Delay-dependent Stability Criteria for Singular Time-delay Systems

FENG Yi-Fu[1, 2] ZHU Xun-Lin[3] ZHANG Qing-Ling[1, 4]

Abstract This note studies the problem of singular time-delay systems. At first, a simplified stability criterion is derived after establishing equivalence among several recently proposed stability criteria. By using a delay decomposition method, a new stability criterion, which is much less conservative than the existing ones, is presented. A numerical example is given to illustrate the effectiveness and less conservatism of the new proposed stability criterion.

Key words Delay-dependent stability, singular systems, linear matrix inequality (LMI), delay decomposition method, equivalence

DOI 10.3724/SP.J.1004.2010.00433

Over the past decades, much attention has been focused on the stability analysis and controller synthesis for singular linear time-delay systems, because the singular system model is a natural presentation of dynamic systems and can describe a larger class of systems than regular ones, such as large-scale systems, power systems, and constrained control systems. Similar to the state-space time-delay systems, the results on stability analysis and stabilization of singular time-delay systems can be classified into two categories, that is, delay independent criteria[1−2] and delay-dependent ones[3−4]. Generally, the delay-dependent case is less conservative than delay-independent ones, especially when the delay is comparatively small.

Recently, many researchers have paid attention to stability analysis of singular systems with time-delay[3, 5−7]. The computational results in [3] show that its stability criterion is less conservative than the one in [5] (see Example 1 in [3]). In fact, the conclusion based on the computational results contains errors.

In this note, we will prove that the stability result proposed in [3] is equivalent to the ones in [5−7], and a simplified version of Theorem 1 in [3] will be derived. Furthermore, by using a delay composition method, a less conservative result will be presented.

1 Problem formulation

Consider the following continuous-time singular system with a time-varying delay in the state[3]:

$$(\Sigma):\quad E\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau),\quad t>0 \tag{1}$$

$$\boldsymbol{x}(t)=\boldsymbol{\phi}(t),\quad t\in[-\tau,\ 0] \tag{2}$$

where $\boldsymbol{x}(t)\in\mathbf{R}^n$ is the state, and $\boldsymbol{\phi}(t)\in\mathcal{C}_{n,\tau}$ is a compatible vector valued initial function. The matrix $E\in\mathbf{R}^{n\times n}$ may be singular and rank $E=p\le n$. A, A_τ are constant matrices with appropriate dimensions. τ is an unknown but constant delay satisfying

$$0<\tau\le\tau_m \tag{3}$$

Without loss of generality, the matrices E, A, and A_τ are assumed to have the forms:

$$E=\begin{bmatrix} I_p & 0\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$$
$$A_\tau=\begin{bmatrix} A_{\tau11} & A_{\tau12}\\ A_{\tau21} & A_{\tau22}\end{bmatrix} \tag{4}$$

For the system (Σ), [3] provided a stability criterion as follows.

Lemma 1[3]. The singular time-delay system (Σ) is regular, impulse free, and asymptotically stable for any constant delay τ satisfying $0<\tau\le\tau_m$, if there exist matrices

$$P=\begin{bmatrix} P_{11} & P_{12}\\ 0 & P_{22}\end{bmatrix},\ P_{11}>0,\quad Q>0$$
$$Z=\begin{bmatrix} Z_{11} & Z_{12}\\ * & Z_{22}\end{bmatrix}>0$$
$$Y=\begin{bmatrix} Y_{11} & 0\\ Y_{21} & 0\end{bmatrix},\quad W=\begin{bmatrix} W_{11} & 0\\ W_{21} & 0\end{bmatrix}$$
$$Y_1=\begin{bmatrix} Y_{11}\\ Y_{21}\end{bmatrix},\quad W_1=\begin{bmatrix} W_{11}\\ W_{21}\end{bmatrix} \tag{5}$$

with appropriate dimensions and $P_{11}\in\mathbf{R}^{p\times p}$, $Z_{11}\in\mathbf{R}^{p\times p}$, $Y_{11}\in\mathbf{R}^{p\times p}$, $W_{11}\in\mathbf{R}^{p\times p}$ satisfying the following LMI:

