Reachability of Affine Systems on Polytopes

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Abstract The paper studies reachability problem of autonomous affine systems on n-dimensional polytopes. Our goal is to obtain both the largest positive invariant set in the polytope and the backward reachable set (the attraction domain) of each facet. Special attention is paid to the largest stable invariant affine subspace. After presenting several useful properties of those sets, a partition procedure is given to determine the largest positive invariant set in the polytope and all the attraction domains of facets.

Key words Reachability, polytope, invariant set, attraction domain, piecewise affine hybrid systems

In modeling and control of engineering systems, one often has to deal with hybrid characterization of a system, which includes a set of continuous dynamics and a set of predicates over the continuous state space. These systems are known as hybrid systems[1] in both control and computer science. Examples of hybrid systems can be found in power networks, robotics, chemical processes, biochemical reactions, etc.

A particular subclass of hybrid systems, called piecewise affine (or piecewise linear) hybrid systems (PWAHS), was first introduced by Sontag[2] in 1981. Recently, this class of hybrid systems regains considerable research attentions since promising new ideas have appeared in the last five years in this area and the computational complexity issues seem relatively simple (see, e.g., [3−9]).

A piecewise affine hybrid system consists of a partition of the state space (corresponding to discrete modes) and a set of affine dynamics. As soon as the continuous state reaches the boundary of each region, a discrete event occurs, transferring the system to a new discrete mode.

In the paper, we focus on one discrete mode, that is, we restrict our attention to an affine system defined on a full-dimensional polytope. The reachability problem of affine systems on polytopes is composed of the following two subproblems. One subproblem is to determine the attraction domain of every facet of the polytope since leaving through a different facet corresponds to a different transition from one mode to the other and may result in a totally different behavior. The other subproblem is to find out the largest positive invariant set in the polytope since if a trajectory enters the invariant set, it will never leave it and no further discrete event can occur. Therefore, these two subproblems are of great importance for reachability analysis of PWAHS.

This work draws inspiration from [10] and extends their work to n-dimension while in [10] the affine system was assumed to be in the plane. For n-dimensional case, we start by showing that the largest positive invariant set lies in the largest stable affine subspace of the system. Therefore, it reduces the complexity of determining the largest positive invariant set in the polytope by looking at only the lower dimensional carrying affine subspace. After introducing the notion of exit sets, we will prove that both the largest positive invariant set and the attraction domain of every exit set are open in their carrying affine subspaces. Finally, a procedure is proposed to partition the polytope and determine the invariant set in the polytope and the attraction domains of facets. The division is based on numerical computation of dividing hypersurfaces.

Also, the paper contributes to the set invariance study for independent interest. Compared with some existing literature, which is mostly based on Nagumo theorem and Lyapunov level set and only gives an approximation of the largest positive invariant set (see [11] for a detailed discussion), our result provides an explicit way to find more accurately the largest positive invariant set in a polytope while the computation complexity is relatively simple and acceptable.

1 Preliminaries

In this section, we provide the background and formulate the problem.

1.1 Terminologies

Notations $\mathbf{R}$ and $\mathbf{C}$ are used to represent the sets of real and complex numbers, respectively. Let $\mathrm{Re}(x)$ denote the real part of a complex number x. A partition of a set S is a collection of disjoint subsets of S whose union is S.

Given a set of m points $V=\{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_m\}$ in $\mathbf{R}^n$, the linear combination of all points in V is denoted by

$$\mathrm{span}(V)=\sum_{i=1}^{m}a_i\boldsymbol{v}_i,\ a_i\in\mathbf{R}$$

The convex hull of V is the convex combination of all points in V, i.e.,

$$\mathrm{cov}(V)=\sum_{i=1}^{m}\alpha_i\boldsymbol{v}_i,\ \alpha_i\in[0,1],\ \sum_{i=1}^{m}\alpha_i=1$$

The affine hull of V is the affine combination of all points in V, i.e.,

$$\mathrm{aff}(V)=\sum_{i=1}^{m}\alpha_i\boldsymbol{v}_i,\ \sum_{i=1}^{m}\alpha_i=1$$

In affine geometry, an affine subspace (or flat) is a subset of $\mathbf{R}^n$ with the property that any affine combination of vectors in the affine subspace also belongs to it. Simply, it can be viewed as translating a corresponding linear subspace from the origin to a point in $\mathbf{R}^n$.

Let S be an m-dimensional set in $\mathbf{R}^n$. We use $\mathrm{int}(S)$ and ∂S to denote the relative interior and relative boundary of S, respectively. Here, the relative topology is used. When $m=n$, these notations are just in the normal sense. Denote $\bar{S}$ as the closure of S.

