Reliable H_∞ Filter Design for Discrete-time Systems with Sector-bounded Nonlinearities: an LMI Optimization Approach

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Abstract This paper is concerned with the reliable H_∞ filtering problem against sensor failures for a class of discrete-time systems with sector-bounded nonlinearities. The resulting design is that the filtering error system is asymptotically stable and meets the prescribed H_∞ norm constraint in normal case as well as in sensor failure case. Sufficient conditions for the existence of the filter are obtained by using appropriate Lyapunov functional and linear matrix inequality (LMI) techniques. Moreover, in order to reduce the design conservativeness and get better performance, we adopt the slack variable method to realize the decoupling between the Lyapunov matrices and the system dynamic matrices. A numerical example is provided to demonstrate the effectiveness of the proposed designs.

Key words H_∞ filtering, reliable filtering, Lyapunov function, sensor failure, linear matrix inequality (LMI)

It has been well recognized that the well-known Kalman fitlering scheme has some limitations in practical applications due to the assumption that the systems under consideration have known dynamics described by a certain well-posed model and have Gaussian noise disturbances with known statistics[1]. H_∞ filtering is introduced as an alternative to classical Kalman filtering when the statistical property of noise sources is unknown or unavailable. Therefore, in the past decade, much research effort has been paid to the H_∞ filter design which makes the worst case H_∞ norm from the process noise to the estimation error minimized. In particular, the linear matrix inequality (LMI) approach to H_∞ filtering[2] is more powerful in numerical computations and suitable for handling the optimization problems with multiple constraints.

Meanwhile, filtering for the nonlinear systems is an important research area that has attracted considerable interest. Recently, a large number of papers about nonlinear filtering problem have published, see, e.g., [1, 3−5], and the references therein. References [3−4] investigated the filtering problem for nonlinear stochastic systems. In particular, the filtering with variance-constrained was designed for uncertain stochastic systems with missing measurements in [4]. Another important type of noises/disturbances described by Brownian motions (or Wiener processes) has seldom been addressed for the filtering problems[5]. Reference [1] presented an H_∞ filtering problem for uncertain stochastic time-delay systems with sector-bounded nonlinearities, and the nonlinear systems with sector-bounded nonlinearities were also investigated in [6−8].

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Note that all the above works are based on a common assumption that the sensors can provide uninterrupted signal measurements. However, contingent failures are possible for all sensors in a system in practice. A large degree of filter performances may degrade and possible hazards may happen, see, e.g., [9]. Therefore, the design of reliable controller and filter have recently received increasing attention, mainly in linear systems[9−11], while the reliable controller and filter for nonlinear systems were investigated in [12−14]. Reference [9] considered the reliable H_∞ controller design for linear systems with sensor or actuator failure via the algebraic Riccati equation (ARE) approach. References [10−11] studied the reliable filtering problem against sensor failures for linear systems and a method of designing adaptive reliable H_∞ filter was proposed by combining the LMI approach with adaptive method. The problem of reliable H_∞ controller design for nonlinear systems was investigated in [12−13] via LMI approach. Moreover, [14] proposed a class of reliable variable structure control laws, which were shown to be able to tolerate the outage of actuators within a prespecified subset of actuators. Unfortunately, to the best of the authors′ knowledge, up to now, the reliable H_∞ filtering problem for nonlinear systems has not been fully investigated.

Motivated by the above points, a reliable H_∞ filter is designed for a class of nonlinear systems with sector-bounded nonlinearities. The paper is organized as follows. First, the sector-bounded nonlinearities and a general sensor failure model, which covers outage cases and the possibility of partial failures, are introduced. Next, the designs that guarantee the asymptotic stability of the estimation errors, and the H_∞ performance of the filtering error system from the exogenous signals to the estimation errors less than a prescribed level are described. In addition, a sufficient condition for the existence of such a reliable H_∞ filter is obtained via appropriate Lyapunov functional and LMI techniques. Then, we adopt the slack variable method[15] to realize the decoupling between the Lyapunov matrices and the filtering error system matrices, which reduces the design conservativeness. Furthermore, by this method, we can adopt different Lyapunov matrices for the normal and the sensor failure cases, respectively. Finally, a numerical example is given to illustrate the effectiveness of the developed techniques.

The notations used throughout this paper are fairly standard. $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, $\mathbf{R}^{m\times n}$ is the set of all $m\times n$ real matrices, and the notation $P = P^{\mathrm{T}} > 0\,(P = P^{\mathrm{T}} \geq 0)$ means that P is a symmetric positive definite (semidefinite) matrix. $\mathrm{diag}\{\rho_1, \rho_2, \cdots, \rho_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $\rho_1, \rho_2, \cdots, \rho_n$. In addition, we use "$*$" as an ellipsis for the terms that are introduced by symmetry.