$$\Phi<0 \tag{6}$$

where

$$\Phi=\begin{bmatrix} \Phi_1 & PA_\tau-Y+W^{\rm T}+\tau_mA^{\rm T}ZA_\tau & -\tau_mY_1\\ * & -Q-W-W^{\rm T}+\tau_mA_\tau^{\rm T}ZA_\tau & -\tau_mW_1\\ * & * & -\tau_mZ_{11}\end{bmatrix}$$
$$\Phi_1=PA+A^{\rm T}P^{\rm T}+Y+Y^{\rm T}+Q+\tau_mA^{\rm T}ZA$$

For convenience of comparison, the stability criteria in[5−7] are listed as the following lemmas.

Lemma 2[5]. Consider the descriptor system (Σ), for given scalars $\tau_m>0$, if there exist matrices $\tilde{P}_1>0$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{Q}>0$, $\tilde{R}>0$, $\tilde{T}_i$, and $\tilde{S}_i$ of appropriate dimensions ($i=1,\ 2,\ 3$) such that

$$\Gamma<0 \tag{7}$$

where

$$\Gamma=\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau_m\tilde{T}_1\\ * & \Gamma_{22} & \Gamma_{23} & \tau_m\tilde{T}_2\\ * & * & \Gamma_{33} & \tau_m\tilde{T}_3\\ * & * & * & -\tau_m\tilde{R}\end{bmatrix}$$
$$\Gamma_{11}=\tilde{Q}+\tilde{T}_1E+E^{\rm T}\tilde{T}_1^{\rm T}-\tilde{S}_1A-A^{\rm T}\tilde{S}_1^{\rm T}$$
$$\Gamma_{12}=-\tilde{T}_1E+E^{\rm T}\tilde{T}_2^{\rm T}-\tilde{S}_1A_\tau-A^{\rm T}\tilde{S}_2^{\rm T}$$
$$\Gamma_{13}=\tilde{P}+\tilde{S}_1+E^{\rm T}\tilde{T}_3^{\rm T}-A^{\rm T}\tilde{S}_3^{\rm T}$$
$$\Gamma_{22}=-\tilde{Q}-\tilde{T}_2E-E^{\rm T}\tilde{T}_2^{\rm T}-\tilde{S}_2A_\tau-A_\tau^{\rm T}\tilde{S}_2^{\rm T}$$
$$\Gamma_{23}=\tilde{S}_2-E^{\rm T}\tilde{T}_3^{\rm T}-A_\tau^{\rm T}\tilde{S}_3^{\rm T}$$
$$\Gamma_{33}=\tau_m\tilde{R}+\tilde{S}_3+\tilde{S}_3^{\rm T}$$
$$P=\begin{bmatrix} \tilde{P}_1 & \tilde{P}_2\\ 0 & \tilde{P}_3\end{bmatrix}$$

Manuscript received December 10, 2008; accepted March 6, 2009
Supported by National Natural Science Foundation of China (60574011)
1. Institute of Systems Science, Northeastern University, Shenyang 110004, P. R. China 2. Jilin Normal University, Siping 136000, P. R. China 3. School of Computer and Communication Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, P. R. China 4. Key Laboratory of Integrated Automation of Process Industry, Ministry of Education, Northeastern University, Shenyang 110004, P. R. China

then the system (Σ) is exponentially stable.

Lemma 3[6]**.** Given scalar $\tau_m > 0$. Then, for any delay $0 < \tau \leq \tau_m$, the singular delay system (Σ) is regular, impulse free, and stable if there exist matrices $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} > 0$, P, Y, and W such that the following LMIs hold:

$$E^{\mathrm{T}}P = P^{\mathrm{T}}E \geq 0 \tag{8}$$

$$\Omega < 0 \tag{9}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \tau_m Y^{\mathrm{T}} & \tau_m A^{\mathrm{T}}Z \\ * & \Gamma_{22} & \tau_m W^{\mathrm{T}} & \tau_m A_\tau^{\mathrm{T}}Z \\ * & * & -\tau_m Z & 0 \\ * & * & * & -\tau_m Z \end{bmatrix}$$