An n-dimensional polytope[12] can be written as an intersection of d half spaces, i.e.,

$$P=\bigcap_{i=1}^{d}\{\boldsymbol{x}\in\mathbf{R}^n|\boldsymbol{n}_i\cdot\boldsymbol{x}\le\gamma_i\}$$

where $\boldsymbol{n}_i$ is the unit normal vector pointing outside of P and γ_i is a constant. The set $\{\boldsymbol{x}\in\mathbf{R}^n|\boldsymbol{n}_i\cdot\boldsymbol{x}=\gamma_i\}$ is called its

Received June 25, 2008; in revised form December 3, 2008
Supported by National Natural Science Foundation of China (60504024), Zhejiang Provincial Natural Science Foundation of China (Y106010), and the Specialized Research Fund for the Doctoral Program of Higher Education of China (20060335022)
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DOI: 10.3724/SP.J.1004.2009.01528

supporting hyperplane. A facet is an $(n-1)$-dimensional intersection of P with one of its supporting hyperplanes, that is,

$$F_i = \{\boldsymbol{x} \in P | \boldsymbol{n}_i \cdot \boldsymbol{x} = \gamma_i\},\ i = 1, \cdots, d$$

Next, consider an affine system defined on a polytope P:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{a},\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ \boldsymbol{x} \in P \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$ and $\boldsymbol{a} \in \mathbf{R}^n$. That is, the above governing dynamics remains valid as long as the state $\boldsymbol{x}$ is in P, and as soon as the state reaches the boundary of the polytope, a discrete event occurs. At this instant, the hybrid system switches from this discrete mode to another, and a different dynamics continues. Because the occurrence of discrete event depends on the facet through which the state leaves the polytope, it is desirable to determine for every initial state in P, which facet will be arrived in finite time, or whether the trajectory will remain in the polytope forever.

Finally, let $\boldsymbol{x}(t, \boldsymbol{x}_0)$ denote the solution trajectory of affine system (1) starting at $\boldsymbol{x}_0$. Denote $\bar{\boldsymbol{x}}$ an equilibrium point of (1) (i.e., $A\bar{\boldsymbol{x}} + \boldsymbol{a} = \boldsymbol{0}$). Hence, if A is nonsingular, there is only one equilibrium point in $\mathbf{R}^n$ (i.e., $\bar{\boldsymbol{x}} = -A^{-1}\boldsymbol{a}$). Otherwise, if A is singular and $\text{rank}[A] = \text{rank}[A, \boldsymbol{a}]$, the equilibrium set is an affine subspace with dimension $n - \text{rank}[A]$. In this paper, we assume that the affine system has a unique equilibrium point and that it is not on the boundary of P. The assumptions here are to make us focus on the systematic analysis instead of putting too much efforts on some complicated arguments for trivial cases (see [8] for similar assumptions and discussions).

1.2 Problem formulation

Next, we introduce a few definitions and then the problem.

Definition 1. Let F_i be a facet of P with its normal vector $\boldsymbol{n}_i$. We define the identifier function on F_i as

$$g_i(\boldsymbol{x}) = \boldsymbol{n}_i \cdot (A\boldsymbol{x} + \boldsymbol{a}),\ \boldsymbol{x} \in F_i \tag{2}$$

Remark 1. From this definition, it can be easily verified that for any point $\boldsymbol{x}_1 \in F_i$ and $g_i(\boldsymbol{x}_1) > 0$, the solution trajectory from $\boldsymbol{x}_1$ leaves the polytope immediately. And if $g_j(\boldsymbol{x}_2) < 0$, then $\boldsymbol{x}_2$ can not be reached from $\text{int}(P)$ (since $-g_j(\boldsymbol{x}_2) > 0$ indicates that the time-backward direction points outside the polytope). Taking this fact into account, we know that for $\boldsymbol{x}_3 \in F_i \cap F_j$, if $g_i(\boldsymbol{x}_3) > 0$ and $g_j(\boldsymbol{x}_3) < 0$, then no trajectory can reach $\boldsymbol{x}_3$ from the interior of the polytope. See Fig. 1 for an example.

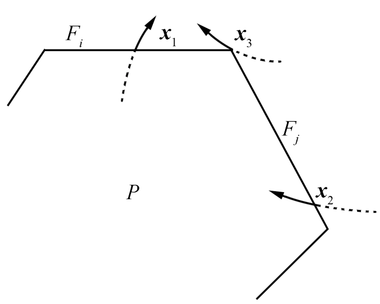


Fig. 1 This figure indicates that the trajectory initiating at $\boldsymbol{x}_1$ goes outside, the trajectory starting at $\boldsymbol{x}_2$ enters P from outside, and the trajectory from $\boldsymbol{x}_3$ is tangent to the surface of polytope. The real line with arrow represents the vector field at that point

Definition 2. The attraction domain of a facet F_i is the set of all interior points of P, from which the solution trajectories reach F_i in the smallest time and then leave the polytope immediately, i.e.,

$$A(F_i) = \{\boldsymbol{x}_0 \in \text{int}(P) | \exists T > 0 \text{ such that } \boldsymbol{x}(t, \boldsymbol{x}_0) \in \text{int}(P) \text{ for } t \in [0, T),\ \boldsymbol{x}(T, \boldsymbol{x}_0) \in F_i \text{ and } g_i(\boldsymbol{x}(T, \boldsymbol{x}_0)) > 0\}$$

Definition 3. Define O as the largest positive invariant set in P, consisting of all interior points of P from which the solutions remain in the interior of P forever, i.e.,

$$O = \{\boldsymbol{x}_0 \in \text{int}(P) | \boldsymbol{x}(t, \boldsymbol{x}_0) \in \text{int}(P) \text{ for } t \in [0, \infty)\}$$

Problem 1. Consider affine system (1) on P, the reachability problem is to determine:

1) the attraction domain of every facet, namely, $A(F_i)$ for $i = 1, \cdots, d$;

2) the largest positive invariant set in P, namely, O.

