Fault Detection Filter Design for Linear Polytopic Uncertain Continuous-time Systems

WANG Heng[1] JU He-Hua[1] YANG Guang-Hong[2, 3]

Abstract The paper studies the problem of fault detection filter design for uncertain linear continuous-time systems. A design procedure dealing with parameter uncertainties is proposed for residual generation, the sensitivity to fault and the robustness against disturbances are both enhanced on residual outputs through satisfying some performance indexes. By the aid of the generalized Kalman-Yakubovich-Popov (GKYP) lemma, the fault sensitivity performance index can be dealt with in the given frequency range directly, which avoids approximations associated with frequency weights of the existing techniques. An iterative algorithm based on linear matrix inequality (LMI) is given to obtain the solutions. A numerical example is given to illustrate the effectiveness of the proposed methods.

Key words Fault detection, polytopic uncertainty, Kalman-Yakubovich-Popov (GKYP) lemma, linear matrix inequality (LMI)
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Fault detection plays an important role in enhancing the reliability and the safety of modern complex dynamic systems, which has attracted more attention. Many approaches have been proposed to detect faults, e.g., the multiple model and generalized likelihood methods, state observer, and parameter estimation approaches[1−7].

In the literature, fault detection filters are usually designed to detect faults, which rely on the use of particular type of state observers and produce the detection residuals. For ideal systems, the classical unknown input observer[8−9], and optimally robust parity relations techniques[10] have already been proposed in the literature to eliminate or minimize the disturbance and modeling error effects on residuals. However, in reality, the system parameters may either be uncertain or time-dependent, resulting in a mismatch between the actual system and the associated mathematical model used for residual generation[11]. For these cases, it is not possible to totally decouple the fault effects from the perturbation effects on the system, and the classical H_∞ control theory has been proved to be an effective tool to tackle these issues[12−18].

Recently, the frequency domain techniques for robust fault detection have received considerable attention. In [19], a frequency domain fault detection and isolation filter were designed so as to make the associated residual more robust to disturbances caused by unknown inputs. In [11], the Kharitonov polynomials and Dasgupta geometry were introduced to design the fault detection filter in the frequency domain. In [20], a tool was developed for the analysis of sampled-data systems in the frequency domain from the fault detection and isolation viewpoint.

In this paper, we consider the fault detection filter design problem for linear uncertain systems in frequency domain with frequency ranges of faults being known beforehand. By satisfying some performance indexes, the sensitivity to fault and the robustness against disturbances are both enhanced on residual outputs. Different from the classical methods which use the weighting matrices to restrict the frequency ranges of faults[21−25], the recently developed Kalman-Yakubovich-Popov (GKYP) lemma[26] is introduced in this study to give direct treatment of the finite frequency performances, completely avoiding approximations associated with frequency weights. An iterative linear matrix inequality (LMI) approach is given to solve the fault detection filter design problem since it is nonconvex in nature. It should be pointed out that the GKYP lemma has already been applied in [27−28] for fault detection and estimation, however, the fault sensitivity performance was not considered in [27] and the approach proposed in [28] cannot deal with the linear continuous-time systems with uncertainties. These will all be investigated in this paper to improve the approaches proposed in [27−28]. This paper is organized as follows. Section 1 presents the problem under consideration and some preliminaries. Section 2 considers the fault detection filter design problem in details, where an iterative linear matrix inequality (LMI) approach is given. Section 3 shows the effectiveness of the proposed design method via an example. Some concluding remarks are given in Section 4.

Notations. For a matrix A, $A^{\rm T}$, and $A^{\perp}$ denote its transpose and orthogonal complement, respectively. I denotes the identity matrix with an appropriate dimension. For a symmetric matrix, $A\ (\geq) > 0$ and $A\ (\leq) < 0$ denote (semi) positive definiteness and (semi) negative definiteness, respectively. The Hermitian part of a square matrix M is denoted by ${\rm He}(M) = M + M^{\rm T}$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. The symbol $*$ within a matrix represents the symmetric entries. $\sigma_{\max}(G)$ and $\sigma_{\min}(G)$ denote maximum and minimum singular values of the transfer matrix G, respectively.

$$\left[\Delta_{ij}\right]_{N\times N} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \dots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \dots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \dots & \Delta_{NN} \end{bmatrix}$$

1 Problem formulation

1.1 System model

Consider a linear time-invariant uncertain system described by

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1. College of Electronic Information and Control Engineering, Beijing University of Technology, Beijing 100124, P. R. China 2. Key Laboratory of Integrated Automation for the Process Industry, Ministry of Education, Shenyang 110004, P. R. China 3. College of Information Science and Engineering, Northeastern University, Shenyang 110004, P. R. China

$$\begin{aligned}\dot{x}(t) &= A(\lambda)x(t) + B_u(\lambda)u(t) + B(\lambda)f(t) + E(\lambda)w(t)\\ y(t) &= C(\lambda)x(t) + D_u(\lambda)u(t) + D(\lambda)f(t) + F(\lambda)w(t)\end{aligned} \quad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state, $x(0) = x_0$, $u(t) \in \mathbf{R}^{n_u}$ is an external input, $f(t) \in \mathbf{R}^p$ is the fault vector which denotes the actuator or component fault, $w(t) \in \mathbf{R}^{n_w}$ is the bounded external disturbance, and $y(t) \in \mathbf{R}^m$ denotes the measured output with $m \geq p$. All matrix dimensions are known, $A(\lambda)$ is assumed to be stable, and it is assumed that matrix $\bar{M}(\lambda)$, which is defined as

$$\bar{M}(\lambda) = \begin{bmatrix} A(\lambda) & B_u(\lambda) & B(\lambda) & E(\lambda) \\ C(\lambda) & D_u(\lambda) & D(\lambda) & F(\lambda) \end{bmatrix} \quad (2)$$

is unknown but belongs to a given convex bounded polyhedral domain D_c. That is, each uncertain matrix in this domain may be written as an unknown convex combination of s given extreme matrices $\bar{M}_1, \bar{M}_2, \cdots, \bar{M}_s$ such that

$$D_c = \left\{ \bar{M}(\lambda) : \bar{M}(\lambda) = \sum_{i=1}^{s} \lambda_i \bar{M}_i, \lambda_i \geq 0, \sum_{i=1}^{s} \lambda_i = 1 \right\} \quad (3)$$

where uncertain parameters, $\lambda_i, i = 1, \cdots, s$ are not available.

To detect fault $f(t)$, we design a fault detection filter which is of the following form:

$$\dot{\hat{x}}(t) = A_f\hat{x}(t) + B_f y(t), \quad \hat{y}(t) = C_f\hat{x}(t) + D_f y(t) \quad (4)$$

where the vector $\hat{x}(t)$ is the filter state vector, and A_f, B_f, C_f, and D_f are real matrices of appropriate dimensions to be determined, and $\hat{y}(t)$ is the output of the fault detection filter. The order of the filter n_f is restricted to be equal to the system order n. Then, we get the residual output $r(t)$ as

$$r(t) = y(t) - \hat{y}(t) \quad (5)$$

Combining (1) and (4), we have the following augmented model:

$$\begin{aligned}\dot{\xi}(t) &= \bar{A}(\lambda)\xi(t) + \bar{B}_u(\lambda)u(t) + \bar{B}(\lambda)f(t) + \bar{B}_w(\lambda)w(t)\\ r(t) &= \bar{C}(\lambda)\xi(t) + \bar{D}_u(\lambda)u(t) + \bar{D}(\lambda)f(t) + \bar{F}(\lambda)w(t)\end{aligned} \quad (6)$$

where $r(t)$ is the estimation error, $\xi(t) = \begin{bmatrix} x(t)^{\mathrm{T}} & \hat{x}(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and

$$\begin{bmatrix} \bar{A}(\lambda) & \bar{B}_u(\lambda) & \bar{B}(\lambda) & \bar{B}_w(\lambda) \\ \bar{C}(\lambda) & \bar{D}_u(\lambda) & \bar{D}(\lambda) & \bar{F}(\lambda) \end{bmatrix} = \left[\begin{array}{cc:c:c:c} A(\lambda) & 0 & B_u(\lambda) & B(\lambda) & E(\lambda) \\ B_fC(\lambda) & A_f & B_fD_u(\lambda) & B_fD(\lambda) & B_fF(\lambda) \\ \hdashline C(\lambda) - D_fC(\lambda) & -C_f & D_u(\lambda) - D_fD_u(\lambda) & D(\lambda) - D_fD(\lambda) & F(\lambda) - D_fF(\lambda) \end{array}\right] \quad (7)$$

1.2 Problem formulation and preliminaries

The fault detection problem can be expressed as follows.

