Finite Horizon H_∞ Preview Control

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Abstract The paper focuses on the H_∞ preview control problem in the finite horizon. Starting with the traditional idea, we found the sticking point and used a suitable linear transformation to eliminate it. Finally, we obtained a sufficient and necessary condition and a simple control-law for the problem. Furthermore, a numerical example is also provided to illustrate that the controller can effectively improve the closed-loop performance.

Key words Preview, H_∞ control, finite horizon, game problem, abstract system

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Controlling and tracking objects more precisely have been a goal unremittingly pursued in many fields including aeronautics, astronautics, navigation, and manufacturing. However, the classical optimal control is over-designed, and the H_∞ control is designed for the worst-case reference signal and may lead to insufficient disturbance attenuation and robustness properties of the closed-loop system.

The development of the infrared and sensor technology renders us access to more or less information in advance, namely, preview information, which can improve the performance effectively when applied appropriately. As shown in the industrial suspension systems[1−3], the closed-loop performance can be improved considerably by using the preview controller. The extensive applications encouraged many works[4−9] with respect to the preview control. So far, the optimal preview theory has been relatively mature[7−8, 10−11]. By contrast, the H_∞ preview control problem is a greater challenge and is stated as Open Problem 51[12].

The preview information may appear in the form of the delay exogenous-input or reference signal. Therefore, preview problems fall into the category of the delay problems, and some methods solving the delay problems are also utilized to deal with the preview control problems[4, 13]. Kojima[4] solved the H_∞ preview problem in the infinite horizon in an abstract space via direct pursuing the explicit positive semi-definite stabilizing solution of the operator Riccati equation. Zhang[13] established the dual relationship between the original problem in the finite horizon and the H_2 fixed-lag smoothing problem for a backward system in Krein space, and thus addressed the control problem via the estimation method. Tadmor[5−6] employed the game theory to give the preview control laws for continuous and discrete systems. Yet, the infinite horizon case[5−6] blurs the difference between the H_∞ preview control and the standard H_∞ control. The paper manages to make it clear in the finite horizon.

The paper is organized as follows. Section 1 proposes the problem, Section 2 solves the problem, Section 3 provides a numerical example, and Section 4 draws a conclusion.

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Notations. Our notations are standard. The transpose of a matrix A is denoted by A^{T}. $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{L_2[a,b]}$ stand for the inner product in $\mathbf{R}^n$ and $L_2[a,b]$, respectively. $\| \cdot \|$ and $\| \cdot \|_{L_2[a,b]}$ are the norms in the appropriate Hilbert space and $L_2[a,b]$.

1 The formulation of the problem

Consider system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t-h) + B_2\boldsymbol{u}(t) && (1)\\ \boldsymbol{z}(t) &= C\boldsymbol{x}(t) + D_2\boldsymbol{u}(t) && (2)\end{aligned}$$

where $\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{u}$ and $\boldsymbol{z}$ are the state, the exogenous input, the control input, and the regulated signal, respectively, and $h (\geq 0)$ is a time delay.

The H_∞ preview control problem in finite horizon is stated as follows: for a given $\gamma > 0$, find the control law like $\boldsymbol{u}(t) = \mathcal{F}(\boldsymbol{x}(t), \boldsymbol{w}(s)|s \in [t-h, t])$ such that

$$\sup_{\boldsymbol{w}} \frac{\|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}} P_{t_f} \boldsymbol{x}(t_f)}{\|\boldsymbol{w}\|^2_{L_2[0,t_f-h]}} < \gamma^2 \tag{3}$$

where t_f is the terminal time, P_{t_f} is a prescribed positive semi-definite matrix, and $\boldsymbol{x}(t_f)$ is the terminal state value.

Denote $\gamma_{\mathrm{opt}} = \inf\{\gamma : \gamma \text{ admits (3)}\}$. Obviously, $\gamma \geq \gamma_{\mathrm{opt}}$ if and only if the problem is solvable.

Remark 1. Different from the general H_∞ control index in finite horizon, (3) only involves the effect of $\boldsymbol{w}(t)$ in the interval $[0, t_f - h]$.

