Global Optimality for Generalized Federated Filter

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Abstract Based on the matrix theory and the information sharing principle, the analytic relation among centralized Kalman filtering, decentralized filtering, and federated filtering is derived. It is proved that the global filtering of federated filters is optimal only when the dimensions of the master filter and the local filters are totally equal. If the dimensions of the master filter and the local filters are different, then only suboptimal solution can be obtained. The structure of a generalized federated filter is proposed. In terms of the information sharing principle, the information matrix of the one-step prediction state error and the one-step prediction state are reset to obtain the suboptimal solution of the global filtering. Furthermore, the suboptimal solution of the global filtering is used as observation feedback to correct the one-step prediction state and yield the optimal solution of the global filtering. The optimal feedback gain matrix is mathematically derived, so the filtering result is theoretically proved to be equivalent to the centralized Kalman filtering. The result of the simulation experiments with a dual-SINS/GPS integrated navigation system demonstrates the validity of the algorithm.

Key words Kalman filter, generalized federated filter, optimality, information sharing, information fusion, feedback correction

With the development of information technology, decentralized filtering[1−3] and federated filtering[4−10] have been widely applied to multisensor information fusion. They have fast computing speed due to their suitable algorithm structure for parallel computing. Fault detection, fault isolation, and system reconfiguration can be achieved with the inherent analytic relation between the input variables and the state variables of the subsystem, so that the reliability of the whole system may be improved. Therefore, they are remarkably superior to the centralized Kalman filtering in computing efficiency and fault tolerance.

In proving the optimality of federated filters, Carlson constructed an augmented system, where the variance upper bound technique was used to eliminate the correlation among several local filters. Then, the global optimal estimation was achieved with an uncorrelated fusion algorithm and the information sharing principle. However, in the high-dimensional situation, the computation amount of the above approach will increase sharply. Another approach is the minimum structure. A master filter only contains the common state, and the information sharing is limited to the common state between the master filter and the local filters. Considering the influence of the common state on the bias state, the common state after the information fusion is used to reset the common state of the local filters and also used as the observation feedback to correct the bias state of the local filters. Reference [7] analyzed the above approach theoretically, and pointed out that the minimum structure approach is based on the hypothetical condition that the state estimated error between the local filters and the master filter is uncorrelated. Usually this hypothetical condition is difficult to satisfy and only the suboptimal solution can be obtained.

In this paper, based on the decentralized filtering algorithm and the information sharing principle, it is proved that the global filtering of federated filters is optimal only when the dimensions of the master filter and the local filters are totally equal. Meanwhile, the generalized federated filter is structured, where the dimensions of the master filter and the local filters are different, and the feedback correction by which the generalized federated filter may realize the global optimality is presented. The validity of the approach is theoretically proved and its effectiveness is demonstrated by its application to an integrated navigation system.

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1 Filter structure

1.1 Centralized Kalman filtering

The system model is

$$\boldsymbol{X}_k = \Phi_{k,k-1}\boldsymbol{X}_{k-1} + \boldsymbol{W}_{k-1} \tag{1}$$

where $\boldsymbol{W}_k$ is a zero-mean white Gaussian noise, whose covariance matrix is Q_k; $\boldsymbol{X}_k$ is the system state; the initial value of $\boldsymbol{X}_k$ is a zero-mean Gaussian random vector which is independent of noise and its covariance matrix is P_0.

The dimension of $\boldsymbol{X}_k$ is n, and $\boldsymbol{X}_k$ is observed by N sensors. Its observation model is

$$\boldsymbol{Z}_k = H_k\boldsymbol{X}_k + \boldsymbol{V}_k \tag{2}$$

where $\boldsymbol{Z}_k = \left[\boldsymbol{Z}_{1k}^{\mathrm{T}}, \boldsymbol{Z}_{2k}^{\mathrm{T}}, \cdots, \boldsymbol{Z}_{Nk}^{\mathrm{T}}\right]^{\mathrm{T}}$, $H_k = [H_{1k}^{\mathrm{T}}, H_{2k}^{\mathrm{T}}, \cdots, H_{Nk}^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{V}_k = \left[\boldsymbol{V}_{1k}^{\mathrm{T}}, \boldsymbol{V}_{2k}^{\mathrm{T}}, \cdots, \boldsymbol{V}_{Nk}^{\mathrm{T}}\right]^{\mathrm{T}}$ is a zero-mean white Gaussian noise, whose covariance matrix is R_k, and $R_k = \mathrm{diag}\{R_{1k}, R_{2k}, \cdots, R_{Nk}\}$.

