Mean-square Exponential Input-to-state Stability of Euler-Maruyama Method Applied to Stochastic Control Systems

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Abstract This paper deals with the mean-square exponential input-to-state stability (exp-ISS) of Euler-Maruyama (EM) method applied to stochastic control systems (SCSs). The aim is to find out the conditions of the exact and EM method solutions to an SCS having the property of mean-square exp-ISS without involving control Lyapunov functions. Second moment boundedness and an appropriate form of strong convergence are achieved under global Lipschitz coefficients and mean-square continuous random inputs. Under the strong convergent condition, it is shown that the mean-square exp-ISS of an SCS holds if and only if that of the EM method is preserved for sufficiently small step size.

Key words Mean-square exponential input-to-state stability (exp-ISS), stochastic control system (SCS), Euler-Maruyama (EM) method, strong convergence

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When investigating stability and designing controller of a system, it is important to characterize the effects of external inputs. The well-known input-to-state stability (ISS) property plays useful role in this regard. The concept of ISS originated in [1] for deterministic systems and has been frequently investigated in recent years[2−6]. Especially, some concepts of exp-ISS for stochastic control systems (SCSs) have been appeared in [7−9]. These concepts are the extension of ISS for deterministic systems (plus exponential stability).

The exp-ISS of a stochastic control system, usually depends on the existence of an appropriate control Lyapunov function. However, in general, there is no very effective method to find such control Lyapunov function. Thus, in the absence of an appropriate control Lyapunov function, we may carry out careful numerical simulations using a numerical method with a small step size Δ. Then, for the mean-square exp-ISS and Euler-Maruyama (EM) method, two key questions follow.

Question 1. If the SCS satisfies mean-square exp-ISS, will the EM method preserve the mean-square exp-ISS for sufficiently small Δ?

Question 2. If the EM method satisfies mean-square exp-ISS for small Δ, can we infer that the underlying SCS also satisfies mean-square exp-ISS?

Results that answer Questions 1 and 2 for uncontrolled stochastic systems can be found in [10−13]. Furthermore, the ISS of Runge-Kutta methods and one-leg methods for deterministic control systems was investigated in [14−15], respectively. However, to the best of the authors′ knowledge, the mean-square exp-ISS of EM method for SCSs remains open, which motivates this paper. In addition, it is of great importance to study control problems by using numerical methods, since the research on control problems (e. g. ISS) becomes much more difficult by using traditional methods with the rapid development of some larger engineering designs and so on.

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Our aim of this study is to give very positive answers to both Questions 1 and 2 without the existence of an appropriate control Lyapunov function. The organization of the paper is as follows. In Section 1, we give our definitions of mean-square exp-ISS for the SCS and EM method. In Section 2, under global Lipschitz coefficients and mean-square continuous random inputs, the second moment boundedness and an appropriate form of strong convergence are obtained in Theorem 1 and Theorem 2, respectively. In Section 3, Theorem 3 shows the equivalence, for sufficiently small step size, of the mean-square exp-ISS of the SCSs and that of the EM method.

Furthermore, it may be noted that the approach used in Theorem 3 allows us to discuss whether an SCS shares mean-square exp-ISS property with other numerical methods (e.g. the stochastic theta method), however, this discussion is not covered here due to space limitation and will be reported in our next paper.

Notation. $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and $\mathbf{Z}$ is the set of all integers. $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathcal{P})$ is a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i. e., it is right continuous and $\mathfrak{F}_0$ contains all $\mathcal{P}$-null sets). $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation. Let $|\cdot|$ denote both the Euclidean norm in $\mathbf{R}^n$ and the trace norm in $\mathbf{R}^{n\times m}$. Denote by $L^2_{\mathfrak{F}_t}(\Omega;\mathbf{R}^n)$ the family of all $\mathfrak{F}_t$-measurable random variables $\boldsymbol{\xi}:\Omega\rightarrow\mathbf{R}^n$ such that $\mathrm{E}|\boldsymbol{\xi}|^2<\infty$. A function $\gamma:\mathbf{R}^+\rightarrow\mathbf{R}^+$ is called a κ-function if it is continuous, strictly increasing, and $\gamma(0)=0$.