$$\Omega_{11} = P^{\mathrm{T}}A + A^{\mathrm{T}}P + Q - Y^{\mathrm{T}}E - E^{\mathrm{T}}Y$$

$$\Omega_{12} = P^{\mathrm{T}}A_\tau + Y^{\mathrm{T}}E - E^{\mathrm{T}}W$$

$$\Omega_{22} = W^{\mathrm{T}}E + E^{\mathrm{T}}W - Q$$

Lemma 4[7]**.** Given scalar $\tau_m > 0$. Then, for any delay $0 < \tau \leq \tau_m$, the singular delay system (Σ) is regular, impulse free, and stable if there exist matrices $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} > 0$, and matrices P_1, P_2, P_3, X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}, Y_1, Y_2, and T_1, such that

$$E^{\mathrm{T}}P_1 = P_1^{\mathrm{T}}E \geq 0 \tag{10}$$

$$\Pi < 0 \tag{11}$$

$$X \geq 0 \tag{12}$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & -Y_1E + P_2^{\mathrm{T}}A_\tau + E^{\mathrm{T}}T_1^{\mathrm{T}} + \tau_m X_{13} \\ * & \Pi_{22} & -Y_2E + P_3^{\mathrm{T}}A_\tau + \tau_m X_{23} \\ * & * & -Q - T_1E - E^{\mathrm{T}}T_1^{\mathrm{T}} + \tau_m X_{33} \end{bmatrix}$$

$$\Pi_{11} = P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + Y_1E + E^{\mathrm{T}}Y_1^{\mathrm{T}} + \tau_m X_{11} + Q$$

$$\Pi_{12} = P_1^{\mathrm{T}} - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + E^{\mathrm{T}}Y_2^{\mathrm{T}} + \tau_m X_{12}$$

$$\Pi_{22} = -P_3 - P_3^{\mathrm{T}} + \tau_m X_{22} + \tau_m Z$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & Y_1 \\ * & X_{22} & X_{23} & Y_2 \\ * & * & X_{33} & T_1 \\ * & * & * & Z \end{bmatrix}$$

In this note, we will prove the equivalence among the above lemmas and give a simplified version of these criteria. Furthermore, by using a delay decomposition method, an improved result is proposed.

2 The equivalence among several stability criteria

In this section, equivalence among the existing stability criteria given in [3, 5−7] will be established, which further shows that the computational results given in [3] are incorrect.

Now, we prove the equivalence among the stability conditions given by Lemmas 1 ∼ 4, and a new stability criterion, which contains fewer decision variables, is also derived.

Theorem 1. The following statements are equivalent:

1) Inequality (6) is feasible;

2) The following inequality is feasible:

$$\Psi < 0 \tag{13}$$

where

$$\Psi = \begin{bmatrix} \Psi_1 & PA_\tau + \tau_m A^{\mathrm{T}}ZA_\tau + \tau_m^{-1}H^{\mathrm{T}}Z_{11}H \\ * & -Q + \tau_m A_\tau^{\mathrm{T}}ZA_\tau - \tau_m^{-1}H^{\mathrm{T}}Z_{11}H \end{bmatrix}$$

$$\Psi_1 = PA + (PA)^{\mathrm{T}} + Q + \tau_m A^{\mathrm{T}}ZA - \tau_m^{-1}H^{\mathrm{T}}Z_{11}H$$

$$H = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

3) Inequality (7) is feasible;

4) Inequality (9) with (8) is feasible;

5) Inequalities (11) and (12) with (10) are feasible.

Proof. 1) ⇔ 2):

Noticing that $Y = Y_1H$ and $W = W_1H$, pre- and post-multiply Φ in (6) on both sides by

$$\begin{bmatrix} I & 0 & \tau_m^{-1}H^{\mathrm{T}} \\ 0 & I & -\tau_m^{-1}H^{\mathrm{T}} \\ 0 & 0 & I \end{bmatrix}$$

and its transpose. From the Schur complement, it follows that $\Phi < 0$ in Lemma 1 is equivalent to

$$\Psi + \begin{bmatrix} -\tau_m Y_1 - H^{\mathrm{T}}Z_{11} \\ -\tau_m W_1 + H^{\mathrm{T}}Z_{11} \end{bmatrix} (\tau_m Z_{11})^{-1} \times \begin{bmatrix} -\tau_m Y_1 - H^{\mathrm{T}}Z_{11} \\ -\tau_m W_1 + H^{\mathrm{T}}Z_{11} \end{bmatrix}^{\mathrm{T}} < 0 \tag{14}$$

So, $\Psi < 0$ holds if $\Phi < 0$ holds.

