A New Delay-dependent Absolute Stability Criterion for Lurie Systems with Time-varying delay

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Abstract The problem of absolute stability analysis for Lurie systems with time-varying delay and norm-bounded parameter uncertainties is considered. By using a new Lyapunov-Krasovskii functional, which splits the whole delay interval into two subintervals and defining a different energy function on each subinterval, some new delay-dependent robust absolute stability criteria are presented in terms of strict linear matrix inequalities (LMIs). The obtained delay-dependent criteria are less conservative than previous ones, which are illustrated by numerical examples.

Key words Lurie system, time-varying delay, absolute stability, nonlinearity, linear matrix inequality (LMI)
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Lurie and his colleagues first studied absolute stability for the plant of automatic pilot in 1940s. Since the pioneering work of Lurie, the problem of absolute stability of a class of nonlinear systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraint has been extensively investigated[1−2]. On the other hand, time delay is frequently encountered in engineering systems, such as nuclear reactors, chemical engineering systems, biological systems, and population dynamic models. Over a long period, the problem of absolute stability for time-delayed Lurie systems has been the subject of considerable research efforts[3−9]. It should be pointed out that most of the existing results in the literature are delay-independent and the considered systems are time-invariant. When the time-delay is small or time-varying, these results are often quite conservative. Recently, many efforts have been made to obtain less conservative delay-dependent criteria. In these criteria, an important index of measuring the conservativeness of the derived criteria is the upper bound of delays guaranteeing the stability of the considered system, which is termed as maximum allowable delay bound (MADB). Thus, considerable research efforts have been paid to the delay-dependent absolute stability of Lurie systems[8, 10−11]. Model transformation method combined with bounding technique for cross terms, such as Park′s or Moon et al.′s inequality, are used to derive the delay-dependent absolute stability conditions in [12−13]. Without using bounding technique for cross terms, some new absolute stability conditions by using Jensen′s inequality or extended Jensen′s inequality are presented subsequently in [8, 11]. Free-weighting matrices method is applied to express the relationships between the terms in the Leibniz-Newton formula in [14], which overcomes the conservativeness of methods involving a fixed model transformation. Nevertheless, all above works concern with the absolute stability of Lurie systems only with constant time delay. Lately, Han et al. proposed a method to analyze the absolute stability of Lurie systems with time-varying delay in [15]. Some new Lyapunov-Krasovskii approaches using delay-decomposition for analysis of time-delay systems are also reported[16−18].

The main contribution of this article is that the delay interval is divided into two subintervals. Based on this, a new Lyapunov-Krasovskii functional, which splits the whole delay interval into two subintervals is used to obtain some new absolute stability conditions. The problems of absolute stability and robust absolute stability for Lurie systems with time-varying delay are discussed by using this idea. The delay-dependent criteria are derived in terms of strict linear matrix inequalities (LMIs) that can be easily solved using the interior point algorithm. It should be pointed out that this delay decomposition method is especially useful for the delay systems whose characteristic of each subinterval is not identical, such as delay caused by different plant components. For example, liquid flows through different pipelines with different delay characteristics of each subinterval. In these subintervals, the full information of delay are used and hence conservativeness has comparatively reduced. Different from the reported approaches, sufficient information of time-delay splitting into two definite subintervals is considered such that much lesser conservative results are obtained. Numerical examples are presented to show the effectiveness and the superiority of our approach.

1 Problem formulation

Consider the following Lurie control system Σ:

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t) + (B+\Delta B)x(t-h(t)) + \\ &\quad (D+\Delta D)w(t) \\ z(t) &= Mx(t) + Nx(t-h(t))\end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^p$, and $z(t) \in \mathbf{R}^q$ are the state vector, input vector, and output vector of the system, respectively. A, B, D, M, and N are real constant matrices with appropriate dimensions. ΔA, ΔB, and ΔD denote real-valued matrix functions representing parameter uncertainties, which are assumed to have the form

$$\begin{bmatrix}\Delta A & \Delta B & \Delta D\end{bmatrix} = GF(t)\begin{bmatrix}E_a & E_b & E_d\end{bmatrix} \tag{2}$$

where G, E_a, E_b, and E_d are known constant matrices with appropriate dimensions, and $F(t) \in \mathbf{R}^{i\times j}$ is an unknown matrix with Lebesgue-measurable elements and satisfies

$$F^{\mathrm{T}}(t)F(t) \le I$$

The nonlinear feedback conjunction is described as

$$w(t) = -\varphi(t, z(t))$$

where $\varphi(t, z(t)) : [0,\infty) \times \mathbf{R}^q \mapsto \mathbf{R}^p$ is a class of memoryless, time-varying, nonlinear vector-valued function that is piecewise continuous in t, globally Lipschitz in $z(t)$,