1 The mean-square exp-ISS

Consider the following n-dimensional Itô SCS:

$$\mathrm{d}\boldsymbol{y}(t)=\boldsymbol{f}(\boldsymbol{y}(t),\boldsymbol{u}(t))\mathrm{d}t+\boldsymbol{g}(\boldsymbol{y}(t),\boldsymbol{u}(t))\mathrm{d}\boldsymbol{w}(t),\quad t\geq 0 \qquad (1)$$

where $\boldsymbol{y}(t)\in\mathbf{R}^n$ and $\boldsymbol{u}(t)\in\mathbf{R}^m$ are the state vector and input vector of the system, respectively. $\boldsymbol{w}(t)$ is a standard p-dimensional Wiener process. The set of admissible inputs is denoted by $\mathcal{F}(\mathbf{R}^m)$ and is the set of all progressively measurable random functions $\boldsymbol{u}:\Omega\times[0,\infty)\rightarrow\mathbf{R}^m$ such that the supremum norm $|\boldsymbol{u}|_{\sup}=\sup\{|\boldsymbol{u}(t)|, t\geq 0, a.s.\}\leq\infty$, where $a.s.$ means almost surely. That is to say, for every $t\geq 0$, the random input $\boldsymbol{u}$ is $\mathfrak{F}_t\times B_t$-measurable

(B_t is the σ-algebra of Borel subsets of $[0,t]$). As a direct consequence, each $\boldsymbol{u}(t)$ is $\mathfrak{F}_t$-adapted.

We always assume that $\boldsymbol{f}:\mathbf{R}^n\times\mathbf{R}^m\rightarrow\mathbf{R}^n$ and $\boldsymbol{g}:\mathbf{R}^n\times\mathbf{R}^m\rightarrow\mathbf{R}^{n\times p}$ are both Borel measurable such that the SCS (1) has a unique solution for any initial data $\boldsymbol{y}(0)=\boldsymbol{\xi}\in L^2_{\mathfrak{F}_0}(\Omega;\mathbf{R}^n)$ and random inputs $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$. We shall denote this solution by $\boldsymbol{y}(t;0,\boldsymbol{\xi},\boldsymbol{u}(t))$. For detailed conditions on the existence and uniqueness of $\boldsymbol{y}(t)=\boldsymbol{y}(t;0,\boldsymbol{\xi},\boldsymbol{u}(t))$, we refer the reader to [9, 16]. For the purpose of stability study in this paper, assume that

$$\boldsymbol{f}(\mathbf{0},\mathbf{0})=\mathbf{0},\quad \boldsymbol{g}(\mathbf{0},\mathbf{0})=\mathbf{0} \tag{2}$$

Definition 1. The SCS (1) is said to satisfy mean-square exp-ISS if there exists a κ-function β and positive constants M and λ such that, for all initial data $\boldsymbol{\xi}\in L^2_{\mathfrak{F}_0}(\Omega;\mathbf{R}^n)$ and random inputs $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$,

$$\mathrm{E}|\boldsymbol{y}(t)|^2\le M\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-\lambda t}+\mathrm{E}\beta(|\boldsymbol{u}|^2_{\sup}) \tag{3}$$

We refer to λ as a rate constant, M as a growth constant, and β as a gain function.

Intuitively, the mean-square exp-ISS property indicates that, with random but almost surely bounded inputs, the behavior of the SCS should remain bounded in mean-square and tend exponentially to the equilibrium in mean-square when inputs almost surely approach to zero.

Remark 1. The mean-square exp-ISS of SCSs is a natural extension of the well-known deterministic ISS introduced by Sontag[2−3]. Furthermore, a concept also named as mean-square exp-ISS for SCSs was introduced in [7, 9]. However, the mean-square exp-ISS employed in [7, 9] essentially is an extension of robust stability for deterministic systems introduced in [2]. The equivalence of the ISS and the robust stability for deterministic systems was proved in [2]. Due to the page limit, the relationship between the mean-square exp-ISS in Definition 1 and that employed in [7, 9] will be investigated and discussed later.

Now, we define the EM method[12, 17] for the SCS (1). The discrete approximation $\boldsymbol{x}_k\approx\boldsymbol{y}(k\Delta)$ with $t_k=k\Delta$, is formed by simulation from $\boldsymbol{x}_0=\boldsymbol{\xi}$; however, in general,

$$\boldsymbol{x}_{k+1}=\boldsymbol{x}_k+\boldsymbol{f}(\boldsymbol{x}_k,\boldsymbol{u}_k)\Delta+\boldsymbol{g}(\boldsymbol{x}_k,\boldsymbol{u}_k)\Delta\boldsymbol{w}_k \tag{4}$$

where $\Delta\boldsymbol{w}_k=\boldsymbol{w}((k+1)\Delta)-\boldsymbol{w}(k\Delta)$ and $\boldsymbol{u}_k=\boldsymbol{u}(k\Delta)$. We introduce the continuous approximation