Conversely, if $\Psi < 0$ holds, by letting

$$Y_1 = -\tau_m^{-1}H^{\mathrm{T}}Z_{11}, \quad W_1 = \tau_m^{-1}H^{\mathrm{T}}Z_{11}$$

it yields that $\Phi < 0$ also holds.

Thus, $\Psi < 0$ is equivalent to $\Phi < 0$.

2) ⇔ 3):

Pre- and post-multiplying Γ in (7) on both sides by

$$\begin{bmatrix} I & 0 & 0 & -\tau_m^{-1}E^{\mathrm{T}} \\ 0 & I & 0 & \tau_m^{-1}E^{\mathrm{T}} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose, it yields that

$$\begin{bmatrix} \Xi & \tilde{T} \\ * & -\tau_m\tilde{R} \end{bmatrix} < 0 \tag{15}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{P} + \tilde{S}_1 - A^{\mathrm{T}}\tilde{S}_3^{\mathrm{T}} \\ * & \Xi_{22} & \tilde{S}_2 - A_\tau^{\mathrm{T}}\tilde{S}_3^{\mathrm{T}} \\ * & * & \tau_m\tilde{R} + \tilde{S}_3 + \tilde{S}_3^{\mathrm{T}} \end{bmatrix}$$

$$\Xi_{11} = \tilde{Q} - \tilde{S}_1A - (\tilde{S}_1A)^{\mathrm{T}} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E$$

$$\Xi_{12} = -\tilde{S}_1A_\tau - (\tilde{S}_2A)^{\mathrm{T}} + \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E$$

$$\Xi_{22} = -\tilde{Q} - \tilde{S}_2A_\tau - (\tilde{S}_2A_\tau)^{\mathrm{T}} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E$$

$$\tilde{T} = \begin{bmatrix} \tau_m\tilde{T}_1 + E^{\mathrm{T}}\tilde{R} \\ \tau_m\tilde{T}_2 - E^{\mathrm{T}}\tilde{R} \\ \tau_m\tilde{T}_3 \end{bmatrix}$$

Similar to the proof of 1) ⇔ 2), it is clear that $\Gamma < 0$ is feasible if and only if $\Xi < 0$ is feasible.

Note that

$$\Xi = \bar{\Xi} + \tilde{S}\mathcal{A} + \mathcal{A}^{\mathrm{T}}\tilde{S}^{\mathrm{T}} \tag{16}$$

where

$$\bar{\Xi} = \begin{bmatrix} \tilde{Q} - \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E & \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E & \tilde{P} \\ * & -\tilde{Q} - \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E & 0 \\ * & * & \tau_m \tilde{R} \end{bmatrix}$$

$$\tilde{S} = \begin{bmatrix} \tilde{S}_1^{\mathrm{T}} & \tilde{S}_2^{\mathrm{T}} & \tilde{S}_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\mathcal{A} = \begin{bmatrix} -A & -A_\tau & I \end{bmatrix}$$

From the elimination lemma[8], it is known that $\Xi < 0$ is equivalent to

$$\tilde{\Xi} = \mathcal{N}_{\mathcal{A}}^{\mathrm{T}} \bar{\Xi} \mathcal{N}_{\mathcal{A}} < 0 \tag{17}$$

where

$$\mathcal{N}_{\mathcal{A}} = \begin{bmatrix} I & 0 \\ 0 & I \\ A & A_\tau \end{bmatrix}$$