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$\varphi(t,0)=0$ and satisfies the following sector condition for $\forall\, t\geq 0, \forall\, z(t)\in \mathbf{R}^q$,

$$\left[\varphi(t,z)-K_1z\right]^{\mathrm{T}}\left[\varphi(t,z)-K_2z\right]\leq 0 \tag{3}$$

$$\varphi^{\mathrm{T}}(t,z)\left[\varphi(t,z)-Kz\right]\leq 0 \tag{4}$$

where K_1 and K_2 are constant real matrices with appropriate dimensions, and K_2-K_1 is a symmetric positive definite matrix. It is well known that the nonlinear function $\varphi(t,z)$ satisfying (3) is said to belong to a sector $[K_1,\ K_2]$. When nonlinear function $\varphi(t,z)$ belongs to a sector $[0,\ K]$, then $\varphi(t,z)$ satisfies (4).

Time-varying delay $h(t)$ is subject to

$$0\leq h(t)\leq h<\infty,\ \dot{h}(t)\leq h_d<\infty,\ \forall\, t\geq 0 \tag{5}$$

The initial condition of system (1) specified on $[-h,0]$ is defined as

$$x(s)=\phi(s),\ \forall s\in[-h,\ 0]$$

where $\phi(s)$ is a first-order continuously differentiable vector-valued function.

For Lurie system (1), the delay-dependent stability problem has been widely studied by using Lyapunov-Krasovskii functional method[5, 8, 11−12, 14−15], whose objective is to obtain an MADB guaranteeing the absolute stability of the considered system. The MADB is an important performance index to measure the conservativeness of criteria. A great number of above results have been reported by choosing the following Lyapunov-Krasovskii functional:

$$V(x_t)=x^{\mathrm{T}}(t)Px(t)+\int_{t-h(t)}^{t}x^{\mathrm{T}}(s)Qx(s)\,\mathrm{d}s+ h\int_{t-h}^{t}(h-t+\alpha)\dot{x}^{\mathrm{T}}(\alpha)Z\dot{x}(\alpha)\,\mathrm{d}\alpha \tag{6}$$

where P, Q, and Z are symmetric positive-definite matrices to be determined. However, these works cannot significantly reduce the conservativeness of the previous results. To further enlarge the MADB, we introduce the following new Lyapunov-Krasovskii functional:

$$V(x_t)=\sum_{i=1}^{3}V_i(x_t) \tag{7}$$

where

$$\begin{aligned}
V_1(x_t)&=x^{\mathrm{T}}(t)Px(t)\\
V_2(x_t)&=\frac{h}{2}\int_{-\frac{h}{2}}^{0}\int_{t+\beta}^{t}\dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta+\\
&\quad\frac{h}{2}\int_{-h}^{-\frac{h}{2}}\int_{t+\beta}^{t}\dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta\\
V_3(x_t)&=\int_{t-h(t)}^{t-\frac{h}{2}}x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s+\int_{t-\frac{h}{2}}^{t}x^{\mathrm{T}}(s)Q_2x(s)\mathrm{d}s
\end{aligned}$$

with P, Q_1, Q_2, Z_1, and Z_2 are symmetric positive-definite matrices to be determined. Compared with the conventional Lyapunov functional (6), it is observed that $V_2(x_t)$ and $V_3(x_t)$ in (6) split the whole interval $[-h,0]$ into two subintervals, that is, $[-h,-h/2]$ and $[-h/2,0]$ such that each half subinterval has a different Lyapunov matrix. This Lyapunov-Krasovskii functional will give some new delay-dependent stability criteria for Lurie system (1). It should be mentioned that if we choose $Q_1=Q_2$ and $Z_1=Z_2$, that is, the same Lyapunov matrices are designed for two subintervals. Lyapunov functional (7) reduces to (6) and the results in this article reduce to the reported stability criteria in literatures.

Before proceeding further, the following technical lemmas and definition are introduced, which will be used in the proof of the main results.