$$\boldsymbol{x}(t)=\boldsymbol{\xi}+\int_0^t\boldsymbol{f}(\boldsymbol{z}(s),\boldsymbol{U}(s))\mathrm{d}s+\int_0^t\boldsymbol{g}(\boldsymbol{z}(s),\boldsymbol{U}(s))\mathrm{d}\boldsymbol{w}(s) \tag{5}$$

where

$$\boldsymbol{z}(t)=\sum_{k=0}^{\infty}\boldsymbol{x}_k\mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

$$\boldsymbol{U}(t)=\sum_{k=0}^{\infty}\boldsymbol{u}_k\mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

with $\mathbf{1}_G$ denoting the indicator function for the set G. It is easily shown that $\boldsymbol{x}(k\Delta)=\boldsymbol{x}_k$, and hence, $\boldsymbol{x}(t)$ is an interpolant to the discrete EM method solution.

Following Definition 1, we may define the mean-square exp-ISS for the continuous EM method.

Definition 2. For a given step size $\Delta>0$, the continuous EM method is said to satisfy mean-square exp-ISS, if there exists a κ-function γ and positive constants N and l such that, for all initial data $\boldsymbol{\xi}\in L^2_{\mathfrak{F}_0}(\Omega;\mathbf{R}^n)$ and random inputs $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$,

$$\mathrm{E}|\boldsymbol{x}(t)|^2\le N\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-lt}+\mathrm{E}\gamma(|\boldsymbol{u}|^2_{\sup}) \tag{6}$$

We refer to l as a rate constant, N as a growth constant, and γ as a gain function.

To be simple, we denote $a\beta(x)+bx$ by $(a\beta+b)(x)$ and denote $a\beta(x)+bx\le cx$ for all $x\in\mathbf{R}^+$ by $a\beta+b\le c$, where β is any κ-function and a,b,c are any positive constants. Clearly, it implies that $(b\beta+a)$ is also a κ-function.

Furthermore, we must declare that the questions addressed, results proved, as well as style of analysis use much of the work [10] for reference.

2 Strong convergence

Our aim is to find conditions under which the EM method reproduces the stability behavior of the underlying problem, for sufficiently small Δ. In order to do this, we introduce some conditions and perform preliminary analysis that establishes second moment boundedness and an appropriate form of strong convergence under global Lipschitz coefficients and mean-square continuous random inputs.

Assumption 1. Assume that both $\boldsymbol{f}$ and $\boldsymbol{g}$ are globally Lipschitz continuous, that is,

$$|\boldsymbol{f}(\boldsymbol{y},\boldsymbol{u})-\boldsymbol{f}(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}})|^2\le K_1(|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2+|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2) \tag{7}$$

and

$$|\boldsymbol{g}(\boldsymbol{y},\boldsymbol{u})-\boldsymbol{g}(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}})|^2\le K_2(|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2+|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2) \tag{8}$$

for all $\boldsymbol{y},\bar{\boldsymbol{y}}\in\mathbf{R}^n$ and $\boldsymbol{u},\bar{\boldsymbol{u}}\in\mathcal{F}(\mathbf{R}^m)$, where K_1 and K_2 are positive constants. Furthermore, we also assume that, for all sufficiently small $\Delta>0$, $t\ge0$ and $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$,

$$\mathrm{E}|\boldsymbol{u}(t+\Delta)-\boldsymbol{u}(t)|^2\le L\Delta^2\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{9}$$

where L is a positive constant, which implies the random input $\boldsymbol{u}(t)$ is mean-square continuous.

Theorem 1. Under (2) and the global Lipschtiz conditions (7) and (8), for sufficiently small Δ, the continuous EM method solution (5) satisfies,

$$\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2\le \mathrm{B}_{\boldsymbol{\xi},T,|\boldsymbol{u}|_{\sup}},\ \ T\ge0 \tag{10}$$

where

$$B_{\boldsymbol{\xi},T,|\boldsymbol{u}|_{\sup}}=3(\mathrm{E}|\boldsymbol{\xi}|^2+T(TK_1+K_2)\mathrm{E}|\boldsymbol{u}|^2_{\sup})\mathrm{e}^{3(TK_1+K_2)T}$$

Proof. From (2) and the global Lipschtiz conditions (7) and (8), we note that

$$|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})|^2\le K_1(|\boldsymbol{x}|^2+|\boldsymbol{u}|^2),\quad |\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u})|^2\le K_2(|\boldsymbol{x}|^2+|\boldsymbol{u}|^2)$$