To solve the problem, we further introduce several notions of exit set and attraction domain of exit set. The definitions are drawn from [10] with some modifications.

Definition 4. We say a point $\boldsymbol{x} \in \partial P$ satisfies exit condition if $g_i(\boldsymbol{x}) > 0$ holds for all the facets F_i that $\boldsymbol{x}$ belongs to.

As an example, in Fig. 1 the point $\boldsymbol{x}_1$ satisfies exit condition while $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ do not.

Definition 5. A total exit set U_{tot} contains all the points in ∂P that satisfy the exit condition.

We divide the total exit set U_{tot} into a collection of K disjoint sets $U_1, \cdots, U_K$ so that each U_i is connected. We call each U_i an exit set. Notice that every facet F_i is partitioned into at most two subsets by the identifier function on F_i. One of them (which, if exists, is of $(n-1)$-dimensions and convex) belongs to U_{tot} while the other does not. So each exit set U_i may consist of just a subset of one facet (see, e.g., Fig. 2) or several subsets from different facets, which are connected through the intersection of facets (see, e.g., Fig. 3).

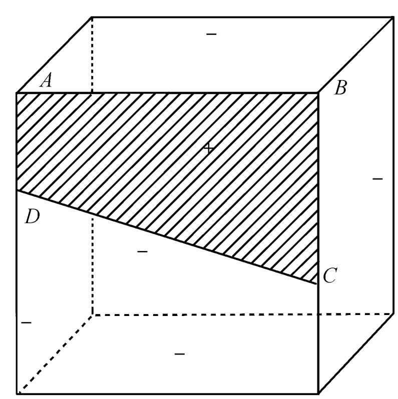


Fig. 2 The sign of identifier function on each facet is marked with $+/-$. In this case, an exit set U_i is the shaded set excluding the relative boundaries $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$

Following the method in [10], we will start by computing the attraction domain of exit set U_i. We define the attraction domain of an exit set in a similar way.

Definition 6. The attraction domain of U_i is defined as

$$A_i = \{\boldsymbol{x}_0 \in \text{int}(P) | \exists T > 0 \text{ such that } \boldsymbol{x}(t, \boldsymbol{x}_0) \in \text{int}(P) \text{ for } t \in [0, T), \text{and } \boldsymbol{x}(T, \boldsymbol{x}_0) \in U_i\}$$

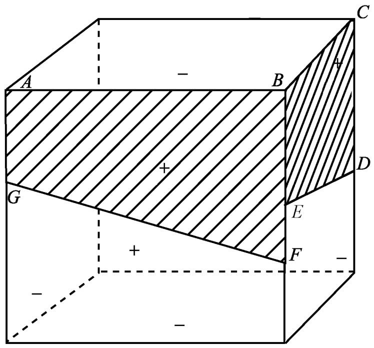


Fig. 3 According to Remark 1, it is clear that the points whose neighborhood (in the topological sense) has both positive and negative signs do not belong to exit sets. So in this case, an exit set U_i consists of two pieces from two facets, which is the shaded set excluding the points on $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}, \overline{FG}, \overline{GA}$

Definition 7. Let D be a subset of $\mathrm{int}(P)$ such that the trajectory from each point in D will reach ∂P in a finite time, and on that occasion the vector field is tangent to at least one facet of P, i.e.,

$$D = \{\boldsymbol{x}_0 \in \mathrm{int}(P) | \exists T > 0 \text{ such that } \boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathrm{int}(P) \text{ for } t \in [0, T),\ \boldsymbol{x}(T, \boldsymbol{x}_0) \in \partial P \text{ and } \exists i : g_i(\boldsymbol{x}(T, \boldsymbol{x}_0)) = 0\}$$

2 Main results

2.1 Characterization

Firstly, we provide several known results on the reachability problem, namely, a partition of the polytope P in terms of reachability and a non-existence condition of invariant set.

Lemma 1[10]**.** Consider affine system (1) on P. The collection of sets $A_1, A_2, \cdots, A_K, O$, and D is a partition of $\mathrm{int}(P)$.

Lemma 2[13]**.** For affine system (1) on P, if P has no equilibrium point, then O is an empty set.

Secondly, we present some properties of the largest positive invariant set when it exists. That is, the equilibrium point $\bar{\boldsymbol{x}}$ is inside P. It will be shown in the following theorem that the largest positive invariant set lies in the largest stable affine subspace (See Theorem 1).