Lemma 1[19]**.** Given matrices Γ, Ξ, and Ω with Ω symmetric,

$$\Omega+\Gamma F(\sigma)\Xi+\Xi^{\mathrm{T}}F^{\mathrm{T}}(\sigma)\Gamma^{\mathrm{T}}<0$$

holds for any $F(\sigma)$ satisfying $F^{\mathrm{T}}(\sigma)F(\sigma)\leq I$, if and only if there exists a scalar $\varepsilon>0$ such that

$$\Omega+\varepsilon^{-1}\Gamma\Gamma^{\mathrm{T}}+\varepsilon\Xi^{\mathrm{T}}\Xi<0$$

Lemma 2 (Jensen′s inequality[20]**).** For any symmetric positive-definite matrix Z, scalar $0\leq h(t)\leq h$, and vector-valued function $x(\cdot):[-h(t),0]\rightarrow\mathbf{R}^n$ with first-order continuous-derivative entries such that the following integral inequality is well defined, then

$$-h\int_{-h(t)}^{0}\dot{x}^{\mathrm{T}}(t+\alpha)Z\dot{x}(t+\alpha)\,\mathrm{d}\alpha\leq \begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-Z & Z\\ * & -Z\end{bmatrix}\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix} \tag{8}$$

Remark 1. Zhang et al.[21] extended the Jensen′s inequality to the following form: Given any symmetric positive-definite matrix Z, matrices M_1, M_2, scalar $0\leq h(t)\leq h$, and vector value function $\dot{x}(t):[-h,\ 0]\rightarrow\mathbf{R}^n$ such that the following integration is well defined, then

$$\begin{aligned}
-h\int_{-h(t)}^{0}\dot{x}^{\mathrm{T}}(t+\alpha)Z\dot{x}(t+\alpha)\,\mathrm{d}\alpha\leq \begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^{\mathrm{T}}\times\\
\begin{bmatrix}M_1^{\mathrm{T}}+M_1 & -M_1^{\mathrm{T}}+M_2\\ * & -M_2^{\mathrm{T}}-M_2\end{bmatrix}\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}+\\
\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}M_1^{\mathrm{T}}\\M_2^{\mathrm{T}}\end{bmatrix}Z^{-1}\begin{bmatrix}M_1^{\mathrm{T}}\\M_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}
\end{aligned} \tag{9}$$

It can be easily verified that Jensen′s inequality (8) can be regarded as a special case of extended inequality (9) with $M_1=-Z$ and $M_2=Z$. Then, it is natural to ask if this extension brings extra freedom and gives less conservative results. The answer is no. It will be shown in Remark 2 and Appendix that the stability criterion obtained using (9) is equivalent to that using (8) in the sense of conservativeness, though more free variables are involved in (9).

Definition 1. The Lurie system (1) with $\Delta A=\Delta B=0$ is said to be absolutely stable in the given sector $[K_1,\ K_2]$ or $[0,\ K]$, if this system is globally uniformly asymptotically stable for any nonlinear function $\varphi(t,z)$ satisfying (3) or (4).

2 Main results

The robust stability criteria for system (1) will be given in this section. To this end, we first give an absolute stability criterion for nominal Lurie system (1):

$$\begin{aligned}
\dot{x}(t)&=Ax(t)+Bx(t-h(t))+Dw(t)\\
z(t)&=Mx(t)+Nx(t-h(t))
\end{aligned} \tag{10}$$

2.1 Stability issue

Theorem 1. Given a scalar $h > 0$, the nominal Lurie system (10) with nonlinear function satisfying (4) is absolutely stable, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2, and a positive scalar ε such that LMI (11) holds,

$$\Xi = \begin{bmatrix} \Xi_{11} & Z_1 & PB & \Xi_{14} & \frac{h}{2}A^{\mathrm{T}}(Z_1+Z_2) \\ * & \Xi_{22} & Z_2 & 0 & 0 \\ * & * & \Xi_{33} & \varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & \frac{h}{2}B^{\mathrm{T}}(Z_1+Z_2) \\ * & * & * & -2\varepsilon I & \frac{h}{2}D^{\mathrm{T}}(Z_1+Z_2) \\ * & * & * & * & -(Z_1+Z_2) \end{bmatrix} < 0 \tag{11}$$

where

$\Xi_{11} = A^{\mathrm{T}}P + PA - Z_1 + Q_2$, $\Xi_{14} = PD + \varepsilon M^{\mathrm{T}}K^{\mathrm{T}}$

$\Xi_{22} = -Z_1 - Z_2 + Q_1 - Q_2$, $\Xi_{33} = -Z_2 - (1-h_d)Q_1$

Proof. Choose a Lyapunov-Krasovskii functional candidate $V(x_t)$ as (7). Taking the derivative of $V_i(x_t)$, $i = 1, 2, 3$ with respect to t along the trajectory of system (10) yields

$$\dot{V}_1(x_t) = 2x^{\mathrm{T}}(t)P\Big[Ax(t) + Bx(t-h(t)) + Dw(t)\Big] \tag{12}$$