The reason excluding $\boldsymbol{w}(t)$ in $[-h, 0)$ or $(t_f - h, t_f]$ in performance index (3) is displayed as follows:

1) Although $\boldsymbol{w}(t)$ in $[-h, 0)$ impacts the evolution of the system and thus the regulated signal $\boldsymbol{z}(t)$, the H_∞ control here concentrates on measuring the influence produced by the undetermined factor, while $\boldsymbol{w}(t)$, $t \in (-h, 0]$ is known and deterministic.

2) Until t_f instant, $\boldsymbol{w}(t)$ in $(t_f - h, t_f]$ cannot affect the system and thus the regulated signal, so it should not be introduced in the index (3).

Without loss of generality, we make a orthogonal assumption: $D_2^{\mathrm{T}}[\begin{array}{cc} C & D_2 \end{array}] = [\begin{array}{cc} 0 & I \end{array}]$, which considerably facilitates the discussion of the problem.

2 To solve the problem

Let us outline our main idea first. Introduce the problem $\max\min J(\boldsymbol{u}, \boldsymbol{w})$, where

$$J(\boldsymbol{u}, \boldsymbol{w}) = \| \boldsymbol{z} \|^2 + \boldsymbol{x}(t_f)^{\mathrm{T}} P_{t_f} \boldsymbol{x}(t_f) - \| \boldsymbol{w} \|^2 \tag{4}$$

subject to (1) and (2), $\boldsymbol{u}$ is the minimizing player, and $\boldsymbol{w}$ is the maximizing player. The H_∞ optimal control problem is equivalent to finding both the "smallest" value of $\gamma_{\mathrm{opt}} > 0$ under which the optimal game value of $\min\max J(\boldsymbol{u}, \boldsymbol{w})$ is bounded above by zero and the corresponding controller that achieves the optimal value. If the problem is solvable, i.e. $\gamma > \gamma_{\mathrm{opt}}$, then $\min\max J(\boldsymbol{u}, \boldsymbol{w})$ is solvable and has the finite upper value. By virtue of the projection theorem, $\min\max J(\boldsymbol{u}, \boldsymbol{w})$ is solvable and has the finite lower value as $\gamma > \gamma_{\mathrm{opt}}$, which will be verified below. Whereby we can derive the Riccati equations and the property of its solution thereof. As for the proof of the solvability, we will resort to the abstract space theory. More details will be found in the following sections.

2.1 The ground to solve

It is necessary to show the solvability of the problem

$$\max_{\boldsymbol{w}}\{\min_{\boldsymbol{u}}(\|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}} P_{t_f} \boldsymbol{x}(t_f)) - \gamma^2 \|\boldsymbol{w}\|^2_{L_2[0,t_f-h]}\} \tag{5}$$

Finally, we introduce the projection theorem in Hilbert space.

Lemma 1. Let V_1 and V_2 be the Hilbert spaces with bounded linear operators $J : V_2 \to V_2$ and $S : V_1 \to V_2$. Suppose $J^{\mathrm{T}} = J$ and $S^{\mathrm{T}} J S > \epsilon I$ for some $\epsilon > 0$. Then, given any $\boldsymbol{v}_2 \in V_2$, there exists a unique solution to the optimization problem

$$\min_{\boldsymbol{v}_1 \in V_1} \| S\boldsymbol{v}_1 - \boldsymbol{v}_2 \|^2_J = \min_{\boldsymbol{v}_1 \in V_1} \langle S\boldsymbol{v}_1 - \boldsymbol{v}_2, J(S\boldsymbol{v}_1 - \boldsymbol{v}_2) \rangle \tag{6}$$

Denote the optimal value $\max\min J(\boldsymbol{u}, \boldsymbol{w})$ by g_{opt}. On the basis of Lemma 1, we achieve the following conclusion.

Lemma 2. If $\gamma > \gamma_{\mathrm{opt}}$, the game problem (5) is solvable. Moreover, g_{opt} is nonnegative for any initial data.