Then, we derive from (5) that, for $0\le t\le T$,

$$\mathrm{E}|\boldsymbol{x}(t)|^2\le 3\mathrm{E}|\boldsymbol{\xi}|^2+3T\mathrm{E}\int_0^t|\boldsymbol{f}(\boldsymbol{z}(s),\boldsymbol{U}(s))|^2\mathrm{d}s+$$
$$3\mathrm{E}\int_0^t|\boldsymbol{g}(\boldsymbol{z}(s),\boldsymbol{U}(s))|^2\mathrm{d}s\le$$
$$3\mathrm{E}|\boldsymbol{\xi}|^2+3(TK_1+K_2)\times$$
$$\left(\int_0^t\mathrm{E}|\boldsymbol{z}(s)|^2\mathrm{d}s+T\mathrm{E}|\boldsymbol{u}|^2_{\sup}\right)$$

Since the right-hand side term is nondecreasing in t, we have

$$\sup_{0\le t\le t_1} \mathrm{E}|\boldsymbol{x}(t)|^2 \le 3\mathrm{E}|\boldsymbol{\xi}|^2 + 3T(TK_1+K_2)\mathrm{E}|\boldsymbol{u}|^2_{\sup} + 3(TK_1+K_2)\int_0^{t_1}\mathrm{E}|\boldsymbol{z}(s)|^2\mathrm{d}s \le 3\mathrm{E}|\boldsymbol{\xi}|^2 + 3T(TK_1+K_2)\mathrm{E}|\boldsymbol{u}|^2_{\sup} + 3(TK_1+K_2)\int_0^{t_1}(\sup_{0\le t\le s}\mathrm{E}|\boldsymbol{x}(t)|^2)\mathrm{d}s$$

for any $t_1 \in [0,T]$. The continuous Gronwall inequality hence yields

$$\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le 3(\mathrm{E}|\boldsymbol{\xi}|^2 + T(TK_1+K_2)\mathrm{E}|\boldsymbol{u}|^2_{\sup})\mathrm{e}^{3(TK_1+K_2)T}$$

which is the required assertion (10). □

Lemma 1. Under (2) and the global Lipschitz conditions (7) and (8), for sufficiently small Δ, the continuous EM method solution (5) satisfies

$$\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{z}(t)|^2 \le (2K_2+1)\Delta\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + (2K_2+1)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{11}$$

for all $T>0$.

Proof. Given any $0\le t\le T$, let $k=[t/\Delta]$ be the integer part of t/Δ, this seems incomplete; so $k\Delta\le t\le(k+1)\Delta$. It follows from (5) that

$$\boldsymbol{x}(t)-\boldsymbol{z}(t) = \boldsymbol{f}(\boldsymbol{x}_k,\boldsymbol{u}_k)(t-k\Delta) + \boldsymbol{g}(\boldsymbol{x}_k,\boldsymbol{u}_k)(\boldsymbol{w}(t)-\boldsymbol{w}(k\Delta)) \tag{12}$$

Then, we have

$$\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{z}(t)|^2 \le 2(K_1\Delta+K_2)\Delta\mathrm{E}(|\boldsymbol{x}_k|^2+|\boldsymbol{u}_k|^2) \le (2K_2+1)\Delta\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + (2K_2+1)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}$$

if $2K_1\Delta\le 1$. Hence the assertion (11) follows. □

Theorem 2. Under (2) and Assumption 1, for sufficiently small Δ, the continuous EM method solution (5) satisfies

$$\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le C_T\Delta\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + D_T\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{13}$$

where

$$C_T = 4T(K_1T+K_2)(2K_2+1)\mathrm{e}^{4T(K_1T+K_2)}$$

and

$$D_T = 2T(K_1T+K_2)(4K_2+3)\mathrm{e}^{4T(K_1T+K_2)}$$

Proof. It follows from (1) and (5) that for any $0\le t\le T$,

$$\boldsymbol{x}(t)-\boldsymbol{y}(t) = \int_0^t[\boldsymbol{f}(\boldsymbol{z}(s),\boldsymbol{U}(s))-\boldsymbol{f}(\boldsymbol{y}(s),\boldsymbol{u}(s))]\mathrm{d}s + \int_0^t[\boldsymbol{g}(\boldsymbol{z}(s),\boldsymbol{U}(s))-\boldsymbol{g}(\boldsymbol{y}(s),\boldsymbol{u}(s))]\mathrm{d}\boldsymbol{w}(s)$$