After some manipulation, one can get

$$\tilde{\Xi} = \begin{bmatrix} \tilde{P}A + A^{\mathrm{T}}\tilde{P}^{\mathrm{T}} + \tilde{Q} - \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E + \tau_m A^{\mathrm{T}} \tilde{R} A & \tilde{P}A_\tau + \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E + \tau_m A^{\mathrm{T}} \tilde{R} A_\tau \\ * & -\tilde{Q} - \tau_m^{-1} E^{\mathrm{T}} \tilde{R} E + \tau_m A_\tau^{\mathrm{T}} \tilde{R} A_\tau \end{bmatrix}$$

By letting $P = \tilde{P}$, $Q = \tilde{Q}$, and $Z = \tilde{R}$, it is easy to know that Ψ in (13) is the same as $\tilde{\Xi}$, so $\Psi < 0$ if and only if $\tilde{\Xi} < 0$.

Thus, from the above analysis, one can get that $\Psi < 0$ if and only if $\Gamma < 0$.

2) $\Leftrightarrow$ 4): Similar to the proof of 1) $\Leftrightarrow$ 2), the equivalence between 2) and 4) can be easily obtained, and the details are omitted here.

2) $\Leftrightarrow$ 5): Similar to the proofs of 1) $\Leftrightarrow$ 2) and Theorem 2 in [9], the equivalence between 2) and 5) can also be derived, and it is omitted here. □

Remark 1. Theorem 1 establishes the equivalence among several stability criteria reported in [3, 5 − 7], which implies that the computation-based assertion in [3] claiming that the stability criterion of [3] is less conservative than the one in [5], is incorrect. Compared with Lemma 1[3], 2) of Theorem 1 involves less decision variables. Hence, from a mathematical point of view, 2) of Theorem 1 is more "powerful".

3 An improved stability criterion

In this section, an improved stability criterion will be proposed by using a delay decomposition method[10].

Theorem 2. The singular time-delay system (Σ) is regular, impulse free, and asymptotically stable for a given positive integer N and any constant delay τ satisfying $0 < \tau \le \tau_m$, if there exist matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\quad Q_i > 0$$

$$Z_i > 0\ ,\quad i = 1, 2, \cdots, N \tag{18}$$

with appropriate dimensions and $P_{11} \in \mathbf{R}^{p\times p}$ satisfying the following LMI:

$$\Theta < 0 \tag{19}$$

where

$$\Theta = \begin{bmatrix} \Theta_1 & \frac{N}{\tau_m} E^{\mathrm{T}} Z_1 E & 0 & \cdots & 0 & PA_\tau + \frac{\tau_m}{N} A^{\mathrm{T}} \tilde{Z} A_\tau \\ * & \Theta_2 & \frac{N}{\tau_m} E^{\mathrm{T}} Z_2 E & \cdots & 0 & 0 \\ * & * & \Theta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \Theta_N & \frac{N}{\tau_m} E^{\mathrm{T}} Z_N E \\ * & * & * & \cdots & * & \Theta_{N+1} \end{bmatrix}$$

$$\Theta_1 = PA + A^{\mathrm{T}} P^{\mathrm{T}} + Q_1 + \frac{\tau_m}{N} A^{\mathrm{T}} \tilde{Z} A - \frac{N}{\tau_m} E^{\mathrm{T}} Z_1 E$$

$$\Theta_i = -Q_{i-1} + Q_i - \frac{N}{\tau_m} E^{\mathrm{T}} (Z_{i-1} + Z_i) E,\quad i = 2, 3, \cdots, N$$

$$\Theta_{N+1} = -Q_N - \frac{N}{\tau_m} E^{\mathrm{T}} Z_N E + \frac{\tau_m}{N} A_\tau^{\mathrm{T}} \tilde{Z} A_\tau$$

$$\tilde{Z} = \sum_{i=1}^{N} Z_i$$

Proof. From (19), it follows that

$$PA + A^{\mathrm{T}} P^{\mathrm{T}} + Q_1 - \frac{N}{\tau_m} E^{\mathrm{T}} Z_1 E < 0 \tag{20}$$