$$\begin{aligned} \dot{V}_2(x_t) = {} & \Big(\frac{h}{2}\Big)^2 \dot{x}^{\mathrm{T}}(t)Z_1\dot{x}(t) + \Big(\frac{h}{2}\Big)^2 \dot{x}^{\mathrm{T}}(t)Z_2\dot{x}(t) - \\ & \frac{h}{2}\int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\,\mathrm{d}\alpha - \\ & \frac{h}{2}\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{V}_3(x_t) = {} & x^{\mathrm{T}}\Big(t-\frac{h}{2}\Big)Q_1x\Big(t-\frac{h}{2}\Big) - \\ & (1-h_d)x^{\mathrm{T}}(t-h(t))Q_1x(t-h(t)) + \\ & x^{\mathrm{T}}(t)Q_2x(t) - x^{\mathrm{T}}\Big(t-\frac{h}{2}\Big)Q_2x\Big(t-\frac{h}{2}\Big) \end{aligned} \tag{14}$$

Applying Jensen′s inequality (8), the following inequalities are true:

$$\begin{aligned} & -\frac{h}{2}\int_{t-\frac{h}{2}}^{t} \dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\mathrm{d}\alpha \le \\ & \begin{bmatrix} x(t) \\ x(t-\frac{h}{2}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -Z_1 & Z_1 \\ Z_1 & -Z_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{h}{2}) \end{bmatrix} \end{aligned} \tag{15}$$

and

$$\begin{aligned} & -\Big(h-\frac{h}{2}\Big)\int_{t-h}^{t-\frac{h}{2}} \dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha \le \\ & \begin{bmatrix} x(t-\frac{h}{2}) \\ x(t-h(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -Z_2 & Z_2 \\ Z_2 & -Z_2 \end{bmatrix} \begin{bmatrix} x(t-\frac{h}{2}) \\ x(t-h(t)) \end{bmatrix} \end{aligned} \tag{16}$$

From (1) and (4), it follows

$$-w^{\mathrm{T}}(t)w(t) + w^{\mathrm{T}}(t)K(Mx(t) + Nx(t-h(t))) \ge 0 \tag{17}$$

Combining (12) ∼ (17) gives that

$$\begin{aligned} \dot{V}(x_t) = {} & \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) \le \\ & \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) + \\ & 2\varepsilon\big[-w^{\mathrm{T}}(t)w(t) + w^{\mathrm{T}}(t)K(Mx(t)+ \\ & Nx(t-h(t)))\big] = \xi^{\mathrm{T}}(t)\Theta\xi(t) \end{aligned} \tag{18}$$

where

$$\Theta = \begin{bmatrix} \Xi_{11} & Z_1 & PB & PD+\varepsilon M^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Xi_{22} & Z_2 & 0 \\ * & * & \Xi_{33} & \varepsilon N^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & * & -2\varepsilon I \end{bmatrix} + \Big(\frac{h}{2}\Big)^2 \begin{bmatrix} A \\ 0 \\ B \\ D \end{bmatrix} (Z_1+Z_2) \begin{bmatrix} A \\ 0 \\ B \\ D \end{bmatrix}^{\mathrm{T}}$$

and $\Xi(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\frac{h}{2}) & x^{\mathrm{T}}(t-h(t)) & \omega^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$.

Using Schur complement lemma, it is shown that $\Xi < 0$ implies $\Theta < 0$, and then $\dot{V}(x_t) < 0$. Hence, there must exists a scalar $\rho > 0$ such that $\dot{V}(x_t) \le -\rho\|x(t)\|^2$, $\rho = \lambda_{\min}(-\Theta) > 0$, which implies the absolute stability of system (10). □

Remark 2. If the extended Jensen′s inequality (9) is used, an absolute stability criterion equivalent to Theorem 1 is given as: The nominal Lurie system (10) with nonlinear function satisfying (4) is absolutely stable, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2, matrices M_1, M_2, M_3, M_4, and a positive scalar ε such that LMI (19) holds:

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \frac{h}{2}L_1^{\mathrm{T}}Z_1 & \frac{h}{2}L_1^{\mathrm{T}}Z_2 & L_2^{\mathrm{T}} & L_3^{\mathrm{T}} \\ * & -Z_1 & 0 & 0 & 0 \\ * & * & -Z_2 & 0 & 0 \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -Z_2 \end{bmatrix} < 0 \tag{19}$$

where

$$\Gamma_{11} = \begin{bmatrix} \Psi_{11} & -M_1^{\mathrm{T}}+M_2 & PB & \Psi_{14} \\ * & \Psi_{22} & -M_3^{\mathrm{T}}+M_4 & 0 \\ * & * & \Psi_{33} & -\epsilon N^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & * & -2\varepsilon I \end{bmatrix}$$

$L_1 = \begin{bmatrix} A & 0 & B & D \end{bmatrix}$

$L_2 = \begin{bmatrix} M_1 & M_2 & 0 & 0 \end{bmatrix}$

$L_3 = \begin{bmatrix} 0 & M_3 & M_4 & 0 \end{bmatrix}$

$\Psi_{11} = A^{\mathrm{T}}P + PA + M_1^{\mathrm{T}} + M_1 + Q_2$

$\Psi_{22} = Q_1 - Q_2 - M_2^{\mathrm{T}} - M_2 + M_3^{\mathrm{T}} + M_3$

$\Psi_{33} = -(1-h_d)Q_1 - M_4^{\mathrm{T}} - M_4$

$\Psi_{14} = PD + \varepsilon M^{\mathrm{T}}K^{\mathrm{T}}$

The proof is given in Appendix.