Hence, for sufficiently small Δ,

$$\begin{aligned}
&\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le \\
&2(K_1T+K_2)\mathrm{E}\int_0^t(|\boldsymbol{z}(s)-\boldsymbol{y}(s)|^2+|\boldsymbol{U}(s)-\boldsymbol{u}(s)|^2)\mathrm{d}s \le \\
&4(K_1T+K_2)\mathrm{E}\int_0^t(|\boldsymbol{z}(s)-\boldsymbol{x}(s)|^2+|\boldsymbol{x}(s)-\boldsymbol{y}(s)|^2)\mathrm{d}s + \\
&2T(K_1T+K_2)L\Delta^2\mathrm{E}|\boldsymbol{u}|^2_{\sup} \le \\
&4T(K_1T+K_2)(2K_2+1)\times \\
&\Delta\left(\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2+\mathrm{E}|\boldsymbol{u}|^2_{\sup}\right) + 2(K_1T+K_2)\times \\
&\left(2\mathrm{E}\int_0^t|\boldsymbol{x}(s)-\boldsymbol{y}(s)|^2\mathrm{d}s + TL\Delta^2\mathrm{E}|\boldsymbol{u}|^2_{\sup}\right) \le \\
&4(K_1T+K_2)\mathrm{E}\int_0^t|\boldsymbol{x}(s)-\boldsymbol{y}(s)|^2\mathrm{d}s + \\
&4T(K_1T+K_2)(2K_2+1)\Delta\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \\
&2T(K_1T+K_2)(4K_2+3)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}
\end{aligned}$$

if $L\Delta\le 1$. From an application of the continuous Gronwall Lemma, we obtain a bound of the form

$$\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le C_T\Delta\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + D_T\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}$$

Since this holds for any $t\in[0,T]$, the assertion (13) must hold. □

Remark 2. The inequality (13) implies that the EM method has a strong finite-time convergence order of at least 1/2 with a "squared error constant" that is linearly proportional to $\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2$ and $\mathrm{E}|\boldsymbol{u}|^2_{\sup}$. Obviously, for uncontrolled stochastic systems, it implies that the strong convergence order of EM method also is greater than 1/2.

3 Main results

It is of interest to ask whether the EM method shares mean-square exp-ISS with the SCS (1). The results below answer this question positively.

Lemma 2. Assume that the SCS (1) satisfies mean-square exp-ISS with rate constant λ, growth constant M, and gain β. Under (2) and Assumption 1, there exists a $\Delta^*>0$ such that for every $0<\Delta<\Delta^*$, the continuous EM method satisfies mean-square exp-ISS of the SCS (1) with rate constant $l=\frac{1}{2}\lambda$, growth constant $N=2M\mathrm{e}^{\frac{1}{2}\lambda T}$ and gain $\gamma=(\beta+1)/(1-\mathrm{e}^{-\frac{1}{2}\lambda T})$.

Proof. Choose $T=1+(4\log M)/\lambda$, so that

$$M\mathrm{e}^{-\lambda T} \le \mathrm{e}^{-\frac{3}{4}\lambda T} \tag{14}$$

Now, for any $\tau>0$,

$$\mathrm{E}|\boldsymbol{x}(t)|^2 \le (1+\tau)\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 + (1+\frac{1}{\tau})\mathrm{E}|\boldsymbol{y}(t)|^2 \tag{15}$$

From (2), Assumption 1 and Theorem 2, we have (13). Then, using (13) and (3), we see that

$$\sup_{0\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le (1+\tau)(C_{2T}\Delta\sup_{0\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 + D_{2T}\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}) + ((1+\frac{1}{\tau})(M\mathrm{E}|\boldsymbol{\xi}|^2+\mathrm{E}\beta(|\boldsymbol{u}|^2_{\sup}))$$

If we take Δ sufficiently small, this rearranges to

$$\sup_{0\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le \frac{(1+\frac{1}{\tau})M}{1-(1+\tau)C_{2T}\Delta}\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}\frac{(1+\frac{1}{\tau})\beta+(1+\tau)D_{2T}\Delta}{1-(1+\tau)C_{2T}\Delta}(|\boldsymbol{u}|^2_{\sup}) \quad (16)$$

Now, taking the supremum over $[T,2T]$ in (15), using (13) and the bound (16), and also the stability condition (3), we have

$$\sup_{T\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le (1+\tau)\sup_{0\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 + (1+\frac{1}{\tau})\sup_{T\le t\le 2T}\mathrm{E}|\boldsymbol{y}(t)|^2 \le \frac{(1+\tau)(1+\frac{1}{\tau})M\mathrm{E}|\boldsymbol{\xi}|^2}{1-(1+\tau)C_{2T}\Delta}C_{2T}\Delta + (1+\frac{1}{\tau})M\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-\lambda T} + \mathrm{E}\frac{(1+\tau)(1+\frac{1}{\tau})\beta+(1+\frac{1}{\tau})D_{2T}\Delta}{(1+\frac{1}{\tau})C_{2T}\Delta}C_{2T}\Delta(|\boldsymbol{u}|^2_{\sup}) + (1+\frac{1}{\tau})D_{2T}\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup} + (1+\frac{1}{\tau})\mathrm{E}\beta(|\boldsymbol{u}|^2_{\sup})$$