Proof. We only prove the case $P_{t_f} = 0$, and the case $P_{t_f} \neq 0$ is similar except that the operators S and J are more complicated. Now the roles of S, J, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ in Lemma 1 are taken by the input-output operator of the system (1) and (2), identity operator, the control input, and the output driven by initial data and the disturbance, respectively. For convenience, we will omit the interval of the norms in $L_2[a,b]$. It is clear that the optimal problem $\min_{\boldsymbol{u}} \| \boldsymbol{z} \|^2_{L_2} - \gamma^2 \| \boldsymbol{w} \|^2_{L_2}$ is solvable. As $\gamma > \gamma_{\mathrm{opt}}$, there exists a $\delta > 0$ such that $\gamma^2 \| \boldsymbol{w} \|^2_{L_2} - \| \boldsymbol{z} \|^2_{L_2} < \delta^2 \| \boldsymbol{w} \|^2_{L_2}$. With the similar line again, the roles of S, J, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ in Lemma 1 are taken by the input-output operator (after applying the optimal control input) from the disturbance $\boldsymbol{w}$ to $\boldsymbol{z}$, $\mathrm{diag}\{I, -\gamma^2 I\}$, $\boldsymbol{w}$, and the output driven by initial data, respectively. That the optimal value is nonnegative follows from that $\max\min J(\boldsymbol{u}, \boldsymbol{w}) \geq \min J(\boldsymbol{u}, 0) \geq 0$. □

2.2 Minimization on $\boldsymbol{u}$

With the solvability, the following will take a traditional idea to solve the inner minimization problem in (5).

Let $H = \frac{1}{2}\boldsymbol{z}(t)^{\mathrm{T}}\boldsymbol{z}(t) + \boldsymbol{p}_1(t)^{\mathrm{T}}(A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t-h) + B_2\boldsymbol{u}(t))$ be the Hamilton function related with $\min_{\boldsymbol{u}} \|\boldsymbol{z}\|^2_{L_2}$ and (1), naturally, the optimal $\boldsymbol{u}$ should satisfy

$$\begin{aligned} \dot{\boldsymbol{p}}_1(t) &= -A^{\mathrm{T}}\boldsymbol{p}_1(t) - C^{\mathrm{T}}C\boldsymbol{x}(t) && (7)\\ \boldsymbol{u}(t) &= -B_2^{\mathrm{T}}\boldsymbol{p}_1(t) && (8)\end{aligned}$$

where $\boldsymbol{p}_1(t)$ is the so-called costate.

Combine system (1) and (2) with (7) and (8), a Hamilton-Jacobi type original differential equation (ODE) is given as

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{p}}_1(t) \end{bmatrix} = \begin{bmatrix} A & -B_2B_2^{\mathrm{T}} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{p}_1(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \boldsymbol{w}(t-h) \tag{9}$$

From (1) and (9), for any $t_0, t_1 \in [0, t_f]$, and $t_0 < t_1$,

$$\|\boldsymbol{z}\|^2_{L_2[t_0,t_1]} = -\langle \boldsymbol{x}, \boldsymbol{p}_1 \rangle|^{t_1}_{t_0} + \langle \boldsymbol{w}(t-h), B_1^{\mathrm{T}}\boldsymbol{p}_1(t) \rangle_{L_2[t_0,t_1]} \tag{10}$$

If $\boldsymbol{w} \equiv 0$, the game problem (5) is only a standard linear quadratic regulated (LQR) one. Given that $\boldsymbol{u}$ should be casual, we assume

$$\boldsymbol{p}_1(t) = X(t)\boldsymbol{x}(t) \tag{11}$$

$X(t)$ in (11) will be characterized further.

Substituting (11) into (9), we can find that $X(t)$ satisfies the relationship

$$-\dot{X} = XA + A^{\mathrm{T}}X + C^{\mathrm{T}}C - XB_2B_2^{\mathrm{T}}X, \quad X(t_f) = P_{t_f} \quad (12)$$

For a compact arrangement, differential Riccati equation (12) omits the time index t. We proceed to study the property of $X(t)$. In view of (10), we have

$$\min_{\boldsymbol{u}} \|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) = \quad (13)$$
$$\langle \boldsymbol{x}(0), X(0)\boldsymbol{x}(0)\rangle - \langle \boldsymbol{x}(t_f), X(t_f)\boldsymbol{x}(t_f)\rangle + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f)$$

for any initial value $\boldsymbol{x}(0)$. If $X(t_f) = P_{t_f}$, the minimal value in the above is simplified as

$$\min_{\boldsymbol{u}} \|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) = \langle \boldsymbol{x}(0), X(0)\boldsymbol{x}(0)\rangle \geq 0 \quad (14)$$

which together with Lemma 2 shows that $X(0)$ is positive semi-definite. For any $t > 0, t \in [0, t_f]$, consider the minimization problem $\min_{\boldsymbol{u}} \|\boldsymbol{z}\|^2_{L_2[t,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f)$. Likewise, we will verify that $X(t)$ is positive semi-definite.