holds, which implies that

$$P_{22} A_{22} + A_{22}^{\mathrm{T}} P_{22}^{\mathrm{T}} < 0 \tag{21}$$

So, A_{22} is nonsingular. Pre- and post-multiplying $\underbrace{[I\ \ I\ \ \cdots\ \ I\ \ I]}_{N+1}$ and its transpose on both sides of Θ in (19), it yields that

$$P(A + A_\tau) + (A + A_\tau)^{\mathrm{T}} P^{\mathrm{T}} - \frac{N}{\tau_m} \sum_{i=1}^{N} E^{\mathrm{T}} Z_i E < 0 \tag{22}$$

which implies that $A_{22} + A_{\tau 22}$ is also nonsingular. Thus, the pairs $(E,\ A)$ and $(E,\ A + A_\tau)$ are regular and impulse free.

Construct the Lyapunov-Krasovskii functional for system (Σ) as

$$V(\boldsymbol{x}_t) = \boldsymbol{x}^{\mathrm{T}}(t) P E \boldsymbol{x}(t) + \sum_{i=1}^{N} \Big(\int_{t-\tau_i}^{t-\tau_{i-1}} \boldsymbol{x}^{\mathrm{T}}(s) Q_i \boldsymbol{x}(s) \mathrm{d}s + \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) E^{\mathrm{T}} Z_i E \dot{\boldsymbol{x}}(s) \mathrm{d}s \mathrm{d}\theta \Big) \tag{23}$$

where $\boldsymbol{x}_t = \boldsymbol{x}(t + \theta)$, $-\tau_m \le \theta \le 0$ and $\tau_i = i/N \times \tau$, $i = 0, 1, 2, \cdots, N$.

Taking the time derivative of $V(\boldsymbol{x}_t)$ along with the solution of (Σ) yields

$$\dot{V}(\boldsymbol{x}_t) = 2\boldsymbol{x}^{\mathrm{T}}(t)PE\dot{\boldsymbol{x}}(t)+$$
$$\sum_{i=1}^{N}\Big(\boldsymbol{x}^{\mathrm{T}}(t-\tau_{i-1})Q_i\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)Q_i\boldsymbol{x}(t-\tau_i)\Big)+$$
$$\sum_{i=1}^{N}\Big(\frac{\tau}{N}\dot{\boldsymbol{x}}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_iE\dot{\boldsymbol{x}}(t)-\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_iE\dot{\boldsymbol{x}}(s)ds\Big)\le$$
$$2\boldsymbol{x}^{\mathrm{T}}(t)P[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]+$$
$$\sum_{i=1}^{N}\Big(\boldsymbol{x}^{\mathrm{T}}(t-\tau_{i-1})Q_i\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)Q_i\boldsymbol{x}(t-\tau_i)\Big)+$$
$$\sum_{i=1}^{N}\Big(\frac{\tau_m}{N}[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]^{\mathrm{T}}Z_i[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]\Big)-$$
$$\frac{N}{\tau_m}\sum_{i=1}^{N}([\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}(t-\tau_i)]^{\mathrm{T}}E^{\mathrm{T}}$$
$$Z_iE[\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}(t-\tau_i)])=$$
$$\boldsymbol{\xi}^{\mathrm{T}}(t)\Theta\boldsymbol{\xi}(t) \tag{24}$$

where

$$\boldsymbol{\xi}(t)=\begin{bmatrix}\boldsymbol{x}^{\mathrm{T}}(t) & \boldsymbol{x}^{\mathrm{T}}(t-\tau_1) & \cdots & \boldsymbol{x}^{\mathrm{T}}(t-\tau)\end{bmatrix}^{\mathrm{T}}$$

Therefore, by (19), it is easy to see that $\dot{V}(\boldsymbol{x}_t)<0$. □

Remark 2. In the proof of Theorem 2, the delay interval $[0,\ \tau_m]$ is divided into N segments of equal length τ_m/N, such that the information of delayed states $\boldsymbol{x}(t-i\tau_m/N)$, $i=1,2,\cdots,N$ are all taken into account. It is clear that the Lyapunov function defined in Theorem 2 is more general than the ones in [3, 5−7], etc.

The following theorem shows the relationship between Theorem 2 and 2) of Theorem 1.