We write this as

$$\sup_{T\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le R(\Delta)\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}S(\Delta)(|\boldsymbol{u}|^2_{\sup}) \quad (17)$$

where

$$R(\Delta) = \frac{(1+\tau)(1+\frac{1}{\tau})}{1-(1+\tau)C_{2T}\Delta}C_{2T}\Delta M + (1+\frac{1}{\tau})M\mathrm{e}^{-\lambda T}$$

$$S(\Delta) = \frac{(1+\tau)[(1+\frac{1}{\tau})\beta+(1+\tau)D_{2T}\Delta]}{1-(1+\tau)C_{2T}\Delta}C_{2T}\Delta + (1+\frac{1}{\tau})D_{2T}\Delta + (1+\frac{1}{\tau})\beta$$

Let $\tau = 1/\sqrt{\Delta}$. Note that $R(\Delta)$ and $S(\Delta)$ increase monotonically with Δ. It implies that $S(\Delta_1)(x) < S(\Delta_2)(x)$ for all $x\in\mathbf{R}^+$ if $0<\Delta_1<\Delta_2$. Hence, by using (14), and taking Δ sufficiently small, we may ensure that

$$R(\Delta)\le \mathrm{e}^{-\frac{1}{2}\lambda T},\quad S(\Delta)\le\beta+1 \quad (18)$$

In (17) this gives

$$\sup_{T\le t\le 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le \mathrm{e}^{-\frac{1}{2}\lambda T}\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup})$$

Now, let $\hat{\boldsymbol{y}}(t)$ be the solution to the SCS (1) for $t\in[T,\infty)$, with the initial condition $\hat{\boldsymbol{y}}(T)=\boldsymbol{x}(T)$. Following the previous analysis, we have

$$\mathrm{E}|\boldsymbol{x}(t)|^2 \le (1+\tau)\mathrm{E}|\boldsymbol{x}(t)-\hat{\boldsymbol{y}}(t)|^2 + (1+\frac{1}{\tau})\mathrm{E}|\hat{\boldsymbol{y}}(t)|^2 \quad (19)$$

Taking the supremum over $[T,3T]$, and using the Markov property for the SCS (1), we can shift (3) and (13) to $[T,3T]$, obtaining

$$\sup_{T\le t\le 3T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le \frac{(1+\frac{1}{\tau})M}{1-(1+\tau)C_{2T}\Delta}\mathrm{E}|\boldsymbol{x}(T)|^2 + \mathrm{E}\frac{(1+\frac{1}{\tau})\beta+(1+\tau)D_{2T}\Delta}{1-(1+\tau)C_{2T}\Delta}(|\boldsymbol{u}|^2_{\sup})$$

Note that $\mathrm{E}|\hat{\boldsymbol{y}}(t)|^2 \le M\mathrm{E}|\boldsymbol{x}(T)|^2\mathrm{e}^{-\lambda(t-T)} + \mathrm{E}\beta(|\boldsymbol{u}|^2_{\sup})$ for all $t\ge T$. Then, taking the supremum over $[2T,3T]$ in (19), in place of (17), we arrive at

$$\sup_{2T\le t\le 3T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le R(\Delta)\mathrm{E}|\boldsymbol{x}(T)|^2 + \mathrm{E}S(\Delta)(|\boldsymbol{u}|^2_{\sup})$$

Continuing this approach and using (18) gives

$$\sup_{(i+1)T\le t\le (i+2)T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le \mathrm{e}^{-\frac{1}{2}\lambda T}\mathrm{E}|\boldsymbol{x}(iT)|^2 + \mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \quad (20)$$

for $i\ge 0$. From (20) we see that

$$\sup_{(i+1)T\le t\le (i+2)T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le \mathrm{e}^{-\frac{1}{2}\lambda T}\sup_{iT\le t\le (i+1)T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \le \mathrm{e}^{-\frac{1}{2}\lambda T}\mathrm{e}^{-\frac{1}{2}\lambda T}\sup_{(i-1)T\le t\le iT}\mathrm{E}|\boldsymbol{x}(t)|^2 + (\mathrm{e}^{-\frac{1}{2}\lambda T}+1)\mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \le\cdots\le \mathrm{e}^{-\frac{1}{2}\lambda T(i+1)}\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \frac{1-\mathrm{e}^{-\frac{1}{2}\lambda T(i+1)}}{1-\mathrm{e}^{-\frac{1}{2}\lambda T}}\mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \quad (21)$$