Fault detection filter design problem. Given system (1), design a fault detection filter (4) such that the augmented model (6) is stable, and the fault effects on residual are maximized while the disturbance and reference input effects on residual are minimized. More specifically, we will find a filter such that the following performance indexes are satisfied

$$1)\ \sup_{\omega \in \Omega_1} \sigma_{\max}(G_{ru}(\mathrm{j}\omega)) < \gamma_u,\ \Omega_1 = [\varpi_1, \varpi_2] \quad (7)$$

$$2)\ \sup_{\omega} \sigma_{\max}(G_{rw}(\mathrm{j}\omega)) < \gamma_w \quad (8)$$

$$3)\ \inf_{\omega \in \Omega_2} \sigma_{\min}(G_{rf}(\mathrm{j}\omega)) > \beta,\ \Omega_2 = [-\varpi_l, \varpi_l] \quad (9)$$

where

$$\begin{aligned} G_{ru}(\mathrm{j}\omega) &= \bar{C}(\lambda)(\mathrm{j}\omega + \bar{D}_u(\lambda)I - \bar{A}(\lambda))^{-1}\bar{B}_u(\lambda)\\ G_{rw}(\mathrm{j}\omega) &= \bar{C}(\lambda)(\mathrm{j}\omega I - \bar{A}(\lambda))^{-1}\bar{B}_w(\lambda) + \bar{F}(\lambda)\\ G_{rf}(\mathrm{j}\omega) &= \bar{C}(\lambda)(\mathrm{j}\omega I - \bar{A}(\lambda))^{-1}\bar{B}(\lambda) + \bar{D}(\lambda)\end{aligned}$$

are the transfer function matrices from reference input $u(t)$, disturbance $w(t)$, and fault $f(t)$ to residual output $r(t)$, respectively. ϖ_1, ϖ_2, and ϖ_l are given scalars which reflect the frequency ranges of external input and faults, respectively. The minus sign of $-\varpi_l$ denotes the direction of rotation, which does not affect the physical meaning of frequency ω.

Remark 1. Conditions (8) and (9) are used to minimize the effects of reference input and disturbance on residual outputs. Condition (10) is used to maximize the effects of faults on residual outputs in finite frequency ranges. Note that the fault $f(t)$ is restricted to the low frequency range in condition (10). This usually occurs in practice, e.g., the stuck fault ($\omega = 0$) considered in [29−30] and the incipient fault stated in [3] both belonged to the low frequency domain. The frequency range of input $u(t)$ is assumed to be in certain frequency range $[\varpi_1, \varpi_2]$, which is known beforehand.

The following preliminaries are essential for later developments.

Lemma 1 (GKYP Lemma[26]). Given system (A, B, C, D), if symmetric matrix Π is of appropriate dimension, the following statements are equivalent:

1) The finite frequency inequality

$$\begin{bmatrix} G(\mathrm{j}\omega) \\ I \end{bmatrix}^{\mathrm{T}} \Pi \begin{bmatrix} G(\mathrm{j}\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in \Omega_\ell \quad (10)$$

where $G(\mathrm{j}\omega) = C(\mathrm{j}\omega I - A)^{-1}B + D$, $\ell = 1, 2$.

2) There exist matrices P and Q satisfying $Q > 0$, and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^{\mathrm{T}} \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^{\mathrm{T}} \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \quad (11)$$

where $\Xi = \begin{bmatrix} -Q & P - \mathrm{j}\varpi_c Q \\ P - \mathrm{j}\varpi_c Q & -\varpi_1\varpi_2 Q \end{bmatrix}$, $\omega_c = (\varpi_1 + \varpi_2)/2$, for middle frequency range $\omega \in \Omega_1$, $\Xi = \begin{bmatrix} -Q & P \\ P & \varpi_l^2 Q \end{bmatrix}$ for low frequency range $\omega \in \Omega_2$

Lemma 2 (Finsler's Lemma[31]). If $\mathcal{Q} \in \mathbf{R}^{n\times n}$ and $\mathcal{U} \in \mathbf{R}^{n\times m}$, and if $\mathcal{U}^{\perp}$ be any matrix such that $\mathcal{U}^{\perp}\mathcal{U} = 0$, then the following statement are equivalent:

1) $\mathcal{U}^{\perp}\mathcal{Q}\mathcal{U}^{\perp^{\mathrm{T}}} < 0$;

2) $\exists \mathcal{Y} \in \mathbf{R}^{m\times n} : \mathcal{Q} + \mathcal{U}\mathcal{Y} + \mathcal{Y}^{\mathrm{T}}\mathcal{U}^{\mathrm{T}} < 0$.

The following lemma provides an alternative condition for inequality (12). We define $J, \bar{H}$, and $\bar{L}$ of appropriate dimensions as

$$J = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} 0 & 0 \\ C^{\mathrm{T}} & 0 \\ D^{\mathrm{T}} & I \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} -I \\ A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}$$

Lemma 3. For matrix variables $P, Q \in \mathbf{R}^{n\times n}$ and $Q > 0$, the condition in (12) holds if there exists $\mathcal{X}$ of appropriate dimension such that

$$J\Xi J^{\mathrm{T}} + \bar{H}\Pi\bar{H}^{\mathrm{T}} < \mathrm{He}(\bar{L}\mathcal{X}) \tag{12}$$

holds, where Ξ and Π are the same as those defined in Lemma 1.

Proof. Notice that the null space of $\bar{L}$ is $\begin{bmatrix} A^{\mathrm{T}} & I & 0 \\ B^{\mathrm{T}} & 0 & I \end{bmatrix}$. Using Lemma 2, the result is immediate. □

Lemma 4. Inequality condition

$$\begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix}^{\mathrm{T}} \sum_{k=1}^{s} \lambda_k J^k \begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix} \geq 0 \tag{13}$$

holds if there exist symmetric matrices $J_{ii}^{j}, j \neq i, 1 \leq i \leq s$, $1 \leq j \leq s$, and $J_{ij}^{k}, 1 \leq i < j \leq s$, $1 \leq k \leq s$ such that the following LMIs hold:

$$J_{ii}^{j} + J_{ij}^{i} + (J_{ij}^{i})^{\mathrm{T}} > 0, 1 \leq i < j \leq s \tag{14}$$

$$J_{jj}^{i} + J_{ij}^{j} + (J_{ij}^{j})^{\mathrm{T}} > 0, 1 \leq i < j \leq s \tag{15}$$

$$\mathrm{He}(J_{ij}^{k} + J_{ik}^{j} + J_{jk}^{i}) > 0, 1 \leq i < j < k \leq s \tag{16}$$

where $J^k = \left[J_{ij}^{k}\right]_{s\times s}, J_{ii}^{i} = 0, J_{ij}^{k} = (J_{ji}^{k})^{\mathrm{T}}, 1 \leq i < j \leq s$.