Theorem 3. Inequality (19) is feasible if inequality (13) is feasible.

Proof. If inequality (13) is feasible, then there exists a scalar $\varepsilon>0$ such that

$$\tilde{\Psi}<0 \tag{25}$$

where

$$\tilde{\Psi}=\begin{bmatrix}\tilde{\Psi}_1 & 0 & 0 & \cdots & 0 & PA_\tau+\tau_mA^{\mathrm{T}}ZA_\tau+\tau_m^{-1}H^{\mathrm{T}}Z_{11}H\\ * & -\varepsilon I & 0 & \cdots & 0 & 0\\ * & * & -\varepsilon I & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & * & \cdots & -\varepsilon I & 0\\ * & * & * & \cdots & * & -Q+\tau_mA_\tau^{\mathrm{T}}ZA_\tau-\tau_m^{-1}H^{\mathrm{T}}Z_{11}H\end{bmatrix}$$

$$\tilde{\Psi}_1=PA+(PA)^{\mathrm{T}}+Q+(N-1)\varepsilon I+\tau_mA^{\mathrm{T}}ZA-\tau_m^{-1}H^{\mathrm{T}}Z_{11}H$$
$$H=\begin{bmatrix}I_p & 0\end{bmatrix}$$

Letting $Z_i=Z$, $i=1,2,\cdots,N$, $Q_N=Q$, $Q_{N-1}=Q+\varepsilon I$, $\cdots$, $Q_1=Q+(N-1)\varepsilon I$, and denoting $\Delta=\Theta-\tilde{\Psi}$, it yields that

$$\Delta=\begin{bmatrix}-\frac{N-1}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE & 0 & \cdots & 0 & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE\\ * & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE & \cdots & 0 & 0\\ * & * & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & * & \cdots & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE\\ * & * & * & \cdots & * & -\frac{N-1}{\tau_m}E^{\mathrm{T}}ZE\end{bmatrix} \tag{26}$$

Next, we prove that $\Delta\le 0$ holds.

When $N=1$, it is obvious that $\Delta=0$, so $\Theta<0$ is also feasible.

If $N=2$, then Δ becomes

$$\Lambda=\begin{bmatrix}-\frac{1}{\tau_m}E^{\mathrm{T}}ZE & \frac{2}{\tau_m}E^{\mathrm{T}}ZE & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE\\ * & -\frac{4}{\tau_m}E^{\mathrm{T}}ZE & \frac{2}{\tau_m}E^{\mathrm{T}}ZE\\ * & * & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE\end{bmatrix} \tag{27}$$

Pre- and post-multiplying $\begin{bmatrix}I & I & I\\ 0 & I & 0\\ 0 & \frac{1}{2}I & I\end{bmatrix}$ and its transpose on both sides of Λ, it follows

$$\tilde{\Lambda}=\begin{bmatrix}0 & 0 & 0\\ * & -\frac{4}{\tau_m}E^{\mathrm{T}}ZE & 0\\ * & * & 0\end{bmatrix} \tag{28}$$

It is obvious that $\tilde{\Lambda}\le 0$, which implies that $\Delta\le 0$ holds.

The proof of $N>2$ is similar to the case of $N=2$, and thus, it is omitted here. □

Remark 3. From Theorem 3, it is easy to see that Theorem 2 is less conservative than 2) of Theorem 1. As N increases, the conservatism of Theorem 2 decreases. An example in the next section will verify this fact.

4 Example

Example 1[3]. Consider a singular delay system that is in the form of (1) with

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0.5 & 0\\ -1 & -1\end{bmatrix},\quad A_\tau=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}$$

Table 1 lists the comparison of the computation results obtained by the stability criteria of [3, 5−7, 11] and this note.

It is worth pointing out that the maximum τ_m obtained by Theorem 3.5 in [11] should be 1.1547 and not 1.1612, which was given in [3].

Certainly, the maximum τ_m obtained by Theorem 1 in [3] should be 1.1547 and not 1.2011, which was provided in [3].