Now, using $\tau=1/\sqrt{\Delta}$ in (16), for sufficiently small Δ we see that

$$\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le 2M\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \quad (22)$$

It follows from (21) and (22) that

$$\sup_{(i+1)T\le t\le (i+2)T}\mathrm{E}|\boldsymbol{x}(t)|^2 \le 2M\mathrm{e}^{-\frac{1}{2}\lambda T(i+1)}\mathrm{E}|\boldsymbol{\xi}|^2 + \frac{1-\mathrm{e}^{-\frac{1}{2}\lambda T(i+2)}}{1-\mathrm{e}^{-\frac{1}{2}\lambda T}}\mathrm{E}(\beta+1)(|\boldsymbol{u}|^2_{\sup}) \le 2M\mathrm{e}^{\frac{1}{2}\lambda T}\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-\frac{1}{2}\lambda t} + \mathrm{E}\frac{\beta+1}{1-\mathrm{e}^{-\frac{1}{2}\lambda T}}(|\boldsymbol{u}|^2_{\sup})$$

Hence, the continuous EM method satisfies mean-square exp-ISS with $l=\frac{1}{2}\lambda$, $N=2M\mathrm{e}^{\frac{1}{2}\lambda T}$, and $\gamma=(\beta+1)/(1-\mathrm{e}^{-\frac{1}{2}\lambda T})$. □

The next lemma gives a positive answer to Question 2.

Lemma 3. Under (2) and Assumption 1, assume that for a step $\Delta>0$, the continuous EM method satisfies mean-square exp-ISS with rate constant l, growth constant N and gain γ. If Δ satisfies

$$C_{2T}\mathrm{e}^{lT}(\Delta+\sqrt{\Delta})+1+\sqrt{\Delta}\le\mathrm{e}^{\frac{1}{4}lT},\quad C_T\Delta\le 1 \quad (23)$$

and

$$(D_{2T}+C_{2T}\gamma)(\Delta+\sqrt{\Delta})+\gamma\sqrt{\Delta}\le 1,\quad (D_T+C_T\gamma)\Delta\le 1 \quad (24)$$

where $T=1+(4\log N)/l$, then the SCS (1) satisfies mean-square exp-ISS with rate constant $\lambda=\frac{1}{2}l$, growth constant $M=2N\mathrm{e}^{\frac{1}{2}lT}$, and gain $\beta=(\gamma+1)/(1-\mathrm{e}^{-\frac{1}{2}lT})$.

Proof. First, note that

$$\mathrm{e}^{-\frac{3}{4}lT}N\le\mathrm{e}^{-\frac{1}{2}lT} \quad (25)$$

For any $\tau > 0$, we have

$$\mathrm{E}|\boldsymbol{y}(t)|^2 \leq (1+\tau)\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 + (1+\frac{1}{\tau})\mathrm{E}|\boldsymbol{x}(t)|^2 \quad (26)$$

Using (13) and (6) in (26), we obtain

$$\sup_{T\leq t\leq 2T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq (1+\tau)\sup_{0\leq t\leq 2T}\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 + (1+\frac{1}{\tau})\sup_{T\leq t\leq 2T}\mathrm{E}|\boldsymbol{x}(t)|^2 \leq [(1+\tau)C_{2T}\Delta + (1+\frac{1}{\tau})\mathrm{e}^{-lT}]N\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}[(1+\tau)(D_{2T}+C_{2T}\gamma)\Delta + (1+\frac{1}{\tau})\gamma](|\boldsymbol{u}|^2_{\sup}) \quad (27)$$

Setting $\tau = 1/\sqrt{\Delta}$ gives

$$\sup_{T\leq t\leq 2T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq [C_{2T}\mathrm{e}^{lT}(\Delta+\sqrt{\Delta})+1+\sqrt{\Delta}]N\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-lT} + \mathrm{E}[(D_{2T}+C_{2T}\gamma)(\Delta+\sqrt{\Delta}) + (1+\sqrt{\Delta})\gamma](|\boldsymbol{u}|^2_{\sup}) \quad (28)$$

Using (23) $\sim$ (25), we then have

$$\sup_{T\leq t\leq 2T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq \mathrm{e}^{-\frac{3}{4}lT}N\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup}) \leq \mathrm{e}^{-\frac{1}{2}lT}\sup_{0\leq t\leq T}\mathrm{E}|\boldsymbol{y}(t)|^2 + \mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup}) \quad (29)$$