Let $\lambda_1, \cdots, \lambda_m$ be m $(m \le n)$ distinct eigenvalues of A, and μ_i and ν_i be the algebraic and geometric multiplicity of λ_i, respectively. Hence, $\sum_{i=1}^{m} \mu_i = n$. If $\mathrm{Re}(\lambda_i) < 0$, then let V_i be the set of eigenvectors (including generalized eigenvectors) of λ_i. Therefore, V_i has μ_i vectors if λ_i is real and it has $2\mu_i$ vectors if λ_i is complex. Moreover, if $\mathrm{Re}(\lambda_i) = 0$, then let V_i be the set of eigenvectors (excluding the generalized eigenvectors) of λ_i. Hence, V_i has ν_i vectors for real eigenvalue and $2\nu_i$ for complex eigenvalue. It should be pointed out that $V_i = V_{i+1}$ if λ_i and λ_{i+1} are two conjugate complex eigenvalues.

Let

$$V_Q = \bigcup_{i=1}^{m} V_i \tag{3}$$

and

$$Q = (\bar{\boldsymbol{x}} + \mathrm{span}(V_Q)) \cap \mathrm{int}(P) \tag{4}$$

Theorem 1. For affine system (1) on P, if $\bar{\boldsymbol{x}} \in \mathrm{int}(P)$, then

1) $O \subseteq Q$;
2) $\mathrm{aff}(O) = \mathrm{aff}(Q)$;
3) O is convex.

Proof. 1) Let $\boldsymbol{z} = \boldsymbol{x} - \bar{\boldsymbol{x}}$; then $\dot{\boldsymbol{z}} = A\boldsymbol{z}$. It can be checked that $\mathrm{span}(V_Q)$ is the largest Lyapunov stable subspace[14−15]. Therefore, if $\boldsymbol{z}(0) \notin \mathrm{span}(V_Q)$ then $\boldsymbol{z}(t)$ goes to infinity. That is equivalent to say, when $\boldsymbol{x}(0) \notin \mathrm{aff}(Q)$, the trajectory $\boldsymbol{x}(t)$ goes to infinity and of course leaves the polytope P. Then, from the definition of the positive invariant set, it follows that $O \subset \mathrm{aff}(Q)$. Moreover, since $O \subseteq \mathrm{int}(P)$, we get $O \subseteq Q$.

2) On the one hand, we have $\mathrm{aff}(O) \subseteq \mathrm{aff}(Q)$ from 1). On the other hand, as $\bar{\boldsymbol{x}}$ is in the interior of P, we can select an $\epsilon > 0$ so that the ϵ-ball centered at $\bar{\boldsymbol{x}}$, $B(\bar{\boldsymbol{x}}, \epsilon)$, is entirely in the interior of P, too. Recall that $\mathrm{aff}(Q)$ is a Lyapunov stable affine subspace; so there exists a $\delta > 0$ such that the trajectories starting from any point in $B(\bar{\boldsymbol{x}}, \delta) \cap \mathrm{aff}(Q)$ remain in $B(\bar{\boldsymbol{x}}, \epsilon) \cap \mathrm{aff}(Q)$ and therefore in $\mathrm{int}(P)$. Hence, $B(\bar{\boldsymbol{x}}, \delta) \cap \mathrm{aff}(Q) \subseteq \mathrm{aff}(O)$. Furthermore, since $B(\bar{\boldsymbol{x}}, \delta)$ is full-dimensional, it follows that $\mathrm{aff}(O) \supseteq \mathrm{aff}(Q)$.

3) Consider any two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in O$. Then, we know $\boldsymbol{x}(t, \boldsymbol{x}_1)$ and $\boldsymbol{x}(t, \boldsymbol{x}_2)$, $t \ge 0$, are entirely in $\mathrm{int}(P)$. Let $\boldsymbol{x}_3$ be any convex combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, i.e.,

$$\boldsymbol{x}_3 = \alpha \boldsymbol{x}_1 + (1 - \alpha)\boldsymbol{x}_2,\ \alpha \in [0, 1]$$

It can be easily deduced that the trajectory

$$\boldsymbol{x}(t, \boldsymbol{x}_3) = \alpha \boldsymbol{x}(t, \boldsymbol{x}_1) + (1 - \alpha)\boldsymbol{x}(t, \boldsymbol{x}_2)$$

Combining the fact that P is convex and the fact that $\boldsymbol{x}(t, \boldsymbol{x}_1)$, $\boldsymbol{x}(t, \boldsymbol{x}_2)$ are in $\mathrm{int}(P)$, we have that $\boldsymbol{x}(t, \boldsymbol{x}_3)$ is also in $\mathrm{int}(P)$ for all $t \ge 0$, which means by definition that $\boldsymbol{x}_3 \in O$. So, O is convex. □

Remark 2. Generally, $O \ne Q$. An example showing that O is a strict subset of Q is given in Section 3. Some examples showing $O = Q$ can be found in [10]. In the trivial case, $O = Q = \{\bar{\boldsymbol{x}}\}$ when V_Q is empty.

Next, some properties of attraction domains are investigated.