From Table 1, it is clear that Theorem 1 in [3] may not be less conservative than Theorem 3.5 in [11]. Fortunately, Example 2 in [6] showed that the computation results obtained by Theorem 1 in [6] may be less conservative than the ones obtained by Theorem 3.5 in [11], and that no theoretical proof had been provided in [6].

In summary, 2) of Theorem 1 in this note contains the fewest variables and Theorem 2 in this note is less conservative than those in [3, 5−7].

Table 1 Comparisons of delay-dependent stability conditions of Example 1

Methods	Maximum τ_m allowed	Number of variables
Theorem 1[7]	1.1547	53
Theorem 1[5]	1.1547	33
Theorem 3.5[11]	1.1547	24
Theorem 1[6]	1.1547	17
Theorem 1[3]	1.1547	13
2) of Theorem 1	1.1547	9
Theorem 2, $N=2$	1.1954	15
Theorem 2, $N=3$	1.2025	21
Theorem 2, $N=4$	1.2044	27
Theorem 2, $N=5$	1.2052	33

5 Conclusion

This note studies the stability of singular systems with state delay and theoretically proves the equivalence among several recent results through a technique of eliminating redundant variables. By using the delay decomposition method, a new stability criterion that is much less conservative than the previous relevant ones is obtained, which has been shown by a numerical example.

Acknowledgement

The authors would like to thank Professor YANG Guang-Hong for his guidance.

References

1 Xu S Y, Van Dooren P, Stefan R, Lam J. Robust stability and stabilization for singular systems with state delay and parameter uncertainty. *IEEE Transactions on Automatic Control*, 2002, **47**(7): 1122−1128

2 Feng J, Zhu S, Cheng Z. Guaranteed cost control of linear uncertain singular time-delay systems. In: Proceedings of the 41st IEEE Conference on Decision and Control. Las Vegas, USA: IEEE, 2002. 1802−1807

3 Zhu S, Zhang C, Cheng Z, Feng J. Delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems. *IEEE Transactions on Automatic Control*, 2007, **52**(5): 880−885

4 Yue D, Han Q L. Delay-dependent robust H_∞ controller design for uncertain descriptor systems with time-varying discrete and distributed delays. *IEE Proceedings-Control Theory and Applications*, 2005, **152**(6): 628−638

5 Yue D, Han Q L. A delay-dependent stability criterion of neutral systems and its application to a partial element equivalent circuit model. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 2004, **51**(12): 685−689

6 Xu S Y, Lam J, Zou Y. An improved characterization of bounded realness for singular delay systems and its applications. *International Journal of Robust and Nonlinear Control*, 2008, **18**(3): 263−277

7 Wu Z, Zhou W. Delay-dependent robust H_∞ control for uncertain singular time-delay systems. *IET Control Theory and Applications*, 2007, **1**(5): 1234−1241

8 Boyd S, El Ghaoui L, Feron E, Balakrishnan V. *Linear Matrix Inequality in Systems and Control Theory*. Philadelphia: Society for Industrial and Applied Mathematics, 1994

9 Xu S Y, Lam J, Zou Y. Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems. *IEE Proceedings-Control Theory and Applications*, 2005, **152**(2): 147−151

10 Zhu X L, Yang G H. Jensen integral inequality approach to stability analysis of continuous-time systems with time-varying delay. *IET Control Theory and Applications*, 2008, **2**(6): 524−534

11 Fridman E, Shaked U. H_∞-control of linear state-delay descriptor systems: an LMI approach. *Linear Algebra and Its Applications*, 2002, **351-352**: 271−302

FENG Yi-FU Ph. D. candidate at the Institute of Systems Science, Northeastern University. He is also an associate professor at Jilin Normal University. His research interest covers singular systems and networked control systems.
E-mail: yf19692004@163.com

ZHU Xun-Lin Received his Ph. D. degree in control theory and engineering from Northeastern University in 2008. Currently, he joins Nanyang Technological University as a research fellow. His research interest covers networked control systems and neural networks. Corresponding author of this paper.
E-mail: hntjxx@163.com

ZHANG Qing-Ling Professor at Northeastern University. His research interest covers singular systems, networked control systems, and robust control.
E-mail: qlzhang@mail.neu.edu.cn