However, in most cases, $\boldsymbol{w} \equiv 0$ does not hold. Therefore, more work is needed to solve the H_∞ problem.

2.3 Maximization on $\boldsymbol{w}$

We still resort to the above-mentioned traditional idea to handle the outer maximization problem in (5).

In order to avoid computing $\dfrac{\partial \boldsymbol{w}(t-h)}{\partial \boldsymbol{w}(t)}$, we need to rewrite the game problem as

$$\max_{\boldsymbol{w}}\{\min_{\boldsymbol{u}}(\|\boldsymbol{z}(t)\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f)) -$$
$$\gamma^2\|\boldsymbol{w}(t-h)\|^2_{L_2[h,t_f]}\} \quad (15)$$

Choose the Hamilton function $H = \frac{1}{2}(\boldsymbol{z}(t)^{\mathrm{T}}\boldsymbol{z}(t) - \gamma^2\boldsymbol{w}(t-h)^{\mathrm{T}}\boldsymbol{w}(t-h)) + \boldsymbol{p}_2(t)^{\mathrm{T}}(A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t-h) + B_2\boldsymbol{u}(t))$ connected with (1) and (15). After a straightforward calculation, we note that the optimal $\boldsymbol{w}(t-h)$ admits

$$\dot{\boldsymbol{p}}_2(t) = -A^{\mathrm{T}}\boldsymbol{p}_2(t) - C^{\mathrm{T}}C\boldsymbol{x}(t) \quad (16)$$
$$0 = B_1^{\mathrm{T}}\boldsymbol{p}_2(t) - \gamma^2\boldsymbol{w}(t-h) \quad (17)$$

It is found that (16) is identical with (7), so we can have the Hamilton-Jacobi type ODE

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{p}}_1(t) \end{bmatrix} = \begin{bmatrix} A & -B_2B_2^{\mathrm{T}} + \dfrac{1}{\gamma^2}B_1B_1^{\mathrm{T}} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{p}_1(t) \end{bmatrix} \quad (18)$$

Moreover, (10) clearly still holds for (18).

However, prudential readers will see that (18) holds only in the partial interval $[h, t_f]$, which stems from that $\boldsymbol{w}(t-h), t \in [0, h)$ is known and cannot be chosen. Meanwhile, another fact is that

$$\|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) - \gamma^2\|\boldsymbol{w}\|^2_{L_2[0,t_f-h]} =$$
$$\boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) + \langle \boldsymbol{w}(t-h), B_1^{\mathrm{T}}\boldsymbol{p}_1(t)\rangle_{L_2[0,h]} -$$
$$\langle \boldsymbol{x}(t), \boldsymbol{p}_1(t)\rangle|_0^{t_f} \quad (19)$$

As $h = 0$, (18) holds in the whole interval $[0, t_f]$, and (19) is reduced to

$$\|\boldsymbol{z}\|^2_{L_2[0,t_f]} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) - \gamma^2\|\boldsymbol{w}\|^2_{L_2[0,t_f]} =$$
$$-\langle \boldsymbol{x}(t), \boldsymbol{p}_1(t)\rangle|_0^{t_f} + \boldsymbol{x}(t_f)^{\mathrm{T}}P_{t_f}\boldsymbol{x}(t_f) \quad (20)$$

Assume

$$\boldsymbol{p}_1(t) = Y(t)\boldsymbol{x}(t) \quad (21)$$

We will describe $Y(t)$ further and thus get the controller from (8). Associating (21) with (18), we will find that $Y(t)$ admits Riccati equation

$$-\dot{Y} = YA + A^{\mathrm{T}}Y + C^{\mathrm{T}}C - Y\left(B_2B_2^{\mathrm{T}} - \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}\right)Y \quad (22)$$

By referring to (20), the problem ($h = 0$) is solved when we take $Y(t_f) = P_{t_f}$; in addition, $Y(t) \geq 0$ is ensured by that the optimal game value is nonnegative.