1 Problem formulation

Consider a class of nonlinear discrete-time systems with sector nonlinearities described as

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Ff(\boldsymbol{x}(k)) + B\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + Hh(\boldsymbol{x}(k)) + D\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= L\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k) \in \mathbf{R}^n$ is the state vector, $\boldsymbol{w}(k) \in \mathbf{R}^r$ is the disturbance input, which is assumed to belong to $L_2[0, \infty)$, $\boldsymbol{z}(k) \in \mathbf{R}^q$ is the regulated output, and $\boldsymbol{y}(k) \in \mathbf{R}^p$ is the measured output. The system matrices A, F, B, C, H, D, and L are known constant matrices of appropriate dimensions. The known functions $f(\boldsymbol{x}(k))$ and $h(\boldsymbol{x}(k))$ are the vector-valued nonlinear functions.

Before presenting the main objective of this paper, the following basic assumption is assumed to be valid.

Assumption 1. The vector-valued nonlinear functions $f(\cdot)$ and $h(\cdot)$ are assumed to satisfy the following sector-bounded conditions:

$$\begin{cases} \left[\ f(\boldsymbol{x})-f(\boldsymbol{y})-M_1\boldsymbol{\eta}\ \right]^{\mathrm{T}}\left[\ f(\boldsymbol{x})-f(\boldsymbol{y})-M_2\boldsymbol{\eta}\ \right]\le 0\\ \left[\ h(\boldsymbol{x})-h(\boldsymbol{y})-N_1\boldsymbol{\eta}\ \right]^{\mathrm{T}}\left[\ h(\boldsymbol{x})-h(\boldsymbol{y})-N_2\boldsymbol{\eta}\ \right]\le 0\end{cases} \tag{2}$$

where $\boldsymbol{\eta}=\boldsymbol{x}-\boldsymbol{y}$, $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$, $M_1,M_2\in\mathbf{R}^{n\times n}$ $N_1,N_2\in\mathbf{R}^{n\times n}$ are known constant matrices. For presentation implicity and without loss of generality, we always assume that

$$f(\mathbf{0})=0,\ h(\mathbf{0})=0 \tag{3}$$

Remark 1. It is obvious that the conditions in Assumption 1 are more general than the usual sigmoid functions and the recent commonly used Lipschitz conditions. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see, e.g., [1, 3, 6−8]. In addition, M_1, N_1 and M_2, N_2 are lower and upper slope bounds, respectively.

The sensor outage cases are considered as follows

$$\begin{aligned} &y_{ij}^F(k)=(1-\rho_i^j)y_i(k),\ 0\le \underline{\rho_i^j}\le\rho_i^j\le\overline{\rho_i^j}\le 1,\\ &\qquad i=1,\cdots,p,\ j=1,\cdots,L\end{aligned} \tag{4}$$

where ρ_i^j is an unknown constant. Here, the index j denotes the j-th failure mode, L denotes the total number of the failure modes, and $y_{ij}^F(k)$ represents the measured signal from the i-th sensor that has failed in the j-th failure mode. For every fault mode, $\underline{\rho_i^j}$ and $\overline{\rho_i^j}$ represent the lower and upper bounds of ρ_i^j, respectively. Note that when $\underline{\rho_i^j}=\overline{\rho_i^j}=0$, there is no failure for the i-th sensor y_i in the j-th failure mode. When $\underline{\rho_i^j}=\overline{\rho_i^j}=1$, the i-th sensor y_i is outage in the j-th failure mode. When $0<\underline{\rho_i^j}<\overline{\rho_i^j}<1$, it corresponds to the case of partial failure of y_i. Denote

$$\boldsymbol{y}_j^F(k)=\left[\ y_{1j}^F(k),\, y_{2j}^F(k),\cdots,\, y_{pj}^F(k)\ \right]^{\mathrm{T}}=(I-\rho^j)\boldsymbol{y}(k) \tag{5}$$

where $\rho^j=\mathrm{diag}\{\rho_1^j,\,\rho_2^j,\cdots,\,\rho_p^j\}$ and $j=1,\cdots,L$. The scaling factors ρ^j satisfy

$$\begin{aligned}\boldsymbol{N}_{\rho^j}=\{\rho^j|\rho^j=\mathrm{diag}\{\ \rho_1^j,\rho_2^j,\cdots,\rho_p^j\ \}\in\mathbf{R}^p,\\ 0\le\underline{\rho_i^j}\le\rho_i^j\le\overline{\rho_i^j}\le 1,\ i=1,2,\cdots,p\}\end{aligned} \tag{6}$$

For convenience, in the following sections, for all possible failure modes, we use a uniform sensor failure model

$$\boldsymbol{y}^F(k)=(I-\rho)\boldsymbol{y}(k),\ \rho\in\{\rho^1,\rho^2,\cdots,\rho^L\} \tag{7}$$

where ρ can be described by $\rho=\mathrm{diag}\{\rho_1,\rho_2,\cdots,\rho_p\}$.