Proof. Note that

$$\begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix}^{\mathrm{T}} \sum_{k=1}^{s} \lambda_k J^k \begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix} = \sum_{k=1}^{s} \lambda_k \begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix}^{\mathrm{T}} \times$$

$$\begin{bmatrix} J_{11}^{k} & \cdots & J_{1s}^{k} \\ \vdots & \ddots & \vdots \\ (J_{1s}^{k})^{\mathrm{T}} & \cdots & J_{ss}^{k} \end{bmatrix} \times \begin{bmatrix} \lambda_1 I \\ \vdots \\ \lambda_s I \end{bmatrix} =$$

$$\sum_{i=1}^{s}\sum_{j=i+1}^{s} \lambda_i^2 \lambda_j (J_{ii}^{j} + J_{ij}^{i} + (J_{ij}^{i})^{\mathrm{T}}) +$$

$$\sum_{i=1}^{s}\sum_{j=i+1}^{s} \lambda_i \lambda_j^2 (J_{jj}^{i} + J_{ij}^{j} + (J_{ij}^{j})^{\mathrm{T}}) +$$

$$\sum_{i=1}^{s}\sum_{j=i+1}^{s}\sum_{k=j+1}^{s} \lambda_i \lambda_j \lambda_k \Upsilon_{ijk}$$

where $\Upsilon_{ijk} = \mathrm{He}(J_{ij}^{k} + J_{ik}^{j} + J_{jk}^{i})$. Then, from (15) ~ (17), the result is immediate. □

2 Fault detection filter design

This section considers the fault detection filter design problem. Since it is a nonconvex problem in nature, an iterative LMIs approach is proposed to solve the fault detection filter design problem. This section is arranged as follows. Inequality conditions for performance indexes (8) ~ (10) are formulated in Subsections 2.1 and 2.2, an algorithm is given in Subsection 2.3.

2.1 Conditions for disturbance attenuation objective

The following lemma is essential for the main theorem of this section.

Lemma 5. Given the same matrices $\bar{A}(\lambda)$, $\bar{B}_w(\lambda)$, $\bar{C}(\lambda)$, $\bar{F}(\lambda)$ as stated in (6), the following statements are equivalent:

1) There exist matrix variables A_f, B_f, C_f, $X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$ and positive scalar γ such that

$$\begin{bmatrix} \bar{A}(\lambda)^{\mathrm{T}}X + X\bar{A}(\lambda) & X\bar{B}_w(\lambda) & \bar{C}^{\mathrm{T}}(\lambda) \\ * & -\gamma I & (F(\lambda) - D_f F(\lambda))^{\mathrm{T}} \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{17}$$

holds, where $\bar{A}(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ B_f C(\lambda) & A_f \end{bmatrix}$, $\bar{C}(\lambda) = \begin{bmatrix} C(\lambda) - D_f C(\lambda) & -C_f \end{bmatrix}$.

2) There exist matrix variables A_{fe}, B_{fe}, C_{fe}, $X_a = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ and positive scalar γ such that

$$\begin{bmatrix} \bar{A}_a(\lambda)^{\mathrm{T}}X_a + X\bar{A}_a(\lambda) & X_a\bar{B}_w(\lambda) & \bar{C}^{\mathrm{T}}(\lambda) \\ * & -\gamma I & (F(\lambda) - D_f F(\lambda))^{\mathrm{T}} \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{18}$$

holds, where $\bar{A}_a(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ B_{fe} C(\lambda) & A_{fe} \end{bmatrix}$, $\bar{C}_a(\lambda) = \begin{bmatrix} C(\lambda) - D_f C(\lambda) & -C_{fe} \end{bmatrix}$.

Proof. Define $X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$ with $X_{12}, X_{22} \in \mathbf{R}^{n\times n}$ being nonsingular. Then, we have

$$X_a = \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix} X \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$$

with $Y = X_{11}$ and $N = X_{12}X_{22}^{-1}X_{12}^{\mathrm{T}}$. Let

$$\bar{F} = \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix}, \ \mathcal{F} = \mathrm{diag}\{\bar{F}, I, I\} \tag{19}$$

Then, multiplying the left-hand side of (18) by full rank matrix $\mathcal{F}$ while multiplying the right-hand side of (18) by $\mathcal{F}^{\mathrm{T}}$ produces (19) with

$$\bar{A}_a(\lambda) = \begin{bmatrix} I & 0 \\ 0 & -X_{22}^{-1}X_{12}^{\mathrm{T}} \end{bmatrix}^{-1} \bar{A}(\lambda) \begin{bmatrix} I & 0 \\ 0 & -X_{22}^{-1}X_{12}^{\mathrm{T}} \end{bmatrix} =$$

$$\begin{bmatrix} A(\lambda) & 0 \\ B_{fe}C(\lambda) & A_{fe} \end{bmatrix}$$

$$A_{fe} = (X_{12}^{\mathrm{T}})^{-1}X_{22}A_f X_{22}^{-1}X_{12}^{\mathrm{T}}, B_{fe} = -(X_{12}^{\mathrm{T}})^{-1}X_{22}B_f$$

$$\bar{C}_a = \begin{bmatrix} C(\lambda) - D_f C(\lambda) & -C_{fe} \end{bmatrix}, \ C_{fe} = -C_f X_{22}^{-1}X_{12}^{\mathrm{T}}$$

□

From Lemma 5, it can be concluded that the matrix variable X can be chosen to be $\begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. Then, we have the following theorem, which provides a sufficient condition for performance index (9).

Theorem 1. Consider system model (6) and let real matrices $\bar{A}(\lambda) \in \mathbf{R}^{2n\times 2n}$, $\bar{B}_w(\lambda) \in \mathbf{R}^{2n\times n_w}$, $\bar{C}(\lambda) \in \mathbf{R}^{m\times 2n}$, $\bar{F}(\lambda) \in \mathbf{R}^{m\times n_w}$. The augmented system (6) is stable and the condition

$$\sigma_{\max}(G_{rw}(\mathrm{j}\omega)) < \gamma_w, \ \forall \omega \in \mathbf{R} \tag{20}$$

holds, if there exist matrix variables $Y, N, \mathcal{A} = NA_f, \mathcal{B} = NB_f$, satisfying $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix} > 0$, and

$$\begin{bmatrix} \psi_i & -\mathcal{A} + (-NA_i + \mathcal{B}C_i)^{\mathrm{T}} & YE_i - \mathcal{B}F_i & (C_i - D_f C_i)^{\mathrm{T}} \\ * & \mathcal{A} + \mathcal{A}^{\mathrm{T}} & -NE_i + \mathcal{B}F_i & -C_f^{\mathrm{T}} \\ * & * & -\gamma_w I & (F_i - D_f F_i)^{\mathrm{T}} \\ * & * & * & -\gamma_w I \end{bmatrix} < 0, \quad i = 1, \cdots, s \tag{21}$$

where $\psi_i = YA_i - \mathcal{B}C_i + (YA_i - \mathcal{B}C_i)^{\mathrm{T}}$.