To return to the preview control ($h \neq 0$), when applying the idea to handle the standard H_∞ control problem, we encounter two difficulties: one is whether the assumption $\boldsymbol{p}_1(t) = Y(t)\boldsymbol{x}(t)$ at h is rational or not after h time units evolution of (9); the other is that unlike (20), such an optimal game value in the form of (19) makes no contributions for analyzing the problem further.

Since one of the possible obstacles is aroused by the values of $\boldsymbol{x}(h)$ and $\boldsymbol{p}_1(h)$, we will investigate them below.

From (9), $\boldsymbol{x}$ and $\boldsymbol{p}_1$ are coupled, which is not good for the analysis and calculation of the problem. Hence, let

$$\boldsymbol{y}(t) = \boldsymbol{p}_1(t) - X(t)\boldsymbol{x}(t) \quad (23)$$

where X is as in (12), then (9) is simplified to

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{y}}(t) \end{bmatrix} = \begin{bmatrix} A_2 & -B_2B_2^{\mathrm{T}} \\ 0 & -A_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ -E_2^{\mathrm{T}} \end{bmatrix}\boldsymbol{w}(t-h) \quad (24)$$

with $A_2 = A - B_2B_2^{\mathrm{T}}X(t)$ and $E_2 = B_1^{\mathrm{T}}X(t)$.

According to (24), a straightforward computation yields

$$\boldsymbol{y}(t) = \int_t^{t_f} \Phi(s,t)^{\mathrm{T}}E_2^{\mathrm{T}}\boldsymbol{w}(s-h)\mathrm{d}s \quad (25)$$

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(s,t)B_1\boldsymbol{w}(s-h)\mathrm{d}s - \quad (26)$$
$$\int_0^t \Phi(s,t)B_2B_2^{\mathrm{T}}\int_s^{t_f}\Phi(r,s)^{\mathrm{T}}E_2^{\mathrm{T}}\boldsymbol{w}(r-h)\mathrm{d}r\mathrm{d}s$$

where $\Phi(\cdot,\cdot)$ is the state transition matrix corresponding A_2. Note that the computation uses the boundary value $\boldsymbol{y}(t_f) = \boldsymbol{p}_1(t_f) - X\boldsymbol{x}(t_f) = \boldsymbol{0}$.

Denote

$$G_c(t) = \int_0^t \Phi(r,0)B_2B_2^{\mathrm{T}}\Phi(r,0)\mathrm{d}r \quad (27)$$
$$B(t) = B_1 - G_c(t)E_2^{\mathrm{T}} \quad (28)$$
$$\boldsymbol{m}(h) = \Phi(h,0)\boldsymbol{x}(0) + \int_0^h \Phi(t,r)B(r)\boldsymbol{w}(r-h)\mathrm{d}r \quad (29)$$

Then from (25) and (26), we have the results directly

$$\boldsymbol{y}(h) = \int_h^{t_f} \Phi(s,h)E_2^{\mathrm{T}}\boldsymbol{w}(s-h)\mathrm{d}s \quad (30)$$
$$\boldsymbol{x}(h) = \boldsymbol{m}(h) - G_c(h)\boldsymbol{y}(h) \quad (31)$$

From (29), we know that $\boldsymbol{m}(h)$ (only contains the initial data) qualifies for the initial value. What is more important, we can thus assume

$$\boldsymbol{y}(t) = Z(t)\boldsymbol{m}(t) \quad (32)$$

at h because $\boldsymbol{y}(h)$ is determined completely by the optimal $\boldsymbol{w}(t), t > 0$, which is to be determined.

Base on the above analysis and the linear relationship between variables, we choose the variable pair $\boldsymbol{m}(t)$, $\boldsymbol{y}(t)$ to take the place of the pair $\boldsymbol{x}(t)$, $\boldsymbol{p}_1(t)$ to describe the required optimal trajectories.