Remark 3. For the nonlinearity $\varphi(t, z)$ satisfying more general sector condition (3), by using the well-known loop transformation[22], it can be obtained that the absolute stability of system (10) in the sector $[K_1, K_2]$ is equivalent to that of the following system

$$\begin{aligned} \dot{x}(t) &= (A - DK_1M)x(t) + (B - DK_1N)\times \\ & \quad\; x(t-h(t)) + Dw(t) \\ z(t) &= Mx(t) + Nx(t-h(t)) \end{aligned} \tag{20}$$

in the sector $[0,\ K_2 - K_1]$. Following the similar philosophy as in the proof of Theorem 1, the absolute stability criterion for Lurie system (10) with nonlinear function satisfying (3) can be easily obtained in Corollary 1.

Corollary 1. Given a scalar $h>0$, the nominal Lurie system (10) with nonlinear function satisfying (3) is absolutely stable, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2, and a positive scalar ε such that LMI (21) holds:

$$\Pi=\begin{bmatrix}\Pi_{11} & Z_1 & \Pi_{13} & \Pi_{14} & \Pi_{15}\\ * & \Xi_{22} & Z_2 & 0 & 0\\ * & * & \Xi_{33} & \Pi_{34} & \Pi_{35}\\ * & * & * & -2\varepsilon I & \Pi_{45}\\ * & * & * & * & -(Z_1+Z_2)\end{bmatrix}<0 \quad (21)$$

where Ξ_{22} and Ξ_{33} follow the same definitions as those in (11) and

$$\Pi_{11}=(A-DK_1M)^{\rm T}P+P(A-DK_1M)-Z_1+Q_2$$
$$\Pi_{13}=P(B-DK_1N),\ \Pi_{14}=PD+\varepsilon M^{\rm T}(K_2-K_1)^{\rm T}$$
$$\Pi_{15}=\frac{h}{2}(A-DK_1M)^{\rm T}(Z_1+Z_2),\ \Pi_{34}=\varepsilon N^{\rm T}(K_2-K_1)^{\rm T}$$
$$\Pi_{35}=\frac{h}{2}(B-DK_1N)^{\rm T}(Z_1+Z_2),\ \Pi_{45}=\frac{h}{2}D^{\rm T}(Z_1+Z_2)$$

2.2 Robust stability

Extending Theorem 1 to uncertain Lurie system (1) with time-varying parameter uncertainties yields the following delay-dependent robust absolute stability criterion.

Theorem 2. Given a scalar $h>0$, the uncertain Lurie system (1) with nonlinear function satisfying (4) is robustly absolutely stable, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2, and positive scalars ε, μ, such that LMI (22) holds:

$$\Sigma=\begin{bmatrix}\Xi & \Sigma_{12} & \mu\Sigma_{13}\\ * & -\mu I & 0\\ * & * & -\mu I\end{bmatrix}<0 \quad (22)$$

where Ξ follows the same definition as (11) and

$$\Sigma_{12}=[G^{\rm T}P\quad 0\quad 0\quad 0\quad \frac{h}{2}G^{\rm T}(Z_1+Z_2)]^{\rm T}$$
$$\Sigma_{13}=[E_a\quad 0\quad E_b\quad E_d\quad 0]^{\rm T}$$

Proof. Replacing A, B, and D in (11) with $A+GF(t)E_a$, $B+GF(t)E_b$, and $D+GF(t)E_d$, respectively, then (11) for uncertain Lurie system (1) is equivalent to the following condition:

$$\Xi+\Sigma_{12}F(t)\Sigma_{13}^{\rm T}+\Sigma_{13}F^{\rm T}(t)\Sigma_{12}^{\rm T}<0 \quad (23)$$

where Σ_{12} and Σ_{13} are defined in (22). Using Lemma 1, a sufficient and necessary condition guaranteeing (23) is the existence of a scalar $\mu>0$ such that

$$\Xi+\mu^{-1}\Sigma_{12}\Sigma_{12}^{\rm T}+\mu\Sigma_{13}\Sigma_{13}^{\rm T}<0 \quad (24)$$

Applying the Schur complement shows that (24) is equivalent to (22). □

When the nonlinear function satisfies (3), Corollary 2 is obtained.