Proof. Applying the bounded real lemma[32], we have that condition (21) satisfied if and only if the following inequality holds:

$$\begin{bmatrix} \bar{A}(\lambda)^{\mathrm{T}} X + X\bar{A}(\lambda) & X\bar{B}_w(\lambda) & \bar{C}(\lambda)^{\mathrm{T}} \\ * & -\gamma_w I & \bar{F}(\lambda)^{\mathrm{T}} \\ * & * & -\gamma_w I \end{bmatrix} < 0 \tag{22}$$

Applying Lemma 5, we have that matrix variable X can be chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. Then, (23) becomes

$$\begin{bmatrix} \psi(\lambda) & -\mathcal{A} + (-NA(\lambda) + \mathcal{B}C(\lambda))^{\mathrm{T}} & YE(\lambda) - \mathcal{B}F(\lambda) & (C(\lambda) - D_f C(\lambda))^{\mathrm{T}} \\ * & \mathcal{A} + \mathcal{A}^{\mathrm{T}} & -NE(\lambda) + \mathcal{B}F(\lambda) & -C_f^{\mathrm{T}} \\ * & * & -\gamma_w I & (F(\lambda) - D_f F(\lambda))^{\mathrm{T}} \\ * & * & * & -\gamma_w I \end{bmatrix} < 0 \tag{23}$$

where $\psi(\lambda) = YA(\lambda) - \mathcal{B}C(\lambda) + (YA(\lambda) - \mathcal{B}C(\lambda))^{\mathrm{T}}$.

Since inequality (22) is linearly dependent on A_i, E_i, C_i, F_i, multiplying each inequality in (22) by the uncertain parameter λ_i and then evaluating the sum from $i = 1, \cdots, s$ produces (24). □

Theorem 2. Consider system model (6) and let real matrices $\bar{A}(\lambda) \in \mathbf{R}^{2n\times 2n}$, $\bar{B}_u(\lambda) \in \mathbf{R}^{2n\times n_u}$, and $\bar{C}(\lambda) \in \mathbf{R}^{m\times 2n}$. The augmented system (6) is stable, and the condition

$$\sigma_{max}(G_{ru}(\mathrm{j}\omega)) < \gamma_u,\ \forall \omega \in [\varpi_1, \varpi_2] \tag{24}$$

holds, if there exist matrix variables $Y, N, \mathcal{A} = NA_f, \mathcal{B} = NB_f$, $\bar{P}_i = \begin{bmatrix} P_{ai} & P_{bi} \\ * & P_{ci} \end{bmatrix}$, $\bar{Q}_i = \begin{bmatrix} Q_{ai} & Q_{bi} \\ * & Q_{ci} \end{bmatrix}$ satisfying $\bar{Q}_i > 0$, and

$$\begin{bmatrix} -Q_{ai} & -Q_{bi} & P_{ai} + \mathrm{j}\varpi_c Q_{ai} - Y & P_{bi} + \mathrm{j}\varpi_c Q_{bi} + N & 0 & 0 \\ * & -Q_{ci} & P_{bi}^{\mathrm{T}} + \mathrm{j}\varpi_c Q_{bi} + N & P_{ci} + \mathrm{j}\varpi_c Q_{ci} - N & 0 & 0 \\ * & * & \Upsilon_{1i} & \Upsilon_{2i} & YB_{ui} & C_i^{\mathrm{T}} - C_i^{\mathrm{T}} D_f^{\mathrm{T}} \\ * & * & * & -\varpi_1\varpi_2 Q_{bi} + \mathcal{A} + \mathcal{A}^{\mathrm{T}} & -NB_{ui} & -C_f^{\mathrm{T}} \\ * & * & * & * & -\gamma_u^2 I & D_{ui} - D_f D_{ui} \\ * & * & * & * & * & -I \end{bmatrix} < 0, \quad i = 1, \cdots, s \tag{25}$$

where $\varpi_c = (\varpi_1 + \varpi_2)/2$, $\Upsilon_{1i} = -\varpi_1\varpi_2 Q_{ai} + YA_i - \mathcal{B}C_i + (YA_i - \mathcal{B}C_i)^{\mathrm{T}}$, and $\Upsilon_{2i} = -\varpi_1\varpi_2 Q_{bi} - \mathcal{A} + (-NA_i + \mathcal{B}C_i)^{\mathrm{T}}$.

Proof. Given $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma_u^2 I \end{bmatrix}$, (11) becomes (25). Applying Lemmas 1 and 3, we have that condition (25) is satisfied if the following inequality is feasible for some matrix $\mathcal{X}$ of appropriate dimension:

$$J\Xi J^{\mathrm{T}} + \bar{H}\Pi\bar{H}^{\mathrm{T}} < \mathrm{He}(\bar{L}\mathcal{X}) \tag{26}$$

where $\Xi = \begin{bmatrix} -\bar{Q}(\lambda) & \bar{P}(\lambda) + \mathrm{j}\varpi_c Q(\lambda) \\ \bar{P}(\lambda) - \mathrm{j}\varpi_c Q(\lambda) & -\varpi_1\varpi_2\bar{Q}(\lambda) \end{bmatrix}$, $\bar{H} = \begin{bmatrix} 0 & 0 \\ \bar{C}(\lambda)^{\mathrm{T}} & 0 \\ \bar{D}(\lambda)^{\mathrm{T}} & I \end{bmatrix}$, and $\bar{L} = \begin{bmatrix} -I \\ \bar{A}(\lambda)^{\mathrm{T}} \\ \bar{B}(\lambda)^{\mathrm{T}} \end{bmatrix}$.

By rewriting the matrix $\mathcal{X}$ as $\mathcal{X} = XR$ and setting

$$R = \begin{bmatrix} 0 & -I & 0 \end{bmatrix}$$

using Schur complement, and after some matrix manipulation, (27) becomes

$$\begin{bmatrix} -\bar{Q}(\lambda) & \bar{P}(\lambda) + \mathrm{j}\varpi_c\bar{Q}(\lambda) - X & 0 & 0 \\ * & -\varpi_1\varpi_2\bar{Q}(\lambda) + \bar{A}(\lambda)^{\mathrm{T}} X + X\bar{A}(\lambda) & X\bar{B}_u(\lambda) & \bar{C}(\lambda)^{\mathrm{T}} \\ * & * & -\gamma_u^2 I & \bar{D}_u(\lambda)^{\mathrm{T}} \\ * & * & * & -I \end{bmatrix} < 0 \tag{27}$$

Similar to Theorem 1, X are chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. Multiplying each inequality in (26) by uncertain parameter λ_i and then evaluating the sum from $i = 1, \cdots, s$ produces (28). □

Remark 2. From the proof of Theorem 2, it can be seen that since the matrix R needs to be determined beforehand, only sufficient conditions are obtained for (25) in Theorem 2. A similar treatment was also presented in [33].

2.2 Conditions for fault detection objective

The following theorem provides inequality conditions for performance index (10).

Theorem 3. Consider system model (6) and let real matrices $\bar{A}(\lambda) \in \mathbf{R}^{2n\times 2n}$, $\bar{B}(\lambda) \in \mathbf{R}^{2n\times p}$, $\bar{C}(\lambda) \in \mathbf{R}^{m\times 2n}$, $\bar{D}(\lambda) \in \mathbf{R}^{m\times p}$, and symmetric matrix $\Pi_1 = \begin{bmatrix} -I & 0 \\ 0 & \beta^2 I \end{bmatrix}$ be given. Then, the inequality condition

$$\sigma_{\min}(G_{rf}(\mathrm{j}\omega)) > \beta,\ \forall|\omega| \le \varpi_l \tag{28}$$

holds, if there exist matrix variables $Y, N, \mathcal{A} = NA_f, \mathcal{B} = NB_f$, $\bar{P}_1(\lambda) = \begin{bmatrix} P_1(\lambda) & P_2(\lambda) \\ * & P_3(\lambda) \end{bmatrix}$, $\bar{Q}_1(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ * & Q_3(\lambda) \end{bmatrix}$ satisfying $\bar{Q}_1(\lambda) > 0$, and