Make $\boldsymbol{m}(t)=\boldsymbol{x}(t)+G_c(h)\boldsymbol{y}(t)$ and associate it with (17) in the interval $[h,t_f]$. Equation (24) is transformed to

$$\begin{bmatrix} \dot{\boldsymbol{m}}(t) \\ \dot{\boldsymbol{y}}(t) \end{bmatrix} = \begin{bmatrix} A_r & -R_r \\ -Q_r & -A_r^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{m}(t) \\ \boldsymbol{y}(t) \end{bmatrix} \tag{33}$$

where

$$A_r = A_2+\gamma^{-2}B(h)E_2 \tag{34}$$

$$R_r = \Phi(h,0)B_2B_2^{\mathrm{T}}\Phi(h,0)^{\mathrm{T}}-\gamma^{-2}B(h)B(h)^{\mathrm{T}} \tag{35}$$

$$Q_r = \gamma^{-2}E_2^{\mathrm{T}}E_2 \tag{36}$$

Substituting (32) into (33), we have the Riccati equation as

$$-\dot{Z}=ZA_\gamma+A_\gamma^{\mathrm{T}}Z-ZR_\gamma Z+Q_r \tag{37}$$

It still need to characterize $Z(t)$ and associate it with the optimal game value.

From (19), if we take $Z(t_f)=(P_{t_f}-X(t_f))(I-X(t_f)G_c(h))^{-1}$, the optimal game value is connected closely with $\boldsymbol{p}_1(t)$ in $[0,h]$, whose explicit expression can be computed using the relationship (23) and the evolutions for $\boldsymbol{y}(t)$ in (25) as well as $\boldsymbol{x}(t)$ in (26). In general, the optimal game value is decided by the initial data. Hence, after substituting the explicit expression of $\boldsymbol{p}_1(t)$ into (19), we rearrange the terms in the results. The optimal game value finally is as follows

$$g_{\mathrm{opt}} \triangleq \langle \boldsymbol{m}(h),\boldsymbol{y}(h)\rangle+\langle \boldsymbol{x}(0),X(0)\boldsymbol{x}(0)\rangle + \tag{38}$$

$$2\langle \boldsymbol{x}(0),\int_0^h \Phi(r,0)^{\mathrm{T}}E_2^{\mathrm{T}}\boldsymbol{w}(r-h)\mathrm{d}r\rangle +$$

$$2\int_0^h \langle \boldsymbol{w}(r-h),E_2\int_0^r \Phi(r,s)B(s)\boldsymbol{w}(s-h)\mathrm{d}s\rangle \mathrm{d}r \tag{39}$$

According to Lemma 2, for any initial value, $g_{\mathrm{opt}}\geq 0$; naturally, for $\boldsymbol{w}(t-h)\equiv 0, t\in[0,h)$ and any $\boldsymbol{x}(0)$,

$$g_{opt} = \langle \boldsymbol{x}(0),X(0)\boldsymbol{x}(0)\rangle + \langle \boldsymbol{x}(0),\Phi(h,0)^{\mathrm{T}}Z(h)\Phi(h,0)\boldsymbol{x}(0)\rangle \geq 0 \tag{40}$$

while (5) implies that

$$g_{\mathrm{opt}} \geq \min\|\boldsymbol{z}\|_{L_2[0,t_f]}+\boldsymbol{x}(t_f)^{\mathrm{T}}P(t_f)\boldsymbol{x}(t_f) = \boldsymbol{x}(0),X(0)\boldsymbol{x}(0)\rangle \tag{41}$$

So $Z(h)$ is positive semi-definite via (40) and (41). With the similar lines of $Z(h)$, we can prove that $Z(t)\geq 0, t>0$.

In terms of the above analysis, it is not hard to reach the conclusion.

Theorem 1. Consider the system (1) and (2) and the performance (3). If $\gamma>\gamma_{\mathrm{opt}}$ holds, then the H_∞-like Riccati equation (37) has a positive semi-definite solution.