Theorem 2. Consider affine system (1) on P with exit sets $U_1, \cdots, U_K$. Then, for $i = 1, \cdots, K$,

1) A_i is open;
2) $\mathrm{aff}(A_i) = \mathbf{R}^n$;
3) A_i is connected.

Proof. 1) Let $\boldsymbol{x}_0 \in A_i$. By the definition of A_i, there exists $T > 0$ such that $\boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathrm{int}(P)$ for $t \in [0, T)$ and $\boldsymbol{x}(T, \boldsymbol{x}_0) \in U_i$. Since U_i is relatively open in the topological sense from its definition and solutions of the system depend continuously on the initial values[16], there exists a neighborhood of $\boldsymbol{x}_0$ such that all solution trajectories with initial states in the neighborhood leave the polytope P in a finite time through the exit set U_i. Hence, the neighborhood of $\boldsymbol{x}_0$ is also contained in A_i, which means that A_i is open.

2) Since the neighborhood of $\boldsymbol{x}_0$ is of full dimension and is contained in A_i, we obtain $\mathrm{aff}(A_i) = \mathbf{R}^n$.

3) First, consider the case that U_i lies just in one facet, say F_j (see Fig. 2 for an example). For this case, suppose by contradiction that A_i is not connected. Then, it can be decomposed into a collection of subsets $A_i^1, A_i^2, \cdots$ such that each subset A_i^j is a connected set but no pair is connected. Now, select any two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in A_i^1 and A_i^2, respectively. Then, by the definition of attraction domain, there exist T_1 and $T_2 > 0$ such that $\boldsymbol{x}(T_1, \boldsymbol{x}_1) \in U_i$, $\boldsymbol{x}(T_2, \boldsymbol{x}_2) \in U_i$, $\boldsymbol{x}(t, \boldsymbol{x}_1) \in A_i$ for all $t \in [0, T_1)$, and $\boldsymbol{x}(t, \boldsymbol{x}_2) \in A_i$ for all $t \in [0, T_2)$. Moreover, since A_i^1 is a connected set and $\boldsymbol{x}_1 \in A_i^1$, we obtain

that $\boldsymbol{x}(t,\boldsymbol{x}_1) \in A_i^1$ for all $t \in [0,T_1)$. For the same reason, we get $\boldsymbol{x}(t,\boldsymbol{x}_2) \in A_i^2$ for all $t \in [0,T_2)$. Hence, we can select two points, say $\boldsymbol{x}_1'$ and $\boldsymbol{x}_2'$, on the trajectories $\boldsymbol{x}(t,\boldsymbol{x}_1), t \in [0,T_1)$ and $\boldsymbol{x}(t,\boldsymbol{x}_2), t \in [0,T_2)$, respectively, such that the trajectories starting from $\boldsymbol{x}_1'$ and $\boldsymbol{x}_2'$ reach U_i at the same time instant T. That is, $\boldsymbol{x}(T,\boldsymbol{x}_1') \in U_i$ and $\boldsymbol{x}(T,\boldsymbol{x}_2') \in U_i$. On the other hand, since no pair from the collection of sets $A_i^1, A_i^2, \cdots$ is connected, it follows that there must be a point $\boldsymbol{x}_3' = \alpha\boldsymbol{x}_1' + (1-\alpha)\boldsymbol{x}_2'$ for some $\alpha \in (0,1)$ such that $\boldsymbol{x}_3' \notin A_i$. Hence, one obtains that $\boldsymbol{x}(T,\boldsymbol{x}_3')$ cannot be in U_i by the definition of A_i. However, by the convex argument and the fact that U_i is convex, $\boldsymbol{x}(T,\boldsymbol{x}_3') = \alpha\boldsymbol{x}(T,\boldsymbol{x}_1') + (1-\alpha)\boldsymbol{x}(T,\boldsymbol{x}_2')$ is in U_i, a contradiction. Second, consider the case that U_i lies in several facets, say $F_{i_1}, \cdots, F_{i_l}$ (see Fig. 3 for an example). For this case, it is clear that A_i can be written as

$$A_i = A(F_{i_1}) \cup \cdots \cup A(F_{i_l})$$

By the same argument as above, it can be shown that each $A(F_{i_j})$ is a connected set. Notice that facets $F_{i_1}, \cdots, F_{i_l}$ are connected through intersection points. Say for example, the facets F_{i_1} and F_{i_2} share common intersection points. Thus, $A(F_{i_1})$ and $A(F_{i_2})$ must have common points that can reach the intersection of F_{i_1} and F_{i_2}. Hence, the sets $A(F_{i_1})$ and $A(F_{i_2})$ are connected. By repeating the argument, it then follows that $A(F_{i_1}), \cdots, A(F_{i_l})$ are connected. That is, A_i is a connected set. □

Remark 3. The attraction domain is connected as we showed, but in most cases it is not convex, see Example 2.

2.2 A further partition of Q

Since the largest positive invariant set O lies entirely in set Q as we proved in Theorem 1, we are going to partition set Q and investigate the properties of the partition in order to get O.