$$\Delta(\lambda) < 0 \tag{29}$$

where

$$\Delta(\lambda) = \begin{bmatrix} -Q_1(\lambda) & -Q_2(\lambda) & P_1(\lambda) - Y & P_2(\lambda) + N & YB_0 \\ * & -Q_3(\lambda) & P_2(\lambda)^{\mathrm{T}} + N & P_3(\lambda) - N & -NB_0 \\ * & * & \Gamma_1(\lambda) & \Gamma_2(\lambda) & \Gamma_4(\lambda) \\ * & * & * & \Gamma_3(\lambda) & \Gamma_5(\lambda) \\ * & * & * & * & \Gamma_6(\lambda) \end{bmatrix}$$

with

$$\begin{aligned}\Gamma_1(\lambda) = {} & \varpi_l^2 Q_1(\lambda) + YA(\lambda) - \mathcal{B}C(\lambda) + (YA(\lambda) - \mathcal{B}C(\lambda))^{\mathrm{T}} - \\ & C(\lambda)^{\mathrm{T}}C(\lambda) + C(\lambda)^{\mathrm{T}}D_f^{\mathrm{T}}C(\lambda) - \\ & C(\lambda)^{\mathrm{T}}D_f^{\mathrm{T}}D_{f0}C(\lambda) + C(\lambda)^{\mathrm{T}}D_fC(\lambda) - \\ & C(\lambda)^{\mathrm{T}}D_{f0}^{\mathrm{T}}D_fC(\lambda) + C(\lambda)^{\mathrm{T}}D_{f0}^{\mathrm{T}}D_{f0}C(\lambda) \\ \Gamma_2(\lambda) = {} & \varpi_l^2 Q_2(\lambda) - \mathcal{A} + (-NA(\lambda) + \mathcal{B}C(\lambda))^{\mathrm{T}} + \\ & C(\lambda)^{\mathrm{T}}C_f - C(\lambda)^{\mathrm{T}}D_f^{\mathrm{T}}C_{f0} - C(\lambda)^{\mathrm{T}}D_{f0}^{\mathrm{T}}C_f + \\ & C(\lambda)^{\mathrm{T}}D_{f0}^{\mathrm{T}}C_{f0} \\ \Gamma_3(\lambda) = {} & \varpi_l^2 Q_3(\lambda) + \mathcal{A} + \mathcal{A}^{\mathrm{T}} - C_f^{\mathrm{T}}C_{f0} - C_{f0}^{\mathrm{T}}C_f + C_{f0}^{\mathrm{T}}C_{f0} \\ \Gamma_4(\lambda) = {} & YB(\lambda) - (YA(\lambda) - \mathcal{B}C(\lambda))^{\mathrm{T}}B_0 \\ \Gamma_5(\lambda) = {} & -NB(\lambda) + \mathcal{A}^{\mathrm{T}}B_0 \\ \Gamma_6(\lambda) = {} & \beta^2 I - B_0^{\mathrm{T}}YB(\lambda) - B(\lambda)^{\mathrm{T}}Y^{\mathrm{T}}B_0\end{aligned}$$

where C_{f0} and D_{f0} are auxiliary variables which provide the initial values of C_f and D_f; B_0 is a nominal value of the system matrix $B(\lambda)$.

Proof. Similar to Theorem 2, given $\Pi_1 = \begin{bmatrix} -I & 0 \\ 0 & \beta^2 I \end{bmatrix}$, (11) becomes (29). Applying Lemmas 1 and 3, we have that condition (29) is satisfied if the following inequality is feasible for some matrix $\mathcal{X}$ of appropriate dimension:

$$J\Xi_1 J^{\mathrm{T}} + \bar{H}_1\Pi_1\bar{H}_1^{\mathrm{T}} < \mathrm{He}(\bar{L}_1(\lambda)\mathcal{X}) \tag{30}$$

where $\Xi_1 = \begin{bmatrix} -\bar{Q}_1(\lambda) & \bar{P}_1(\lambda) \\ \bar{P}_1(\lambda) & \varpi_l^2\bar{Q}_1(\lambda) \end{bmatrix}$.

$$\bar{H}_1 = \begin{bmatrix} 0 & 0 \\ \bar{C}(\lambda)^{\mathrm{T}} & 0 \\ \bar{D}(\lambda)^{\mathrm{T}} & I \end{bmatrix},\ \bar{L}_1(\lambda) = \begin{bmatrix} -I \\ \bar{A}(\lambda)^{\mathrm{T}} \\ \bar{B}(\lambda)^{\mathrm{T}} \end{bmatrix}$$

By rewriting the matrix $\mathcal{X}$ as $\mathcal{X} = XR_1$ and setting

$$R_1 = \begin{bmatrix} 0 & -I & \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \end{bmatrix}$$

where B_0 is the nominal value of the system matrix $B(\lambda)$, using Schur complement, and after some matrix manipulation, (31) becomes

$$\Omega(\lambda) - \Theta(\lambda)^{\mathrm{T}}\Theta(\lambda) < 0 \tag{31}$$

where

$$\Omega(\lambda) = \begin{bmatrix} -\bar{Q}_1(\lambda) & \bar{P}_1(\lambda) - X & X\begin{bmatrix} B_0 \\ 0 \end{bmatrix} \\ * & \varpi_l^2\bar{Q}_1(\lambda) + X\bar{A}(\lambda) + \bar{A}(\lambda)^{\mathrm{T}}X & X\bar{B}(\lambda) - \bar{A}(\lambda)^{\mathrm{T}}X\begin{bmatrix} B_0 \\ 0 \end{bmatrix} \\ * & * & \beta^2 I - \bar{B}(\lambda)^{\mathrm{T}}X\begin{bmatrix} B_0 \\ 0 \end{bmatrix} - (\bar{B}(\lambda)^{\mathrm{T}}X\begin{bmatrix} B_0 \\ 0 \end{bmatrix})^{\mathrm{T}} \end{bmatrix}$$

and $\Theta(\lambda) = \begin{bmatrix} 0 & \bar{C}(\lambda) & \bar{D}(\lambda) \end{bmatrix}$ where $\bar{C}(\lambda) = \begin{bmatrix} C(\lambda) - D_fC(\lambda) & -C_f \end{bmatrix}$, $\bar{D}(\lambda) = D(\lambda) - D_fC(\lambda)$.

As is known to all, there exists $\Theta_0(\lambda) = \begin{bmatrix} 0 & \bar{C}_0(\lambda) & \bar{D}_0(\lambda) \end{bmatrix}$ with

$$\bar{C}_0(\lambda) = \begin{bmatrix} C(\lambda) - D_{f0}C(\lambda) - C_{f0} \end{bmatrix},\quad \bar{D}_0(\lambda) = D(\lambda) - D_{f0}C(\lambda)$$

such that

$$(\Theta(\lambda) - \Theta_0(\lambda))^{\mathrm{T}}(\Theta(\lambda) - \Theta_0(\lambda)) \ge 0$$

holds. It can be concluded that if

$$\Omega(\lambda) - \Theta(\lambda)^{\mathrm{T}}\Theta(\lambda) + (\Theta(\lambda) - \Theta_0(\lambda))^{\mathrm{T}}(\Theta(\lambda) - \Theta_0(\lambda)) < 0 \tag{32}$$

holds, (32) readily. On the other hand, if (32) holds, there always exists $\Theta_0(\lambda) = \Theta(\lambda)$ such that (33) becomes (32), so we have that (33) is equivalent to (32).