Then, system (1) with sensor failure (7) is described by

$$\begin{aligned}\boldsymbol{x}(k+1)&=A\boldsymbol{x}(k)+Ff(\boldsymbol{x}(k))+B\boldsymbol{w}(k)\\ \boldsymbol{y}^F(k)&=(I-\rho)(C\boldsymbol{x}(k)+Hh(\boldsymbol{x}(k))+D\boldsymbol{w}(k))\\ \boldsymbol{z}(k)&=L\boldsymbol{x}(k)\end{aligned} \tag{8}$$

The reliable filter is of the form

$$\begin{aligned}\bar{\boldsymbol{x}}(k+1)&=A_f\bar{\boldsymbol{x}}(k)+B_f\boldsymbol{y}^F(k)+F_ff(\bar{\boldsymbol{x}}(k))\\ \bar{\boldsymbol{z}}(k)&=C_f\bar{\boldsymbol{x}}(k)\end{aligned} \tag{9}$$

where $\bar{\boldsymbol{x}}(k)\in\mathbf{R}^n$ is the filter state, $\bar{\boldsymbol{z}}(k)\in\mathbf{R}^q$ is the estimation of $\boldsymbol{z}(k)$, A_f, B_f, C_f, and F_f are the filter parameter matrices to be designed. Here, we assume that the filter is of the same order as the system model.

Applying filter (9) to system (8), we obtain the filtering error system:

$$\begin{aligned}\boldsymbol{\xi}(k+1)&=\bar{A}\boldsymbol{\xi}(k)+\bar{A}_{f1}f(K_1\boldsymbol{\xi}(k))+\bar{A}_{f2}f(K_2\boldsymbol{\xi}(k))+\\ &\quad\ \bar{A}_hh(K_1\boldsymbol{\xi}(k))+\bar{B}\boldsymbol{w}(k)\\ \boldsymbol{e}(k)&=\bar{C}\boldsymbol{\xi}(k)\end{aligned} \tag{10}$$

where $\boldsymbol{\xi}(k)=\begin{bmatrix}\boldsymbol{x}(k)\\ \bar{\boldsymbol{x}}(k)\end{bmatrix}$, $K_1=\begin{bmatrix} I & 0\end{bmatrix}$, $K_2=\begin{bmatrix}0 & I\end{bmatrix}$, $\boldsymbol{e}(k)=\boldsymbol{z}(k)-\bar{\boldsymbol{z}}(k)$ is the estimation error, and $\bar{A}=\begin{bmatrix}A & 0\\ B_f(I-\rho)C & A_f\end{bmatrix}$, $\bar{A}_{f1}=\begin{bmatrix}F\\ 0\end{bmatrix}$, $\bar{A}_{f2}=\begin{bmatrix}0\\ F_f\end{bmatrix}$, $\bar{A}_h=\begin{bmatrix}0\\ B_f(I-\rho)H\end{bmatrix}$, $\bar{B}=\begin{bmatrix}B\\ B_f(I-\rho)D\end{bmatrix}$, and $\bar{C}=\begin{bmatrix}L & -C_f\end{bmatrix}$.

For convenience, in the following sections, we denote the filtering error system without sensor failures, i.e., $\rho=0$, as follows:

$$\begin{aligned}\boldsymbol{\xi}(k+1)&=\tilde{A}\boldsymbol{\xi}(k)+\tilde{A}_{f1}f(K_1\boldsymbol{\xi}(k))+\tilde{A}_{f2}f(K_2\boldsymbol{\xi}(k))+\\ &\quad\ \tilde{A}_hh(K_1\boldsymbol{\xi}(k))+\tilde{B}\boldsymbol{w}(k)\\ \boldsymbol{e}(k)&=\tilde{C}\boldsymbol{\xi}(k)\end{aligned} \tag{11}$$

where $\tilde{A}=\begin{bmatrix}A & 0\\ B_fC & A_f\end{bmatrix}$, $\tilde{A}_{f1}=\begin{bmatrix}F\\ 0\end{bmatrix}$, $\tilde{A}_{f2}=\begin{bmatrix}0\\ F_f\end{bmatrix}$, $\tilde{A}_h=\begin{bmatrix}0\\ B_fH\end{bmatrix}$, $\tilde{B}=\begin{bmatrix}B\\ B_fD\end{bmatrix}$, and $\tilde{C}=\begin{bmatrix}L & -C_f\end{bmatrix}$.

