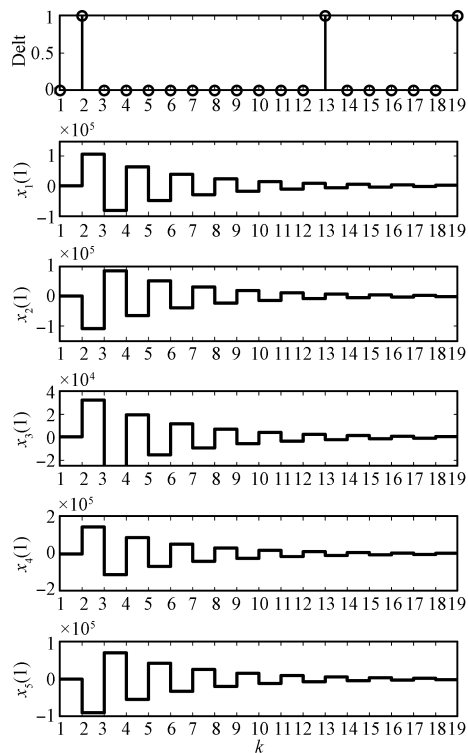Fig. 5   State responses of five generators with $p = 0.3$

Fig. 6   State responses of five generators with $p = 0.8$

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