Similar to Lemma 5, here X is chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. Then, after some matrix manipulation, (33) becomes (30), which completes the proof. □

Remark 3. From the proof of Theorem 3, it can be concluded that matrices C_{f0} and D_{f0} are two auxiliary matrix variables introduced here to provide initial values of matrix variables C_f and D_f for the later iterative algorithm. Matrix B_0 denotes the nominal value of $B(\lambda)$ which corresponds to the case when there is no uncertainty in $B(\lambda)$.

Note that inequality condition (30) cannot be implemented since it is not convex in parameter λ due to the product term $C(\lambda)^{\mathrm{T}}C(\lambda)$. To solve this problem, the following theorem is presented to provide a sufficient condition for inequality (30).

Theorem 4. The condition in (30) holds if there exist matrix variables J_{ii}^k, J_{ij}^k, $Y, N, \mathcal{A}, \mathcal{B}$, $\bar{P}_{1i} = \begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix}$, $\bar{Q}_{1i} = \begin{bmatrix} Q_{1i} & Q_{2i} \\ * & Q_{3i} \end{bmatrix}$ satisfying $\bar{Q}_{1i} > 0$, and the following inequalities:

$$J_{ii}^j + J_{ij}^i + (J_{ij}^i)^{\mathrm{T}} > 0,\quad 1 \le i < j \le s \tag{33}$$

$$J_{jj}^i + J_{ij}^j + (J_{ij}^j)^{\mathrm{T}} > 0,\quad 1 \le i < j \le s \tag{34}$$

$$\mathrm{He}(J_{ij}^k + J_{ik}^j + J_{jk}^i) > 0,\quad 1 \le i < j < k \le s \tag{35}$$

$$\begin{bmatrix}\Delta_{ij}\end{bmatrix}_{s\times s} + \begin{bmatrix}J_{ij}^k\end{bmatrix}_{s\times s} < 0,\quad 1 \le k \le s \tag{36}$$

where

$$\Delta_{ij} = \begin{bmatrix} -Q_{1i} & -Q_{2i} & P_{1i} - Y & P_{2i} + N & YB_0 \\ * & -Q_{3i} & P_{2i}^{\mathrm{T}} + N & P_{3i} - N & -NB_0 \\ * & * & \Gamma_{1ij} & \Gamma_{2i} & \Gamma_{4i} \\ * & * & * & \Gamma_{3i} & \Gamma_{5i} \\ * & * & * & * & \Gamma_{6i} \end{bmatrix}$$

with

$$\begin{aligned}\Gamma_{1ij} =& \varpi_l^2 Q_{1i} + YA_i - \mathcal{B}C_i + (YA_i - \mathcal{B}C_i)^{\mathrm{T}} - C_i^{\mathrm{T}} C_j + \\ & C_i^{\mathrm{T}} D_f^{\mathrm{T}} C_j - C_i^{\mathrm{T}} D_f^{\mathrm{T}} D_{f0} C_j + C_i^{\mathrm{T}} D_f C_j - \\ & C_i^{\mathrm{T}} D_{f0}^{\mathrm{T}} D_f C_j + C_i^{\mathrm{T}} D_{f0}^{\mathrm{T}} D_{f0} C_j \\ \Gamma_{2i} =& \varpi_l^2 Q_{2i} - \mathcal{A} + (-NA_i + \mathcal{B}C_i)^{\mathrm{T}} + C_i^{\mathrm{T}} C_f - \\ & C_i^{\mathrm{T}} D_f^{\mathrm{T}} C_{f0} - C_i^{\mathrm{T}} D_{f0}^{\mathrm{T}} C_f + C_i^{\mathrm{T}} D_{f0}^{\mathrm{T}} C_{f0} \\ \Gamma_{3i} =& \varpi_l^2 Q_{3i} + \mathcal{A} + \mathcal{A}^{\mathrm{T}} - C_f^{\mathrm{T}} C_{f0} C_{f0}^{\mathrm{T}} C_f + C_{f0}^{\mathrm{T}} C_{f0} \\ \Gamma_{4i} =& YB_i - (YA_i - \mathcal{B}C_i)^{\mathrm{T}} B_0 \\ \Gamma_{5i} =& -NB_i + \mathcal{A}^{\mathrm{T}} B_0 \\ \Gamma_{6i} =& \beta^2 I - B_0^{\mathrm{T}} Y B_i - B_i^{\mathrm{T}} Y^{\mathrm{T}} B_0\end{aligned}$$

Proof. Pre- and post-multiplying (37) by $\begin{bmatrix}\lambda_1 I & \cdots & \lambda_s I\end{bmatrix}$ and its transpose, we have

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\lambda_i\lambda_j\Delta_{ij} + \begin{bmatrix}\lambda_1 I\\ \vdots \\ \lambda_s I\end{bmatrix}^{\mathrm{T}} J^k \begin{bmatrix}\lambda_1 I\\ \vdots \\ \lambda_s I\end{bmatrix} < 0 \tag{37}$$

where $J^k = \left[J_{ij}^k\right]_{s\times s}$. Note that $\Delta(\lambda) = \sum_{i=1}^{s}\sum_{j=1}^{s}\lambda_i\lambda_j\Delta_{ij}$. Then, multiplying each inequality in (38) by the uncertain parameter λ_k and then evaluating the sum from $k = 1, \cdots, s$ produces

$$\Delta(\lambda) + \begin{bmatrix}\lambda_1 I\\ \vdots \\ \lambda_s I\end{bmatrix}^{\mathrm{T}} \sum_{k=1}^{s}\lambda_k J^k \begin{bmatrix}\lambda_1 I\\ \vdots \\ \lambda_s I\end{bmatrix} < 0 \tag{38}$$

Applying Lemma 4, the result is immediate. □

2.3 Solutions

Till now, inequality conditions for (8) ∼ (10) have been formulated in Theorems 1, 2, and 4, respectively. Summarily, we have the following theorem.

Theorem 5. Consider the uncertain system model (1), there exists a filter (4) such that the augmented system model (6) is stable and satisfies performance indexes (8) ∼ (10) if inequality conditions (22), (26), and (34) ∼ (37) hold.

Proof. Combining Theorems 1, 2, and 4, it is immediate. □

Note that all the inequalities to be satisfied in Theorem 5 are LMIs except those in (37) because of the product terms between auxiliary variables C_{f0}, D_{f0} and C_f, D_f. To solve this problem, the following algorithm is proposed which gives an integrated design process for appropriate solutions of the fault detection filter parameters A_f, B_f, C_f, D_f.

Algorithm 1. Given system (1), the augmented system model is denoted by (6). Let ϵ_0 be a given large enough constant specifying a stop criterion of this algorithm.

Step 1. Minimize $a_u\gamma_u + a_w\gamma_w$ subject to LMI constraints (22) and (6). The optimal solutions are denoted as $C_{f_{\rm opt}}^0$, $D_{f_{\rm opt}}^0$, and $\gamma_u^{\rm opt}$ and $\gamma_w^{\rm opt}$.

Step 2. Choose $\gamma_u > \gamma_u^{\rm opt}$, $\gamma_w > \gamma_w^{\rm opt}$, $C_f^1 = C_{f_{\rm opt}}^0$, $D_f^1 = D_{f_{\rm opt}}^0$, and maximize β subject to LMI constraints (22), (26), (34) ∼ (37) for $i = 1, \cdots, s$. Let $C_f^v = C_{f_{\rm opt}}^{v-1}$, $D_f^v = D_{f_{\rm opt}}^{v-1}$, where $C_{f_{\rm opt}}^{v-1}$ and $D_{f_{\rm opt}}^{v-1}$ are the solutions of the $(v-1)$-th optimization.

If $\beta^{v_0} < \epsilon_0$ for some $C_{f_{\rm opt}}^{v_0}$, $D_{f_{\rm opt}}^{v_0}$, denote $C_f^{v_0+1} = C_{f_{\rm opt}}^{v_0}$, $D_f^{v_0+1} = D_{f_{\rm opt}}^{v_0}$, and repeat the above optimization, else continue.