Corollary 2. Given a scalar $h>0$, the uncertain Lurie system (1) with nonlinear function satisfying (3) is robustly absolutely stable, if there exist symmetric positive-definite matrices P, Q_1, Q_2, Z_1, Z_2, and positive scalars ε, μ such that LMI (25) holds:

$$\Omega=\begin{bmatrix}\Pi & \Sigma_{12} & \mu\Omega_{13}\\ * & -\mu I & 0\\ * & * & -\mu I\end{bmatrix}<0 \quad (25)$$

where Π is defined in (21), Σ_{12} is defined in (22), and

$$\Omega_{13}=[E_a-E_dK_1M\quad 0\quad E_b-E_dK_1N\quad E_d\quad 0]^{\rm T}$$

Proof. Replacing A, B, and D in (21) with $A+GF(t)E_a$, $B+GF(t)E_b$, and $D+GF(t)E_d$, respectively, then (21) for the uncertain Lurie system (1) is equivalent to

$$\Pi+\Sigma_{12}F(t)\Omega_{13}^{\rm T}+\Omega_{13}F^{\rm T}(t)\Sigma_{12}^{\rm T}<0 \quad (26)$$

Using Lemma 1, a sufficient and necessary condition guaranteeing (26) is that there exists a scalar $\mu>0$ such that

$$\Pi+\mu^{-1}\Sigma_{12}\Sigma_{12}^{\rm T}+\mu\Omega_{13}\Omega_{13}^{\rm T}<0 \quad (27)$$

which is equivalent to (22) in the sense of Schur complement. □

3 Numerical examples

The following numerical examples are presented to illustrate the effectiveness of the proposed theoretical results given in Sections 1 and 2.

Example 1. Consider the nominal Lurie system (10) with nonlinearity satisfying (4). The system′s parameters are the same as in [5, 12], that is,

$$A=\begin{bmatrix}-2 & 0\\ -1 & -2\end{bmatrix},\ B=\begin{bmatrix}-0.2 & -0.5\\ 0.5 & -0.2\end{bmatrix},\ D=\begin{bmatrix}-0.2\\ -0.3\end{bmatrix}$$
$$M=[0.6\quad 0.8],\ N=[0\quad 0],\ K=0.5,\ h_d=0$$

The calculation results for MADB h of Lurie system (10) with different methods are listed in Table 1, which shows that our methods give less conservative result. Obviously, the methods used by Theorem 1 or Remark 2 yield larger MADB h than [5, 12].

Table 1 The MADB h using different methods

Different methods	Nian[5]	He et al.[14]	Theorem 1
The MADB h	0.305	9.989×10^6	9.094×10^9

Example 2. Consider the uncertain Lurie control system (1) with the parameters as follows

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix},\ B=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},\ D=\begin{bmatrix}-0.2\\ -0.3\end{bmatrix}$$
$$M=\begin{bmatrix}0.3\\ 0.1\end{bmatrix}^{\rm T},\ N=\begin{bmatrix}0.1\\ 0.2\end{bmatrix}^{\rm T},\ H=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},\ E_d=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$E_a=E_b=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ K_1=0.2,\ K_2=0.5$$

where the system matrices considered here are of those as in [15]. This problem cannot be solved by the method of [8] which only considers the constant time-delay case. Table 2 lists the MADB h for robust absolute stability of uncertain Lurie system (1).

Table 2 The MADB h for different h_d

h_d	0.00	1.00	2.00	>2.00
Han et al.[15]	3.3057	0.7638	0.7638	0.7638
Corollary 2	4.5549	1.5275	1.5275	1.5275

4 Conclusion

The problem of robust absolute stability for Lurie control system with time-varying delay and norm-bounded parameter uncertainties is investigated. By introducing a new Lyapunov-Krasovskii functional that divides the whole delay interval into two subintervals, some much less conservative conditions for robust absolute stability of Lurie control systems are proposed in terms of strict LMIs. Numerical examples are provided to demonstrate the feasibility and the superiority of the proposed approach.

Appendix. The proof of Remark 1

The LMI condition (19) is equivalent to a much simpler form (11).