Our objective is to develop a filter of the form (9) such that the filtering error systems (10) and (11) satisfy the following requirements:

1) While there is no exogenous disturbance, i.e., $\boldsymbol{w}(k)=\mathbf{0}$, the filtering error systems (10) and (11) are asymptotically stable.

2) For given constants $\gamma_f>\gamma_n>0$, find filter (9) such that

a) The filtering error system (10) in the normal case, i.e., (11), is with an H_∞ performance index no larger than γ_n;

b) The filtering error system (10) in the sensor failure case, i.e., $\rho\in\{\rho^1,\rho^2,\cdots,\rho^L\}$ with $\rho^j\in\boldsymbol{N}_{\rho^j}, j=1,\cdots,L$, is with an H_∞ performance index no larger than γ_f.

The filter of form (9) satisfying the above objective is said to be a reliable H_∞ filter for system (1) with (4) and guarantees that the filtering error systems (10) and (11) are asymptotically stable at the same time.

Now, we first provide some important lemmas, which will be useful in the derivation of our main results.

Lemma 1 (S-procedure)[16]. Let $T_0(\boldsymbol{x}),T_1(\boldsymbol{x}),\cdots,T_p(\boldsymbol{x})$ be quadratic functions of $\boldsymbol{x}\in\mathbf{R}^n$

$$T_i(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}\Psi_i\boldsymbol{x},\ i=0,1,\cdots,p \tag{12}$$

with $\Psi_i=\Psi_i^{\mathrm{T}}$. Then, the implication

$$T_1(\boldsymbol{x})\le 0,\cdots,T_p(\boldsymbol{x})\le 0\Rightarrow T_0(\boldsymbol{x})<0 \tag{13}$$

holds if there exist nonnegative scalars $\tau_1,\cdots,\tau_p$ such that

$$\Psi_0-\sum_{i=1}^p\tau_i\Psi_i<0 \tag{14}$$

Lemma 2. If $f(\cdot)$ is a vector-valued nonlinear function and $M_1,M_2\in\mathbf{R}^{n\times n}$ are known constant matrices, then we have

$$\left[\ f(\boldsymbol{x}) - M_1\boldsymbol{x}\ \right]^{\mathrm{T}} \left[\ f(\boldsymbol{x}) - M_2\boldsymbol{x}\ \right] \leq 0,\ \forall \boldsymbol{x} \in \mathbf{R}^n \tag{15}$$

which implies

$$\left[\ \boldsymbol{x}^{\mathrm{T}} \quad f^{\mathrm{T}}(\boldsymbol{x})\ \right] \left[\begin{array}{cc} \hat{M}_1 & \hat{M}_2 \\ \hat{M}_2^{\mathrm{T}} & I \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ f(\boldsymbol{x}) \end{array}\right] \leq 0 \tag{16}$$

with $\hat{M}_1 = (M_1^{\mathrm{T}} M_2 + M_2^{\mathrm{T}} M_1)/2$ and $\hat{M}_2 = -(M_1^{\mathrm{T}} + M_2^{\mathrm{T}})/2$.

Proof. Due to the limit of the space, the proof is omitted. □

2 Main result

To facilitate the presentation, we denote

$$\begin{aligned}
\hat{M}_1 &= (M_1^{\mathrm{T}} M_2 + M_2^{\mathrm{T}} M_1)/2,\ \hat{M}_2 = -(M_1^{\mathrm{T}} + M_2^{\mathrm{T}})/2 \\
\hat{N}_1 &= (N_1^{\mathrm{T}} N_2 + N_2^{\mathrm{T}} N_1)/2,\ \hat{N}_2 = -(N_1^{\mathrm{T}} + N_2^{\mathrm{T}})/2 \\
\pi &= \tau_1 K_1^{\mathrm{T}} \hat{M}_1 K_1 + \tau_2 K_2^{\mathrm{T}} \hat{M}_1 K_2 + \tau_3 K_1^{\mathrm{T}} \hat{N}_1 K_1 \\
\Omega_1 &= -P_n - \pi,\ \Omega_{11} = -P_f - \pi \\
\Omega_2^{\mathrm{T}} &= \left[\ -\tau_1 K_1^{\mathrm{T}} \hat{M}_2 \quad -\tau_2 K_2^{\mathrm{T}} \hat{M}_2 \quad -\tau_3 K_1^{\mathrm{T}} \hat{N}_2\ \right] \\
\Omega_3 &= \mathrm{diag}\{-\tau_1 I, -\tau_2 I, -\tau_3 I\} \\
\Omega_4 &= \left[\ P_n \tilde{A}_{f1} \quad P_n \tilde{A}_{f2} \quad P_n \tilde{A}_h\ \right] \\
\Omega_{41} &= \left[\ P_f \bar{A}_{f1} \quad P_f \bar{A}_{f2} \quad P_f \bar{A}_h\ \right] \\
\Omega_5 &= \left[\ G_n^{\mathrm{T}} \tilde{A}_{f1} \quad G_n^{\mathrm{T}} \tilde{A}_{f2} \quad G_n^{\mathrm{T}} \tilde{A}_h\ \right] \\
\Omega_{51} &= \left[\ G_f^{\mathrm{T}} \bar{A}_{f1} \quad G_f^{\mathrm{T}} \bar{A}_{f2} \quad G_f^{\mathrm{T}} \bar{A}_h\ \right]
\end{aligned}$$