Step 3. When $\beta^v \geq \epsilon_0$ for any v in Step 2, exit.

Step 4. The filter parameters A_f, B_f are then obtained as $A_f = N^{-1}\mathcal{A}$, $B_f = N^{-1}\mathcal{B}$.

Remark 4. In Algorithm 1, Step 1 corresponds to an LMI optimization problem resulting from Theorems 1 and 2, to satisfy conditions (8) and (9), and to find C_f, D_f which provide the initial solutions for the iterative optimization in Step 2. Step 2 performs an iterative optimization on auxiliary variables C_{f0}, D_{f0} so that the conditions in Theorems 1, 2, and 4 can be satisfied simultaneously for the given performance constraints (8) ∼ (10).

After the fault detection, filter parameter matrices A_f, B_f, C_f, and D_f are designed, the residual evaluation function $J_r(\tau)$ and the threshold J_{th} can be selected as

$$J_r(\tau) = (\tau^{-1}\int_0^{\tau} r^{\mathrm{T}}(t) r(t) \mathrm{d}t)^{\frac{1}{2}}$$

where τ denotes the evaluation time. Under fault-free conditions, the residual output

$$r(s) = G_{ru}(s)u(s) + G_{rw}(s)w(s)$$

Similar to [18], via the Parseval′s theorem, we have that

$$\begin{aligned}\|r(\mathrm{j}\omega)\|_{rms,t,f=0} \leq & \|G_{ru}(\mathrm{j}\omega)\|_\infty \|u(\mathrm{j}\omega)\|_{rms} + \\ & \|G_{rw}(\mathrm{j}\omega)\|_\infty \|w(\mathrm{j}\omega)\|_{rms} = \\ & \gamma_u \|u(\mathrm{j}\omega)\|_{rms} + \gamma_w \bar{w}\end{aligned} \tag{39}$$

where $\bar{w}$ is a convenient upper bound to the rms-norm of the worst disturbance. Then, the threshold can be obtained as

$$J_{\rm th}(t) = \gamma_w \bar{w} + \gamma_u \|u(\mathrm{j}\omega)\|_{rms} \tag{40}$$

Based on this, the occurrence of faults can be detected by the following logic rule:

$$\begin{cases} J_r(\tau) \leq J_{\rm th}, & \text{no alarm} \\ J_r(\tau) > J_{\rm th}, & \text{alarm}\end{cases} \tag{41}$$

3 Numerical example

This section gives two numerical examples to illustrate the effectiveness of our approach.

Example 1. Consider the following system model presented in [18]

$$\begin{aligned} y_1(s) =& \frac{k_3}{s^2 + \theta_1 s + \theta_2}(u(s) + f(s)) + \\ & \frac{k_1 k_3}{(s^2 + \theta_1 s + \theta_2)(T_1 s + 1)} d_1(s) + k_2 \frac{T_2 s + 1}{T_3 s + 1} d_2(s) \\ y_2(s) =& \frac{k_3 s}{s^2 + \theta_1 s + \theta_2}(u(s) + f(s)) + \\ & \frac{k_1 k_3 s}{(s^2 + \theta_1 s + \theta_2)(T_1 s + 1)} d_1(s)\end{aligned}$$

where $T_1 = 0.1\,\mathrm{s}$, $T_2 = 1\,\mathrm{s}$, $T_3 = 0.2\,\mathrm{s}$, $k_1 = 0.3$, $k_2 = 0.2$, $k_3 = 1$, and parameters θ_1 and θ_2 belong to the intervals $0.5 \leq \theta_1 \leq 1.2$, $1 \leq \theta_2 \leq 1.5$. The signals $d_1(s)$ and $d_2(s)$ are assumed to be unitary variance white noises. The frequency range of faults $f(t)$ is known beforehand, i.e., $|\omega| \leq 0.1$. Applying Algorithm 1, firstly, we get the initial values of C_f, D_f to be

$$C_{f0} = \begin{bmatrix} -0.0344 & 0.9979 & 0.8251 & -0.2660 \\ 1.0235 & -0.0415 & 0.9634 & -0.6504 \end{bmatrix}$$

$$D_{f0} = \begin{bmatrix} -0.0057 & -0.0163 \\ -0.0004 & 0.1305 \end{bmatrix}$$

Finally, the fault detection filter parameters are obtained to be

$$A_f = \begin{bmatrix} -6.5407 & 4.9819 & 8.5243 & 16.5591 \\ 2.8526 & -3.5595 & -4.9398 & -8.6023 \\ 3.6148 & -2.0048 & -4.7440 & -5.7387 \\ 8.5232 & -6.9000 & -12.2845 & -23.4843 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -0.1806 & 0.8426 & 0.0934 & -1.4463 \\ 0.8988 & -0.0345 & -0.5393 & -1.6961 \end{bmatrix}$$

$$B_f = \begin{bmatrix} -4.3787 & 7.7769 \\ 3.0648 & -4.6219 \\ 2.2988 & -4.1262 \\ 7.4942 & -10.2365 \end{bmatrix}, \ D_f = \begin{bmatrix} -0.9511 & -0.4885 \\ 0.3859 & 0.5523 \end{bmatrix}$$

The fault sensitivity performance index β is obtained as 1.7, the other performance indexes $\gamma_u = 1.5956$ and $\gamma_w = 0.9502$.

To illustrate the simulation results, assume that a constant fault $f(t) = 1$ ($\omega = 0$) occurs at $t = 50\,\mathrm{s}$, and the reference input $u(t) = 0.5\sin(5t)$.

When $\theta_1 = 1.2$ and $\theta_2 = 1$, the residual output is shown in Fig. 1. From Fig. 1, it can be seen that the robustness against disturbance and the fault sensitivity are both enhanced, and the faults are well discriminated from disturbances.

To illustrate the advantage of our approach, following the approach presented in [18], where the frequency ranges of $f(t)$ and $u(t)$ were restricted by choosing appropriate weights, we get the fault detection filter parameters as

$$A_f = \begin{bmatrix} -1.3994 & 2.2675 & 2.3837 & 1.8453 \\ 0.6967 & -6.8895 & -6.0145 & -7.2073 \\ -2.1301 & 1.2942 & -2.3419 & 1.6826 \\ -0.5576 & -1.5666 & 1.0110 & -5.2763 \end{bmatrix}$$

$$B_f = \begin{bmatrix} 0.5695 & -0.8553 & 0 \\ -5.5605 & -0.7750 & 0 \\ 2.3988 & -0.4542 & 0 \\ 3.3906 & -1.9045 & 0 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -0.0965 & -1.3810 & -0.0712 & -0.0910 \\ -0.3172 & 0.6337 & -0.7169 & 0.8604 \end{bmatrix}$$

$$D_f = \begin{bmatrix} 0.9493 & 0.3479 & -0.2317 \\ -0.1608 & 0.3532 & 0.1150 \end{bmatrix}$$

and the residual output is shown in Fig. 2. From Figs. 1 and 2, it can be concluded that our approach obtains better fault sensitivity.

When $\theta_1 = 1.2$ and $\theta_2 = 1$, with the fault detection filter obtained in this paper, the singular value of the transfer function matrices $G_{rf}(\mathrm{j}\omega)$, $G_{ru}(\mathrm{j}\omega)$, $G_{rw}(\mathrm{j}\omega)$ in certain frequency ranges are plotted in Fig. 3. Fig. 4 shows the singular value plots of $G_{rf}(\mathrm{j}\omega)$, $G_{ru}(\mathrm{j}\omega)$, $G_{rw}(\mathrm{j}\omega)$ using the approach of [22].