Let

$$D^Q \triangleq D \cap Q \text{ and } A_i^Q \triangleq A_i \cap Q, \quad i = 1, \cdots, K$$

Then, we have the following result.

Lemma 3. For affine system (1) on P, the collection of sets $A_1^Q, A_2^Q, \cdots, A_K^Q, O$, and D^Q is a partition of Q.

The lemma can be deduced from Lemma 1 and the fact that $\mathrm{aff}(Q)$ is invariant.

Theorem 3. Consider affine system (1) on P and suppose Q is not empty. Then,

1) $\mathrm{aff}(A_i^Q) = \mathrm{aff}(Q)$ and A_i^Q is open in $\mathrm{aff}(Q)$;

2) if in addition $\bar{\boldsymbol{x}} \in \mathrm{int}(P)$, then O is open in $\mathrm{aff}(Q)$.

Proof. The proof is similar to the one for Theorem 2, so it is omitted. □

Now we are able to extend a result from 2-dimension[10] to a higher dimension, as presented below.

Theorem 4. For affine system(1) on P, the following holds:

1) $\boldsymbol{v} \in \partial A_i^Q \cap \mathrm{int}(P)$ implies that $\boldsymbol{v} \in D^Q$;

2) $\boldsymbol{v} \in \partial O \cap \mathrm{int}(P)$ implies that $\boldsymbol{v} \in D^Q$.

Proof. By Lemma 3, the collection of sets O, A_i^Q, D^Q is a partition of Q. Moreover, both O and A_i^Q are connected and open in $\mathrm{aff}(Q)$. Hence, all boundaries among $A_1^Q, \cdots, A_K^Q$, and O consist of trajectories belonging to set D^Q. □

The above theorem states that D^Q is the hypersurface dividing $A_1^Q, \cdots, A_K^Q$, and O. Therefore, in order to derive the explicit description of the largest positive invariant set O, it is important to obtain D^Q. In the next subsection, we will give a result for the calculation of D.

2.3 The calculation of D

Let the set of points in facet F_i with vector fields tangent to the facet be

$$C_i \triangleq \{\boldsymbol{x} \in F_i | g_i(\boldsymbol{x}) = 0\} \tag{5}$$

Clearly, set C_i is a (lower dimension) polytope, so we denote $\mathrm{vert}(C_i)$ as the set of vertices of C_i. In addition, if $W = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_l\}$ is a collection of finite numbers of points, we denote $\boldsymbol{x}(t,W)$ as the collection of trajectories starting from $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_l$, i.e.,

$$\boldsymbol{x}(t,W) \triangleq \{\boldsymbol{x}(t,\boldsymbol{v}) | \boldsymbol{v} \in W\} \tag{6}$$

Finally, the notation $\mathrm{cov}(\boldsymbol{x}(t,W))$ is used to represent the convex combination of points $\boldsymbol{x}(t,\boldsymbol{v}_1), \cdots, \boldsymbol{x}(t,\boldsymbol{v}_l)$ at time instant. It should be pointed out that it is not a convex combination of these trajectories.

The following theorem gives the computation for D.

Theorem 5. For affine system (1) on P, the set

$$D = \bigcup_{i=1}^{d} D_i \tag{7}$$

where

$$D_i = \left(\bigcup_{t \in (-\infty,0)} \mathrm{cov}\left(\boldsymbol{x}\left(t, \mathrm{vert}(C_i)\right)\right) \right) \bigcap \mathrm{int}(P) \tag{8}$$

Proof. Let $\boldsymbol{x}_0 \in D_i$ for some i. According to (8), we know $\boldsymbol{x}_0 \in \mathrm{cov}(\boldsymbol{x}(-T, \mathrm{vert}(C_i)))$ for some $T > 0$, which is equivalent to say that $\boldsymbol{x}(T,\boldsymbol{x}_0) \in C_i$ by convexity argument. Thus, it follows from the definition of D that $\boldsymbol{x}_0 \in D$.

Let $\boldsymbol{x}_0 \in D$. Then, by the definition of D, there exists $T > 0$ such that $\boldsymbol{x}_1 \triangleq \boldsymbol{x}(T,\boldsymbol{x}_0) \in C_i$ for some i. Notice that $\boldsymbol{x}_1$ can be written as a convex combination of points in $\mathrm{vert}(C_i)$ as C_i is convex. So $\boldsymbol{x}_0 \in \mathrm{cov}(\boldsymbol{x}(-T, \mathrm{vert}(C_i)))$ and thus $\boldsymbol{x}_0 \in D_i$. □

2.4 A procedure to determine O and $A(F_i)$

Finally, a procedure is given to determine the largest positive invariant set O and the attraction domains $A(F_i)$.

Procedure.

1) Compute D (Theorem 5).

2) Compute the equilibrium point $\bar{\boldsymbol{x}}$.

a) If $\bar{\boldsymbol{x}} \notin P$, then $O = \emptyset$ (Lemma 2);

b) If $\bar{\boldsymbol{x}} \in \mathrm{int}(P)$ and $V_Q = \emptyset$, then $O = \{\bar{\boldsymbol{x}}\}$ (Remark 2);

c) If $\bar{\boldsymbol{x}} \in \mathrm{int}(P)$ and $V_Q \neq \emptyset$, then compute Q and the partition of Q (Lemma 3). The set that contains $\bar{\boldsymbol{x}}$ is O.