Before continuing with the solution to the synthesis problem, we present the following theorem which guarantees that the filtering error systems (10) and (11) are asymptotically stable and has H_∞ performance criteria at the same time.

Theorem 1. Given scalars $\gamma_n > 0$, $\gamma_f > 0$, and the known constant matrices $\hat{M}_1, \hat{M}_2, \hat{N}_1$, and $\hat{N}_2$, if there exist matrices $P_n = P_n^{\mathrm{T}} > 0$, $P_f = P_f^{\mathrm{T}} > 0$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ $\bar{A}$, $\bar{B}$, $\bar{C}$ and nonnegative scalars $\tau_i\ (i = 1, 2, 3)$ such that

$$\left[\begin{array}{ccccc} \Omega_1 & * & * & * & * \\ \Omega_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_n^2 I & * & * \\ P_n \tilde{A} & \Omega_4 & P_n \tilde{B} & -P_n & * \\ \tilde{C} & 0 & 0 & 0 & -I \end{array}\right] < 0 \tag{17}$$

holds for $\rho = 0$ and

$$\left[\begin{array}{ccccc} \Omega_{11} & * & * & * & * \\ \Omega_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_f^2 I & * & * \\ P_f \bar{A} & \Omega_{41} & P_f \bar{B} & -P_f & * \\ \bar{C} & 0 & 0 & 0 & -I \end{array}\right] < 0 \tag{18}$$

holds for $\rho \in \{\rho^1, \rho^2, \cdots, \rho^L\}$ with $\rho^j \in \boldsymbol{N}_{\rho^j}, j = 1, \cdots, L$, then the filtering error systems (10) and (11) are asymptotically stable and satisfy the H_∞ performance constraint simultaneously.

Proof. Let us choose a Lyapunov functional candidate as $V(k) = \boldsymbol{\xi}^{\mathrm{T}}(k) P_f \boldsymbol{\xi}(k)$, where P_f is positive and symmetry. Then, after some manipulation including applying Lemmas 1 and 2, we can obtain (17) and (18). Due to the limit of the space, the detail is omitted. □

It is obvious that (17) and (18) are not LMIs, we need to look for a suitable method to change the above inequalities to LMIs. The common method, which can be found in many papers concerning the reliable controlling and filtering problems, see, e.g., [10−11], is to set

$$P_n = P_f \tag{19}$$

before converting all the inequalities to LMIs. However, the constraint (19) introduces significant conservativeness into the design. Another obvious disadvantage is that one unique Lyapunov matrix is adopted for both the normal and the sensor failure cases, which contradicts to our design objective. Moreover, the construct product between the Lyapunov matrices and the filtering error system matrices also introduces conservativeness. For the sake of overcoming the above disadvantages, we design the reliable filter with the following method. First, we give out an equivalent condition to Theorem 1.

Theorem 2. Given scalars $\gamma_n > 0$, $\gamma_f > 0$, and the known constant matrices $\hat{M}_1, \hat{M}_2, \hat{N}_1, \hat{N}_2$, if there exist some matrices $P_n = P_n^{\mathrm{T}} > 0$, $P_f = P_f^{\mathrm{T}} > 0$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ $\bar{A}$, $\bar{B}$, $\bar{C}$, G_n, G_f and nonnegative scalars $\tau_i\ (i = 1, 2, 3)$ such that

$$\left[\begin{array}{ccccc} \Omega_1 & * & * & * & * \\ \Omega_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_n^2 I & * & * \\ G_n^{\mathrm{T}} \tilde{A} & \Omega_5 & G_n^{\mathrm{T}} \tilde{B} & P_n - G_n^{\mathrm{T}} - G_n & * \\ \tilde{C} & 0 & 0 & 0 & -I \end{array}\right] < 0 \tag{20}$$

holds for $\rho = 0$ and

$$\left[\begin{array}{ccccc} \Omega_{11} & * & * & * & * \\ \Omega_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_f^2 I & * & * \\ G_f^{\mathrm{T}} \bar{A} & \Omega_{51} & G_f^{\mathrm{T}} \bar{B} & P_f - G_f^{\mathrm{T}} - G_f & * \\ \bar{C} & 0 & 0 & 0 & -I \end{array}\right] < 0 \tag{21}$$

holds for $\rho \in \{\rho^1, \rho^2, \cdots, \rho^L\}$ with $\rho^j \in \boldsymbol{N}_{\rho^j}, j = 1, \cdots, L$, then the filtering error systems (10) and (11) are asymptotically stable and satisfy the H_∞ performance constraint simultaneously.