Proof. Rewriting $\Gamma < 0$ in (10) as

$$\Gamma = \Gamma|_{M_1=0,M_2=0} + U_1[M_1 \quad M_2]V_1 + V_1^{\mathrm{T}}[M_1 \quad M_2]^{\mathrm{T}}U_1^{\mathrm{T}} \tag{A1}$$

with

$$\begin{aligned} U_1 &= [I \quad -I \quad 0 \quad 0 \quad 0 \quad 0 \quad I \quad 0]^{\mathrm{T}} \\ V_1 &= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{A2}$$

and using the well-known Elimination lemma, $\Gamma < 0$ for some M_1 and M_2 if and only if

$$\Phi_1 = V_{1\perp}^{\mathrm{T}}\Gamma V_{1\perp} < 0,\ \Phi_2 = U_{1\perp}\Gamma U_{1\perp}^{\mathrm{T}} < 0 \tag{A3}$$

where $U_{1\perp}$ and $V_{1\perp}$ are the null matrices of U_1 and V_1, respectively, i.e.,

$$U_{1\perp} = \begin{bmatrix} I & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},\quad V_{1\perp} = \begin{bmatrix} 0_{2\times 6} \\ I_6 \end{bmatrix} \tag{A4}$$

Simplifying inequality (A3), we get

$$\Phi_1 = \begin{bmatrix} \Psi_{33} & -\varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}\bar{Z}_1 & B^{\mathrm{T}}\bar{Z}_2 & 0 & M_4^{\mathrm{T}} \\ * & -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 & 0 & 0 \\ * & * & -Z_1 & 0 & 0 & 0 \\ * & * & * & -Z_2 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix} \tag{A5}$$

$$\Phi_2 = \begin{bmatrix} \Phi_{21} & \Phi_{22} \\ * & \Phi_{24} \end{bmatrix}$$

with

$$\Phi_{21} = \begin{bmatrix} \Phi_{2,11} & \Phi_{2,12} & \Phi_{2,13} \\ * & \Phi_{2,22} & \Phi_{2,23} \\ * & * & \Psi_{33} \end{bmatrix}$$

$$\Phi_{22} = \begin{bmatrix} \Psi_{14} & A^{\mathrm{T}}\bar{Z}_1 & A^{\mathrm{T}}\bar{Z}_2 & M_3^{\mathrm{T}} \\ 0 & 0 & 0 & M_3^{\mathrm{T}} \\ -\varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}\bar{Z}_2 & B^{\mathrm{T}}\bar{Z}_2 & M_4^{\mathrm{T}} \end{bmatrix}$$

$$\Phi_{24} = \begin{bmatrix} -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 & 0 \\ * & -Z_1 & 0 & 0 \\ * & * & -Z_2 & 0 \\ * & * & * & -Z_2 \end{bmatrix}$$

$$\Phi_{2,11} = A^{\mathrm{T}}P + PA + Q_1 + M_3^{\mathrm{T}} + M_3$$

$$\Phi_{2,22} = Q_1 - Q_2 + M_3^{\mathrm{T}} + M_3 - Z_1$$

$$\Phi_{2,12} = Q_1 - Q_2 + M_3^{\mathrm{T}} + M_3$$

$$\Phi_{2,13} = PB - M_3^{\mathrm{T}} + M_4$$

$$\Phi_{2,23} = -M_3^{\mathrm{T}} + M_4, \bar{Z}_1 = \frac{h}{2}Z_1,\ \bar{Z}_2 = \frac{h}{2}Z_2$$

Note that M_3 is present in Φ_2 only, not in Φ_1. Rewriting Φ_2 as

$$\Phi_2 = \Phi_2|_{M_3=0} + U_2M_3V_2 + V_2^{\mathrm{T}}M_3^{\mathrm{T}}U_2^{\mathrm{T}} \tag{A6}$$

where

$$\begin{aligned} U_2 &= [I \quad I \quad -I \quad 0 \quad 0 \quad 0 \quad I]^{\mathrm{T}} \\ V_2 &= [I \quad I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \end{aligned} \tag{A7}$$

and applying the Elimination Lemma again, $\Phi_2 < 0$ for some M_3 if and only if

$$\Phi_3 = V_{2\perp}^{\mathrm{T}}\Phi_2 V_{2\perp} < 0;\ \Phi_4 = U_{2\perp}\Phi_2 U_{2\perp}^{\mathrm{T}} < 0 \tag{A8}$$

where

$$U_{2\perp} = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

$$V_{2\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{A9}$$

Simplifying Φ_3 and Φ_4 gives

$$\Phi_3 = \begin{bmatrix} \Xi_{11} & PB & \Psi_{14} & A^{\mathrm{T}}\bar{Z}_1 & A^{\mathrm{T}}\bar{Z}_1 & 0 \\ * & \Psi_{33} & -\varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}\bar{Z}_1 & B^{\mathrm{T}}\bar{Z}_2 & M_4^{\mathrm{T}} \\ * & * & -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 & 0 \\ * & * & * & -Z_1 & 0 & 0 \\ * & * & * & * & -Z_2 & 0 \\ * & * & * & * & * & -Z_1 \end{bmatrix}$$

$$\Phi_4 = \begin{bmatrix} \Xi_{11} & Z_1 & PB & \Psi_4 & A^{\mathrm{T}}\bar{Z}_1 & A^{\mathrm{T}}\bar{Z}_1 \\ * & \Xi_{22} & Z_2 & 0 & 0 & 0 \\ * & * & -Q_1 - Z_2 & -\varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}\bar{Z}_1 & B^{\mathrm{T}}\bar{Z}_2 \\ * & * & * & -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix} \tag{A10}$$