3) Compute $U_1, U_2, \cdots, U_K$, and the partition of P (Lemma 1). The set that is connected to U_i is the attraction domain A_i.

a) If U_i lies just in one facet, say F_j, then $A(F_j) = A_i$.

b) If U_i lies in more than one facet, say $F_{i_1}, \cdots, F_{i_l}$, then for every pair of adjacent facets F_{i_j} and F_{i_k}, compute

$$H_{jk} = \bigcup_{t \in (-\infty,0)} \mathrm{cov}\left(\boldsymbol{x}\left(t, \mathrm{vert}(F_{i_j} \cap F_{i_k})\right)\right)$$

The hypersurfaces H_{jk} divide the attraction domain A_i into $A(F_{i_1}), \cdots, A(F_{i_l})$.

Remark 4. There is no common point for any pair of attraction domains of exit sets ($A_i, i = 1, \cdots, K$), but there might be common points for some pair of attraction domains of facets ($A(F_1), \cdots, A(F_d)$). As we can see, the common points reach the intersection of facets.

3 Illustrative examples

In this section, two examples are given for illustration. One shows the largest positive invariant set and the other shows the attraction domain of a facet.

Example 1 (The largest positive invariant set). Consider the affine system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0.3980 & -0.2921 & -0.1312 \\ 0.9652 & 0.0567 & 0.8763 \\ -0.4724 & 0.5916 & 0.6590 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0.0253 \\ -1.8982 \\ -0.7782 \end{bmatrix}$$

on a polytope P, where the polytope P is the cube

$$\{\boldsymbol{x} \in \mathbf{R}^3 | -2 \le x_1 \le 2, -2 \le x_2 \le 2, -2 \le x_3 \le 2\}$$

The system matrix in the above system has three eigenvalues:

$$\lambda_{1,2} = -0.0040 \pm 0.1811\mathrm{j}, \quad \lambda_3 = 0.1218$$

The equilibrium point of the system is $\bar{\boldsymbol{x}} = [1,1,1]^{\mathrm{T}}$, which is inside the polytope. By Theorem 1, the largest positive invariant set O belongs to the plane $\mathrm{aff}(Q)$

$$\{\boldsymbol{x} \in \mathbf{R}^3 | \ 0.0082x_1 + 0.0685x_2 + 0.1273x_3 = 0.2040\}$$

In Fig. 4, the quadrangle $ABCD$ is the set Q.

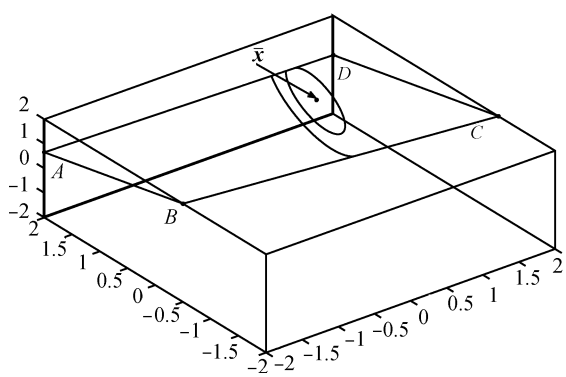


Fig. 4 The largest positive invariant set O in Example 1

The set Q is then divided by D^Q (two spiral curves in Fig. 4) into three parts, and the part that contains the equilibrium point $\bar{\boldsymbol{x}}$ is the largest positive invariant set in P.

Example 2 (Attraction domain of facet $\boldsymbol{F}_1$). Consider the affine system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -0.3727 & 0.8380 & 0.5220 \\ -0.1990 & 0.3455 & 0.2966 \\ -0.4231 & -0.2945 & -0.9401 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -0.9873 \\ -0.4431 \\ 1.6577 \end{bmatrix}$$

on the same polytope P as in Example 1.

The system matrix in this example has three eigenvalues

$$\lambda_1 = -0.4827, \ \lambda_{2,3} = -0.2423 \pm 0.2813\mathrm{j}$$

So $\mathrm{aff}(Q)$ is $\mathbf{R}^3$.

The facet F_1 is the front surface of the cube shown in Fig. 5. The set C_1 (see (5) for its definition) determined by the identifier function on the facet F_1 is the straight line on F_1 in the figure, which divides F_1 into two parts. Applying the procedure, we obtain the attraction domain of F_1 as shown in Fig. 5, where its boundaries are the shaded surface and the bottom surface of the cube.