Proof. Due to the limit of the space, the proof is omitted. □

Remark 2. Theorem 2 provides a new criterion of H_∞ performance which exhibits a kind of decoupling between the Lyapunov matrix P and the filtering error system matrices. This feature is enabled by the introduction of auxiliary slack variables. Furthermore, it is the fact that the extra variables G_n and G_f are full, i.e., they do not present any structural constraint such as symmetry, which is supposed to lead to potentially less conservative results.

Then, instead of imposing constraint $P_n = P_f = P$, we can impose the following artificial constraint, which can be found in many papers concerning the multiobjective problems, with positive scalar parameter λ to be searched before converting all the inequalities to LMIs.

$$G_n = \lambda G_f = \lambda G \tag{22}$$

Consequently, we can adopt different Lyapunov matrices for the normal and the sensor failure cases, respectively. Then, the next theorem gives a solution, expressed in terms of LMI, to the reliable H_∞ filtering problem.

Denote

$$\begin{aligned} W_0 &= -S^{\mathrm{T}} - S,\ W_1 = -S^{\mathrm{T}} - S + N^{\mathrm{T}} + N \\ W_2 &= \tau_1 \hat{M}_1 + \tau_3 \hat{N}_1,\ R_1 = (S-N)^{\mathrm{T}} A + \hat{B}_f C \\ R_2 &= (S-N)^{\mathrm{T}} B + \hat{B}_f D \\ R_3 &= (S-N)^{\mathrm{T}} A + \hat{B}_f (I-\rho) C \\ R_4 &= (S-N)^{\mathrm{T}} B + \hat{B}_f (I-\rho) D \\ R_5 &= (S-N)^{\mathrm{T}} A + N^{\mathrm{T}} B_f (I-\rho) C \\ R_6 &= (S-N)^{\mathrm{T}} B + N^{\mathrm{T}} B_f (I-\rho) D \end{aligned}$$

Theorem 3. Given scalars $\gamma_n > 0$, $\gamma_f > 0$ and the known constant matrices $\hat{M}_1, \hat{M}_2, \hat{N}_1$, and $\hat{N}_2$, a reliable H_∞ filter exists if, for scalar $\lambda > 0$, there exist matrices $0 < P_{1n} = P_{1n}^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $P_{2n} \in \mathbf{R}^{n\times n}$, $0 < P_{3n} = P_{3n}^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $0 < P_{1f} = P_{1f}^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $P_{2f} \in \mathbf{R}^{n\times n}$, $0 < P_{3f} = P_{3f}^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $N \in \mathbf{R}^{n\times n}$, $S \in \mathbf{R}^{n\times n}$, $\hat{A}_f \in \mathbf{R}^{n\times n}$, $\hat{B}_f \in \mathbf{R}^{n\times p}$, $\hat{C}_f \in \mathbf{R}^{q\times n}$ and nonnegative scalars $\tau_i\,(i = 1, 2, 3)$ such that

$$\begin{bmatrix} \Sigma_1 & * & * & * & * \\ \Sigma_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_f^2 I & * & * \\ \Sigma_3 & \Sigma_5 & \Sigma_6 & \Sigma_7 & * \\ \Sigma_4 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{23}$$

holds for $\rho = 0$ and

$$\begin{bmatrix} \Sigma_{11} & * & * & * & * \\ \Sigma_2 & \Omega_3 & * & * & * \\ 0 & 0 & -\gamma_f^2 I & * & * \\ \Sigma_{13} & \Sigma_{15} & \Sigma_{16} & \Sigma_{17} & * \\ \Sigma_4 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{24}$$

holds for $\rho \in \{\rho^1, \rho^2, \cdots, \rho^L\}$ with $\rho^j \in \boldsymbol{N}_{\rho^j}, j = 1, \cdots, L$, where

$$\Sigma_1 = \begin{bmatrix} -\bar{P}_{1n} - W_2 - \tau_2 \hat{M}_1 & * \\ -\bar{P}_{2n} - W_2 & -\bar{P}_{3n} - W_2 \end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix} -\tau_1 \hat{M}_2^{\mathrm{T}} & -\tau_1 \hat{M}_2^{\mathrm{T}} \\ -\tau_2 \hat{M}_2^{\mathrm{T}} & 0 \\ -\tau_3 \hat{N}_2^{\mathrm{T}} & -\tau_3 \hat{N}_2^{\mathrm{T}} \end{bmatrix},\ \Sigma_4 = \begin{bmatrix} L - \hat{C}_f & L \end{bmatrix}$$