Now, let $\hat{\boldsymbol{x}}(t)$ for $t \in [T,\infty)$ denote the approximation that arises from applying the EM method with $\hat{\boldsymbol{x}}(T) = \boldsymbol{y}(T)$. Then, using similar arguments to those that produced (27) and (28), we have

$$\sup_{2T\leq t\leq 3T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq (1+\tau)\sup_{T\leq t\leq 3T}\mathrm{E}|\hat{\boldsymbol{x}}(t)-\boldsymbol{y}(t)|^2 + (1+\frac{1}{\tau})\sup_{2T\leq t\leq 3T}\mathrm{E}|\hat{\boldsymbol{x}}(t)|^2 \leq [(1+\tau)C_{2T}\Delta + (1+\frac{1}{\tau})\mathrm{e}^{-lT}]N\mathrm{E}|\boldsymbol{y}(T)|^2 + [(1+\tau)(D_{2T}+C_{2T}\gamma)\Delta + (1+\frac{1}{\tau})\gamma]\mathrm{E}|\boldsymbol{u}|^2_{\sup} \leq \mathrm{e}^{-\frac{1}{2}lT}\sup_{T\leq t\leq 2T}\mathrm{E}|\boldsymbol{y}(t)|^2 + \mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup})$$

Generally, this approach may be used to show that

$$\sup_{iT\leq t\leq (i+1)T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq \mathrm{e}^{-\frac{1}{2}lT}\sup_{(i-1)T\leq t\leq iT}\mathrm{E}|\boldsymbol{y}(t)|^2 + \mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup})$$

for $i \geq 1$. Hence,

$$\sup_{iT\leq t\leq (i+1)T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq \mathrm{e}^{-\frac{1}{2}liT}\sup_{0\leq t\leq T}\mathrm{E}|\boldsymbol{y}(t)|^2 + \frac{1-\mathrm{e}^{-\frac{1}{2}liT}}{1-\mathrm{e}^{-\frac{1}{2}lT}}\mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup}) \quad (30)$$

Now, using (23) and (24), we see that

$$\sup_{0\leq t\leq T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq 2N\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup})$$

Inserting it into (30), we obtain

$$\sup_{iT\leq t\leq (i+1)T}\mathrm{E}|\boldsymbol{y}(t)|^2 \leq \mathrm{e}^{-\frac{1}{2}l(i+1)T}\mathrm{e}^{\frac{1}{2}lT}2N\mathrm{E}|\boldsymbol{\xi}|^2 + \frac{1-\mathrm{e}^{-\frac{1}{2}l(i+1)T}}{1-\mathrm{e}^{-\frac{1}{2}lT}}\mathrm{E}(\gamma+1)(|\boldsymbol{u}|^2_{\sup}) \leq \mathrm{e}^{-\frac{1}{2}lt}\mathrm{e}^{\frac{1}{2}lT}2N\mathrm{E}|\boldsymbol{\xi}|^2 + \mathrm{E}(\frac{\gamma+1}{1-\mathrm{e}^{-\frac{1}{2}lT}})(|\boldsymbol{u}|^2_{\sup})$$

which proves the required result. □

Lemma 2 and Lemma 3 lead to the following theorem.

Theorem 3. Under (2) and Assumption 1, the SCS (1) satisfies mean-square exp-ISS if and only if there exists a $\Delta > 0$ such that the continuous EM method satisfies mean-square exp-ISS with rate constant l, growth constant N, and gain γ, step size Δ, and global constants C_T, D_T for $T = 1 + (4\log N)/l$ satisfying conditions (23) and (24).

Proof. The "if" part of the theorem follows directly from Lemma 3. To prove the "only if" part, suppose that the SCS (1) satisfies mean-square exp-ISS with rate constant λ, growth constant M, and gain β. Lemma 2 shows that there exists a $\Delta^* > 0$ such that for any step size $0 < \Delta \leq \Delta^*$, the EM method satisfies mean-square exp-ISS with rate constant $l = \frac{1}{2}\lambda$, growth constant $N = 2M\mathrm{e}^{\frac{1}{2}\lambda T}$ and gain $\gamma = (\beta+1)/(1-\mathrm{e}^{-\frac{1}{2}\lambda T})$. Noting that these constants are all independent of Δ, it follows that we may reduce Δ if necessary until (23) and (24) are satisfied. □

4 Conclusion

With the rapid development of scientific research and large scale engineering design many control systems are so complicated that it is very difficult to investigate the systems by using traditional methods such as Lyapunov methods. Thus, it is a natural thought to investigate the control systems by using numerical methods. In this paper, we research the mean-square exp-ISS property of SCSs by EM methods. From Theorem 3, it is feasible to investigate the mean-square exp-ISS of the SCS (1) with careful numerical simulations.

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