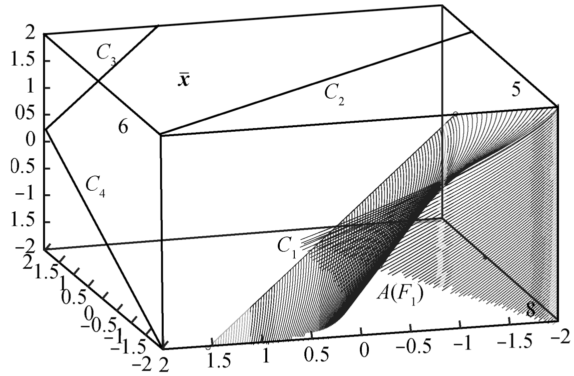


Fig. 5 The attraction domain of F_1 in Example 2

4 Conclusion

In the paper, we first observe that for an autonomous affine system, the largest positive invariant set lies in the largest stable invariant affine subspace. Then, hypersurfaces can be computed to partition the polytope into attraction domains and invariant set. These sets are determined thereafter. As a result, the reachability problem has been solved. In this work, the most numerical computation burden is in computing the dividing hypersurfaces, which requires to calculate the solution trajectories from a finite number of points and then the convex combination of these trajectory points at every time instant.

Two special cases have not been considered in the paper: the case with singular system matrix and the case with the equilibrium point on the boundary of the polytope. But it is possible to extend these results to the special cases. In addition, the attraction domain of each facet is, generally, not convex, which leads to difficulty in solving the reachability problem for piecewise affine hybrid systems. Hence, some "good properties" (such as convexity) of the attraction domain may be required in order to use the reachability result of affine system on one polytope to solve the reachability problem of piecewise affine hybrid systems. Clearly, one condition for attraction domains to have "good properties" is that the system matrix is diagonal, which has been widely used in genetic regulatory networks in bioinfomatics (known as Glass′ model[5, 17]). However, no general condition is given so far. On the other hand, approximate reachability analysis[18−19] also merits attentions.

References

1 Alur R, Courcoubetis C, Halbwachs N, Henzinger T A, Ho P H, Nicollin X. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 1995, **138**(1): 3−34

2 Sontag E D. Nonlinear regulation: the piecewise linear approach. *IEEE Transactions on Automatic Control*, 1981, **26**(2): 346−358

3 Bemporad A, Ferrari-Trecate G, Morari M. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 2000, **45**(10): 1864−1876

4 Lin Z Y, Broucke M E. Reachability and control of affine hypersurface systems on polytopes. In: Proceedings of the 46th IEEE Conference on Decision and Control. New Orleans, USA: IEEE, 2007. 733−738

5 Batt G, Belta C, Weiss R. Temporal logic analysis of gene networks under parameter uncertainty. *IEEE Transactions on Automatic Control*, 2008, **53**(z1): 215−229

6 Azuma S, Yanagisawa E, Imura J. Controllability analysis of biosystems based on piecewise-affine systems approach. *IEEE Transactions on Automatic Control*, 2008, **53**(z1): 139−152

7 Drulhe S, Ferrari-Trecate G, de Jong H. The switching threshold reconstruction problem for piecewise-affine models of genetic regulatory networks. *IEEE Transactions on Automatic Control*, 2008, **53**(z1): 153−165

8 de Jong H, Page M. Search for steady states of piecewise-linear differential equation models of genetic regulatory networks. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 2008, **5**(2): 208−222

9 Roszak B, Broucke M E. Reachability of a set of facets for linear affine systems with $n-1$ inputs. *IEEE Transactions on Automatic Control*, 2007, **52**(2): 359−364

10 Habets L C G J M, van Schuppen J H. Reachability of affine systems on polytopes in the plane. In: Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems. Leuven, Belgium: Katholieke University Leuven, 2004. 1−15

11 Blanchini F. Set invariance in control. *Automatica*, 1999, **35**(11): 1747−1767

12 Ziegler G M. *Lectures on Polytopes*. Berlin: Springer-Verlag, 1995. 5−25

13 Habets L C G J M, Collins P J, van Schuppen J H. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Transactions on Automatic Control*, 2006, **51**(6): 938−948

14 Hirsch M W, Smale S. *Differential Equations, Dynamical Systems, and Linear Algebra*. New York: Academic Press, 1974. 50−67

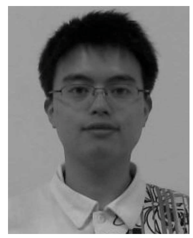
15 Arnold V I. *Ordinary Differential Equaions*. Cambridge: The MIT Press, 1973. 11−37

16 Khalil H K. *Nonlinear Systems*. New Jersey: Prentice Hall, 2001. 86−97

17 de Jong H. Modeling and simulation of genetic regulatory systems: a literature review. *Journal of Computational Biology*, 2002, **9**(1): 69−105

18 Asarin E, Dang T, Frehse G, Girard A, Le Guernic C, Maler O. Recent progress in continuous and hybrid reachability analysis. In: Proceedings of IEEE International Symposium on Computer Aided Control Systems Design. Munich, Germany: IEEE, 2006. 1582−1587

19 Asarin E, Dang T, Maler O, Bournez O. Approximate reachability analysis of piecewise-linear dynamical systems. In: Proceedings of the 3rd International Workshop on Hybrid Systems: Computation and Control. London, UK: Springer-Verlag, 2000. 20−31



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