$$\Sigma_3 = \begin{bmatrix} \lambda S^{\mathrm{T}} A & \lambda S^{\mathrm{T}} A \\ \lambda (R_1 + \hat{A}_f) & \lambda R_1 \end{bmatrix},\ \Sigma_6 = \begin{bmatrix} \lambda S^{\mathrm{T}} B \\ \lambda R_2 \end{bmatrix}$$

$$\Sigma_5 = \begin{bmatrix} \lambda S^{\mathrm{T}} F & 0 & 0 \\ \lambda (S-N)^{\mathrm{T}} F & \lambda \hat{F}_f & \lambda \hat{B}_f H \end{bmatrix}$$

$$\Sigma_7 = \begin{bmatrix} \bar{P}_{1n} + \lambda W_0 & * \\ \bar{P}_{2n} + \lambda W_0 & \bar{P}_{3n} + \lambda W_1 \end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix} -\bar{P}_{1f} - W_2 - \tau_2 \hat{M}_1 & * \\ -\bar{P}_{2f} - W_2 & -\bar{P}_{3f} - W_2 \end{bmatrix}$$

$$\Sigma_{13} = \begin{bmatrix} S^{\mathrm{T}} A & S^{\mathrm{T}} A \\ R_3 + \hat{A}_f & R_3 \end{bmatrix},\ \Sigma_{16} = \begin{bmatrix} S^{\mathrm{T}} B \\ R_4 \end{bmatrix}$$

$$\Sigma_{15} = \begin{bmatrix} S^{\mathrm{T}} F & 0 & 0 \\ (S-N)^{\mathrm{T}} F & \hat{F}_f & \hat{B}_f (I-\rho) H \end{bmatrix}$$

$$\Sigma_{17} = \begin{bmatrix} \bar{P}_{1f} + W_0 & * \\ \bar{P}_{2f} + W_0 & \bar{P}_{3f} + W_1 \end{bmatrix}$$

Moreover, if there exist solutions of these inequalities, then the reliable filter can be given by

$$A_f = N^{-\mathrm{T}} \hat{A}_f,\ B_f = N^{-\mathrm{T}} \hat{B}_f,\ F_f = N^{-\mathrm{T}} \hat{F}_f,\ C_f = \hat{C}_f \tag{25}$$

Proof. Due to the limit of the space, the proof is omitted. □

Remark 3. It is noted that for given λ, the conditions in Theorem 3 are LMI conditions with respect to the scalars γ_n and γ_f. Therefore, γ_n and γ_f can be minimized by using convex optimization algorithms. Then, the problem of reliable H_∞ filter can be converted to the following optimization problem:

$$\begin{aligned} &\min_{\substack{\tau_i,\, \bar{P}_{in},\, \bar{P}_{if}\,(i=1,2,3),\\ S,\, N,\, \hat{A}_f,\, \hat{B}_f,\, \hat{C}_f,\, \theta_n,\, \theta_f}} \quad \alpha\theta_n^2 + \beta\theta_f^2 \\ &\text{s.t.} \quad (23), (24), \theta_n = \gamma_n, \theta_f = \gamma_f \end{aligned} \tag{26}$$

The minimal disturbance attenuation $\gamma_n^* = \theta_n^*$, $\gamma_f^* = \theta_f^*$, and α and β are weighting coefficients. Usually, we can choose $\alpha > \beta$ in (26), since systems operate under the normal condition most of the time. In addition, the parameters of the designed filter can be obtained by (25).

Remark 4. The sufficient conditions expressed in LMIs are presented in Theorem 3, where sensor failures exist. When the sensor failures are not considered, i.e., $\rho = 0$, the problem reduces to a standard H_∞ filter design, where (23) should be satisfied.

3 Numerical simulation

To illustrate the validity and effectiveness of the reliable H_∞ filter, a numerical simulation is carried out to provide a comparison of the approaches proposed in this paper.