Choosing the residual evaluation function and determining the threshold according to (41), we get the residual evaluation outputs and the thresholds as shown in Fig. 5, where the threshold is 0.6592. When the stuck fault occurs, it can readily be detected through the fault detection filters designed in this paper.

The following example includes the case when there is parameter uncertainty in system matrix C.

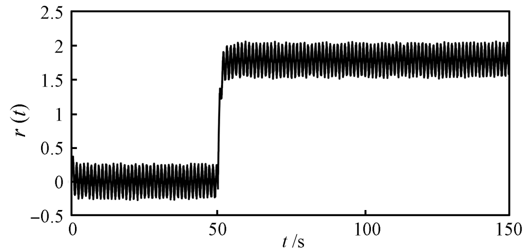

Fig. 1 Residual output $r(t)$ of Example 1 using the approach of this paper

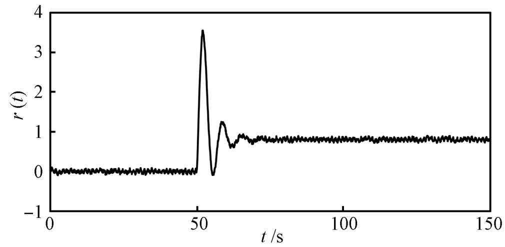

Fig. 2 Residual output $r(t)$ of Example 1 using the existing techniques

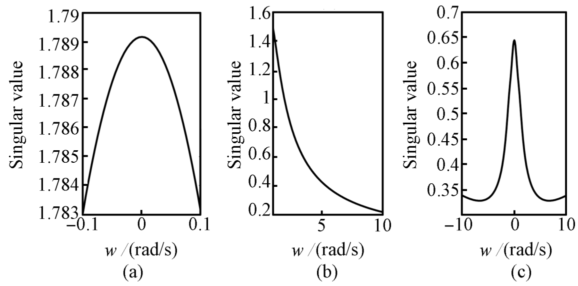

Fig. 3 Singular value plots of this paper ((a) The singular value plot of $\sigma_{\min}(G_{rf}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$; (b) The singular value plot of $\sigma_{\max}(G_{ru}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$; (c) The singular value plot of $\sigma_{\max}(G_{rw}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$)

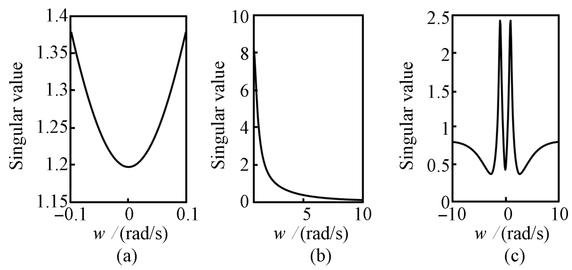

Fig. 4 Singular value plots of existing techniques ((a) The singular value plot of $\sigma_{\min}(G_{rf}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$; (b) The singular value plot of $\sigma_{\max}(G_{ru}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$; (c) The singular value plot of $\sigma_{\max}(G_{rw}(\mathrm{j}\omega))$ when $\theta_1 = 1.2$, $\theta_2 = 1$)

Example 2. Consider the system model

$$\dot{x}(t) = \begin{bmatrix} 0 & -0.8 \\ 1-\theta_1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + f(t)) + \begin{bmatrix} -0.45 \\ 0.35 \end{bmatrix} w(t)$$
$$y(t) = \begin{bmatrix} 0.5+\theta_2 & -1.5 \end{bmatrix} x(t) + 0.05w(t) \tag{42}$$

where $0 \leq \theta_1 \leq 0.2$, $0 \leq \theta_2 \leq 0.2$, and the frequency range of faults is $|\omega| \leq 0.1$.

To detect fault $f(t)$, a fault detection filter

$$\dot{x}(t) = A_f x_f(t) + B_f y(t)$$
$$\hat{y}(t) = C_f x_f(t)$$

is designed. Firstly, the initial value of C_f is obtained through Step 1 of Algorithm 1 as $C_{f0} = \begin{bmatrix} 0.4853 & -1.4183 \end{bmatrix}$, filter parameter matrices are obtained as

$$A_f = \begin{bmatrix} -0.7073 & 1.1811 \\ 0.6134 & -1.4369 \end{bmatrix}, B_f = \begin{bmatrix} 0.5969 \\ 0.7559 \end{bmatrix}$$
$$C_f = \begin{bmatrix} 0.4462 & -1.7461 \end{bmatrix}$$

performance index $\gamma_w = 0.8367, \gamma_u = 0.8944$, and $\beta = 1.0877$.

When $\theta_1 = 0.1$, $\theta_2 = 0.15$, we get the residual output as shown in Fig. 6, the residual evaluation outputs and the thresholds as shown in Fig. 7, where the threshold is 0.4. When a stuck fault occurs, it can readily be detected.

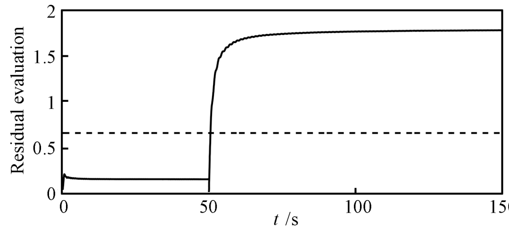

Fig. 5 The residual evaluation output of Example 1 (solid line) and the threshold (dashed line)

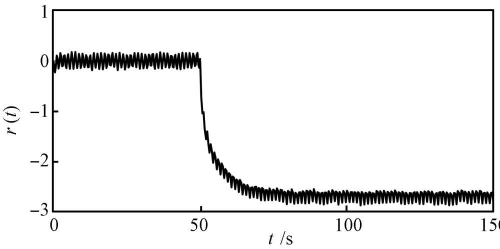

Fig. 6 Residual output $r(t)$ of Example 2 using the approach of this paper

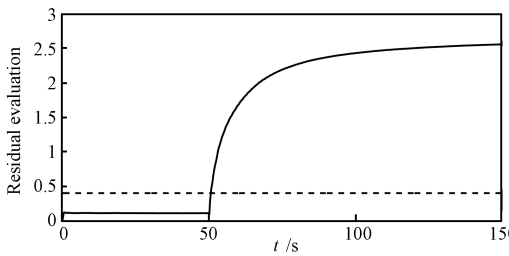

Fig. 7 The residual evaluation output of Example 2 (solid line) and the threshold (dashed line)

4 Conclusions

In this paper, the problem of fault detection filter design for uncertain linear continuous-time systems has been investigated. By the aid of the GKYP lemma and the bounded real lemma, inequality conditions for the finite frequency fault sensitivity performance and the full frequency disturbance robustness performance are both formulated. LMI conditions and iterative algorithm based on linear matrix inequality have been proposed, respectively. By comparing with existing techniques, the numerical example has illustrated the effectiveness of the proposed approach.

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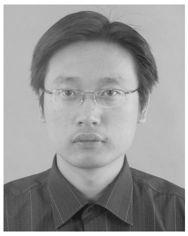

WANG Heng Lecturer at Beijing University of Technology. He received his Ph. D. degree from Northeastern University in 2008. His research interest covers fault detection and isolation, and robust control. Corresponding author of this paper. E-mail: wangheng@bjut.edu.cn

JU He-Hua Professor at Beijing University of Technology. His research interest covers robot motion planning and control.
E-mail: Juhehua@bjut.edu.cn

YANG Guang-Hong Professor at Northeastern University. His research interest covers fault-tolerant control, fault detection and isolation, and robust control.
E-mail: yangguanghong@ise.neu.edu.cn