2.4 The H_∞ preview controller

Similar to the general time-delay systems, the system (1) and (2) can also be reformed as an equivalent abstract delay-free system[5, 15]. The H_∞ preview control (3) for the system (1) and (2) can thus be converted into a standard H_∞ control. Denote $\boldsymbol{f}(t)=(\boldsymbol{x}(t),\breve{\boldsymbol{w}}_t)$ as the abstract state, where $\breve{\boldsymbol{w}}_t=\{\boldsymbol{w}(t+s), s\in[-h,0]\}$. Then, the H_∞ control for the abstract system generates the optimal game value like $\langle(\boldsymbol{x}(0),\breve{\boldsymbol{w}}_0),\mathcal{X}(\boldsymbol{x}(0),\breve{\boldsymbol{w}}_0)\rangle$ with positive semi-definite operator kernel $\mathcal{X}$. $\mathcal{X}$ satisfies operator Riccati equation

$$-\dot{\mathcal{X}}=\mathcal{X}\mathcal{A}+\mathcal{A}^{\mathrm{T}}\mathcal{X}+\mathcal{X}\left(\frac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}_1^{\mathrm{T}}-\mathcal{B}_2\mathcal{B}_2^{\mathrm{T}}\right)+\mathcal{C}^{\mathrm{T}}\mathcal{C} \tag{42}$$

where

$$\mathcal{A}(\boldsymbol{\eta},\boldsymbol{\phi})=\left(A\boldsymbol{\eta}+B_1\boldsymbol{\phi}(-h),\frac{\mathrm{d}}{\mathrm{d}\theta}\boldsymbol{\phi}(\theta)\right)$$

$$\mathcal{D}(\mathcal{A})=\left\{(\boldsymbol{\eta},\boldsymbol{\phi})\in M_2,\boldsymbol{\phi}(\theta)=\int_\theta^0 \boldsymbol{\psi}(\sigma)\mathrm{d}\sigma,\phi\in L_2[-h,0]\right\}$$

$$\mathcal{B}_1\boldsymbol{w}=(0,\delta(\cdot))\boldsymbol{w},\quad \mathcal{B}_2\boldsymbol{u}=(B_2\boldsymbol{u},0)$$

$$\mathcal{C}(\boldsymbol{\eta},\boldsymbol{\phi})=C\boldsymbol{\eta},\quad \mathcal{D}_2\boldsymbol{u}=D_2\boldsymbol{u}$$

Immediately, the γ-suboptimal H_∞ control-law can be given as

$$\boldsymbol{u}(t)=-\mathcal{B}_2^{\mathrm{T}}\mathcal{X}\left(\boldsymbol{x}(t),\breve{\boldsymbol{w}}_t\right) \tag{43}$$

As (43) is combined with (38), the following theorem holds.

Theorem 2. Considering the system (1) and (2), if given $\gamma>\gamma_{\mathrm{opt}}$, then a suboptimal H_∞ control law satisfying (3) is given as

for $0\leq t\leq t_f-h$,

$$\boldsymbol{u}(t) = -B_2^{\mathrm{T}}\big((X(t)+\Phi(h,0)^{\mathrm{T}}Z(t+h)\Phi(h,0))\boldsymbol{x}(t)+ \int_0^h (\Phi(r,0)^{\mathrm{T}}E_2^{\mathrm{T}}+\Phi(h,0)^{\mathrm{T}}Z(t+h)\Phi(h,r)B(r))\times \boldsymbol{w}(t-h+r)\mathrm{d}r\big) \tag{44a}$$

for $t_f-h<t\leq t_f$,

$$\boldsymbol{u}(t)=-B_2^{\mathrm{T}}\left(X(t)\boldsymbol{x}(t)+\int_t^{t_f}\Phi(s,t)^{\mathrm{T}}E_2^{\mathrm{T}}\boldsymbol{w}(r-h)\mathrm{d}r\right) \tag{44b}$$

where $B(\cdot)$ is defined as in (28), and $X(\cdot)$ and $Z(\cdot)$ are solutions to (12) and (37), respectively.

It is easy to observe that the controller in (44) takes advantage of not only the state but also the preview information $\boldsymbol{w}(t+s), s\in[-h,0]$ at t instant. Meanwhile, compared with the general control law, the gain matrix in (44) is unusual and involves the two Riccati equations (12) and (37).

We proceed to discuss the sufficient condition for $\gamma>\gamma_{\mathrm{opt}}$, namely, the solvable conditions for the H_∞ preview control problem.