Consider the nonlinear discrete-time system (1) with the following parameters:

$$A = \begin{bmatrix} 0.6 & -0.3 & -0.3 \\ 0 & 0.2 & -0.7 \\ -0.1 & 0.3 & 0.7 \end{bmatrix},\ F = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.2 & 0.4 \\ 0.4 & -0.5 \\ -0.6 & 0.3 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & -8 \\ 1 & -3 & 2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.5 & -0.1 & 0 \\ 0.2 & 0 & 0.1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 1 \\ 0 & -0.8 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 0.2 & 0.1 & -0.3 \end{bmatrix}$$

$$M_1 = N_1 = \begin{bmatrix} 0.01 & 0.01 & -0.01 \\ 0.01 & 0.02 & 0.04 \\ -0.02 & 0.01 & 0.02 \end{bmatrix},\ M_2 = N_2 =$$

$$\begin{bmatrix} -0.01 & 0.01 & -0.01 \\ -0.03 & -0.02 & -0.02 \\ 0.02 & -0.03 & -0.04 \end{bmatrix},\ f(\boldsymbol{x}(k)) = h(\boldsymbol{x}(k)) =$$

$$\begin{bmatrix} 0.02x_1(k)\sin^2(x_1(k)) - 0.01(x_1(k) - x_2(k) + x_3(k)) \\ -0.01(x_1(k) - x_3(k)) \\ -0.01(x_2(k) + x_3(k)) \end{bmatrix}$$

where $f(\boldsymbol{x}(k))$ and $h(\boldsymbol{x}(k))$ satisfy (2).

Here, the following four possible sensor failure modes are considered:

Normal mode 1: Both of the two sensors are normal, that is, $\rho^1 = \mathrm{diag}\{\rho_1^1, \rho_2^1\} = \mathrm{diag}\{0, 0\}$.

Sensor failure mode 2: The first sensor is normal and the second is outage, that is, $\rho^2 = \mathrm{diag}\{\rho_1^2, \rho_2^2\} = \mathrm{diag}\{0, 1\}$.

Sensor failure mode 3: The first sensor is outage and the second is normal, that is, $\rho^3 = \mathrm{diag}\{\rho_1^3, \rho_2^3\} = \mathrm{diag}\{1, 0\}$.

Sensor failure mode 4: The two sensors are partial failure, that is, $\rho^4 = \text{diag}\{\rho_1^4, \rho_2^4\}$, where $0 \le \rho_1^4 \le 0.4$ and $0 \le \rho_2^4 \le 0.5$.

Then, by applying Remark 3 and the *fminsearch* with the initial value of $\lambda = 1$, the optimal reliable H_∞ performances are 0.3175 (normal) and 0.4867 (failure) when $\alpha = 10$, $\beta = 1$, and $\lambda = 1.2478$. Due to the limit of the space, the filter gain matrices are omitted.

Table 1 presents a comparison between Theorem 3 and Remark 4. Table 1 shows that the standard filter has the best performance in the normal case. However, the optimal H_∞ performance of the standard filter is seriously deteriorative in the sensor failure case, while the reliable H_∞ filter performs well.

Table 1 Comparison between Theorem 3 and Remark 4

Design	Methods	Normal	Sensor failures
Theorem 3	Reliable filter	$\gamma_n^* = 0.3175$	$\gamma_f^* = 0.4867$
Remark 4	Standard filter	$\gamma_n^* = 0.2440$	$\gamma_f^* = 1.0312$

In order to show the effectiveness of our method more clearly, a simulation is also performed. In the following simulation, let the system initial state be $\boldsymbol{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and the filter initial state be $\bar{\boldsymbol{x}}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. In addition, we assume the disturbance input $\boldsymbol{w}(k) = \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}$ as follows:

$$w_1(k) = w_2(k) = \begin{cases} 0.2\,(0.1 + \cos(1.7k)), & 5 \le k \le 10 \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

Fig. 1 shows the estimation error $\boldsymbol{e}(k)$ responses of the filters designed by the proposed methods for the reliable filter, the standard filter, and the standard filter with (4), respectively. From Fig. 1, we can easily find that the standard filter performs well in the normal case, while it is seriously deteriorative in the sensor failure case. This phenomenon shows the effectiveness of our design methods.

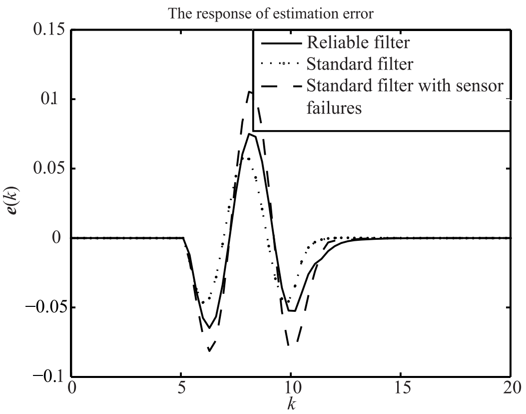

Fig. 1 Comparison of different design methods

4 Conclusion

The problem of reliable H_∞ filtering is discussed for a class of discrete time nonlinear systems with sector-bounded nonlinearities and sensor failures. Sufficient conditions for the existence of the filter are obtained to ensure the asymptotic stability with sensor failures and H_∞ performance. A new LMI formulation is also proposed to reduce the conservativeness in the design. Finally, a numerical example is given to illustrate the effectiveness of the main results.

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