Note that $\Phi_1 < 0$ if $Z_1 > 0$ and $\Phi_3 < 0$. But $Z_1 > 0$ is implied by $\Phi_3 < 0$. Hence, $\Gamma < 0$ for some M_1, M_2, and M_3 if and only if $\Phi_3 < 0$ and $\Phi_4 < 0$.

Since M_4 is present only in Φ_3, not in Φ_4. Rewriting Φ_3 as

$$\Phi_3 = \Phi_3|_{M_4=0} + U_3M_4V_3 + V_3^{\mathrm{T}}M_4^{\mathrm{T}}U_3^{\mathrm{T}} \tag{A11}$$

with

$$\begin{aligned} U_3 &= [0 \quad -I \quad 0 \quad 0 \quad 0 \quad I]^{\mathrm{T}} \\ V_3 &= [0 \quad I \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \end{aligned} \tag{A12}$$

and applying the Elimination Lemma again, we get $\Phi_3 < 0$ for some M_4 if and only if

$$\Phi_5 = V_{3\perp}^{\mathrm{T}}\Phi_3 V_{3\perp} < 0,\ \Phi_6 = U_{3\perp}\Phi_3 U_{3\perp}^{\mathrm{T}} < 0 \tag{A13}$$

where

$$U_{3\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} V_{3\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{A14}$$

Simplifying the above gives

$$\Phi_5 = \begin{bmatrix} \Xi_{11} & \Psi_{14} & A^{\mathrm{T}}\bar{Z}_1 & A^{\mathrm{T}}\bar{Z}_1 & 0 \\ * & -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 & 0 \\ * & * & -Z_1 & 0 & 0 \\ * & * & * & -Z_2 & 0 \\ * & * & * & * & -Z_2 \end{bmatrix}$$

$$\Phi_6 = \begin{bmatrix} \Xi_{11} & PB & \Psi_{14} & A^{\mathrm{T}}\bar{Z}_1 & A^{\mathrm{T}}\bar{Z}_1 \\ * & -Q_1 - Z_2 & -\varepsilon N^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}\bar{Z}_1 & B^{\mathrm{T}}\bar{Z}_2 \\ * & * & -2\varepsilon I & D^{\mathrm{T}}\bar{Z}_1 & D^{\mathrm{T}}\bar{Z}_2 \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -Z_2 \end{bmatrix} \tag{A15}$$

We see that the left-top 4×4 block of Φ_5 can be obtained from Φ_4 by removing its second and third rows and columns and that $Z_2 > 0$ is implied by $\Phi_4 < 0$. Therefore, $\Phi_5 < 0$ if $\Phi_4 < 0$. Similarly, $\Phi_6 < 0$ can be obtained from Φ_4 by removing second row and column. So again, $\Phi_6 < 0$ if $\Phi_4 < 0$. Hence, $\Gamma < 0$ for some M_1, M_2, M_3, and M_4 if and only if $\Phi_4 < 0$.

Finally, using Schur complement, $\Phi_4 < 0$ if and only if $\hat{\Phi} < 0$, $Z_1 > 0$ and $Z_2 > 0$. And the latter two conditions are assumed.

The equivalence between (19) and (11) raises the following question: Supposing (11) holds, how do we choose M_1, M_2, M_3, and M_4 to satisfy (19)? The answer turns out to be rather simple:

$$M_1 = -Z_1,\ M_2 = Z_1,\ M_3 = -Z_2,\ M_4 = Z_4 \tag{A16}$$

Indeed, using Schur complement, (19) is equivalent to

$$\Phi_{11} + \left(\frac{h}{2}\right)^2 L_1^{\rm T}(Z_1+Z_2)L_1 + L_2^{\rm T}Z_1^{-1}L_2 + L_3^{\rm T}Z_2^{-1}L_3 < 0 \tag{A17}$$

which becomes (11) by taking (A16). □

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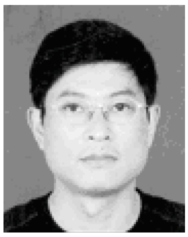

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