Theorem 3. Considering the system (1) and (2), if there exist positive semi-definite solutions to (12) and (37), respectively, then $\gamma>\gamma_{\mathrm{opt}}$.

Proof. The proof mainly depends on the standard H_∞ control conclusions[15] of the equivalent abstract system and is omitted. □

Theorem 3 together with Theorems 1 and 2 in effect results in the following theorem.

Theorem 4. Considering the system (1) and (2) and the performance (3), $\gamma>\gamma_{\mathrm{opt}}$ if and only if that there exist positive semi-definite solutions to (12) and (37), respectively. Moreover, if $\gamma>\gamma_{\mathrm{opt}}$ holds, (44) provides a suboptimal H_∞ full-information controller.

3 Numerical example

We have already proposed the control law for the H_∞ preview control problem in the preceding section. In order to illustrate that it is effective, let us consider the H_∞ preview control problem for the scalar system (1) and (2) with $A = 2, B_1 = 1, B_2 = 1, C = 0$, and $D = 1$, where the choices of C and D follow the aforementioned orthogonal assumption.

Because the finite horizon problem is considered, we will compute the left-side of the performance index (3) instead of the standard γ-iteration in our numerical experiment. For prescribed $\gamma = 2.5$, the state responses using the standard H_∞ control law and our H_∞ preview control law (11) are displayed in Fig. 1, and the control performance, namely, the exact ratio of the left-side in (3) is shown in Fig. 2.

The state x_1 in the solid line in Fig. 1 is generated by the H_∞ preview control law, and the state x_2 in the dash-dot line is yielded by the standard H_∞ control law. Given their amplitudes and response-speeds, the former, by contrast, is more favorable.

From Fig. 2, it is as expected that the H_∞ preview control law ($h \neq 0$) leads to better performance than the standard H_∞ control law ($h = 0$). Another fact is also seen that the H_∞ preview control performance improves as the preview length h increases, it keeps nearly close to certain level when the preview length h is larger than 1. In other words, it is not the case that the more information, the better performance. Such phenomenon is in line with Mirkin′s research[16] and is the so-called saturation of performance.

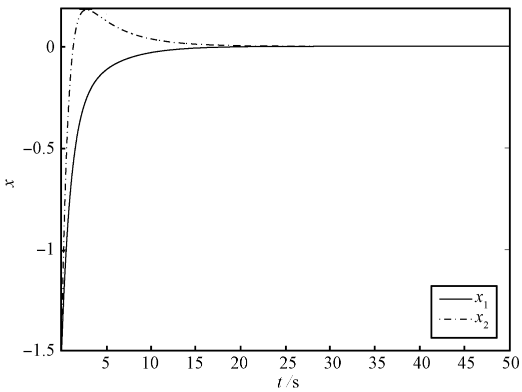

Fig. 1 The state responses

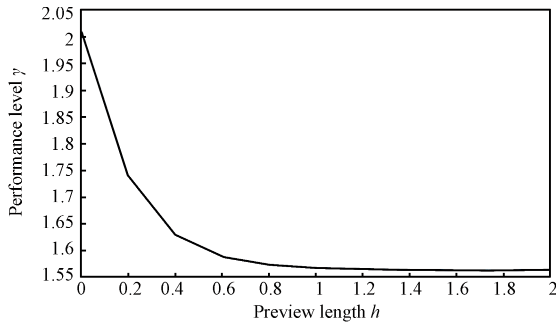

Fig. 2 The performance improvement

4 Conclusion

The paper investigates the H_∞ preview control problem in the finite horizon, and obtains a sufficient and necessary condition for the solvability of the H_∞ preview problem. Meanwhile, a simple control law is formulated. It utilizes the future information of exogenous input so that it is able to preempt the effect of $\boldsymbol{w}$ when $\boldsymbol{u}$ and $\boldsymbol{w}$ are introduced, and the performance is thus improved.

As a matter of fact, the H_∞-like Riccati equation is achieved by the similitude transformation of a delay-free H_∞ Riccati equation from of a mathematical viewpoint. The transformation may change the original stable subspace and also the solution of the Riccati equation, so the paper analyzes the rationality of the transformation, and verifies that the solution of the H_∞-like Riccati equation is positive semi-definite.

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