Thus

$$\boldsymbol{Z}_{ik} = H_{ik}\boldsymbol{X}_k + \boldsymbol{V}_{ik}, \quad i = 1, 2, \cdots, N \tag{3}$$

Define that $\boldsymbol{V}_k$ and $\boldsymbol{W}_k$ are uncorrelated. The centralized Kalman filtering of the above system can be expressed as

$$\hat{\boldsymbol{X}}_{k|k-1} = \Phi_{k,k-1}\hat{\boldsymbol{X}}_{k-1} \tag{4a}$$

$$K_k = P_k H_k^{\mathrm{T}} R_k^{-1} \tag{4b}$$

$$P_{k|k-1}^{-1} = \left[\Phi_{k,k-1}P_{k-1}\Phi_{k,k-1}^{\mathrm{T}} + Q_{k-1}\right]^{-1} \tag{4c}$$

$$\hat{\boldsymbol{X}}_k = P_k P_{k|k-1}^{-1}\hat{\boldsymbol{X}}_{k|k-1} + P_k H_k^{\mathrm{T}} R_k^{-1}\boldsymbol{Z}_k \tag{5}$$

$$P_k^{-1} = P_{k|k-1}^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k \tag{6}$$

1.2 Decentralized Kalman filtering

Suppose the state equation and observation equation of the subsystem are

$$\boldsymbol{X}_{ik} = \Phi_{i;k,k-1}\boldsymbol{X}_{i,k-1} + \boldsymbol{W}_{i,k-1}, \quad i = 1, 2, \cdots, N \tag{7}$$

$$\boldsymbol{Z}_{ik} = A_{ik}\boldsymbol{X}_{ik} + \boldsymbol{V}_{ik}, \quad i = 1, 2, \cdots, N \tag{8}$$

where $\boldsymbol{W}_{ik}$ and $\boldsymbol{V}_{ik}$ are both zero-mean white Gaussian noises whose covariance matrices are Q_{ik} and R_{ik}, respectively.

Similarly, the local Kalman filtering can be expressed as

$$\hat{\boldsymbol{X}}_{i,k|k-1} = \Phi_{i;k,k-1}\hat{\boldsymbol{X}}_{i,k-1} \tag{9a}$$

$$K_{ik} = P_{ik}A_{ik}^{\mathrm{T}}R_{ik}^{-1} \tag{9b}$$

$$P_{i,k|k-1}^{-1} = \left[\Phi_{i;k,k-1}P_{i,k-1}\Phi_{i;k,k-1}^{\mathrm{T}} + Q_{i,k-1}\right]^{-1} \tag{9c}$$

$$\hat{\boldsymbol{X}}_{ik} = P_{ik}P_{i,k|k-1}^{-1}\hat{\boldsymbol{X}}_{i,k|k-1} + P_{ik}A_{ik}^{\mathrm{T}}R_{ik}^{-1}\boldsymbol{Z}_{ik} \tag{10}$$

$$P_{ik}^{-1} = P_{i,k|k-1}^{-1} + A_{ik}^{\mathrm{T}}R_{ik}^{-1}A_{ik} \tag{11}$$

Suppose there exists a matrix M_i between the subsystem state $\boldsymbol{X}_{ik}$ and the general system state $\boldsymbol{X}_k$, and that $\boldsymbol{X}_{ik} = M_i\boldsymbol{X}_k$. With the above substitution into (8) and the comparison to (3), we obtain $H_{ik} = A_{ik}M_i$. Substituting (10) and (11) into (5) and (6), respectively, we obtain

$$\begin{cases} \hat{\boldsymbol{X}}_k = P_kP_{k|k-1}^{-1}\hat{\boldsymbol{X}}_{k|k-1} + \sum\limits_{i=1}^{N} P_kM_i^{\mathrm{T}}P_{ik}^{-1}\hat{\boldsymbol{X}}_{ik} - \\ \quad \sum\limits_{i=1}^{N} P_kM_i^{\mathrm{T}}P_{i,k|k-1}^{-1}\hat{\boldsymbol{X}}_{i,k|k-1} \\ P_k^{-1} = P_{k|k-1}^{-1} + \sum\limits_{i=1}^{N} M_i^{\mathrm{T}}P_{ik}^{-1}M_i - \sum\limits_{i=1}^{N} M_i^{\mathrm{T}}P_{i,k|k-1}^{-1}M_i \end{cases} \tag{12}$$

Equation (12) describes the relation between the global filtering and local filtering in decentralized filters. The global filtering is equivalent to centralized Kalman filtering, and local filtering is also optimal. The signal flow diagram is shown in Fig. 1 with $N = 2$, where $B_k = P_kP_{k|k-1}^{-1}\Phi_{k,k-1}$; $B_{1k} = P_{1k}P_{1,k|k-1}^{-1}\Phi_{1;k,k-1}$; $B_{2k} = P_{2k}P_{2,k|k-1}^{-1}\Phi_{2;k,k-1}$; $K_{1k} = P_{1k}A_{1k}^{\mathrm{T}}R_{1k}^{-1}$; $K_{2k} = P_{2k}A_{2k}^{\mathrm{T}}R_{2k}^{-1}$; $F_{1k} = P_kM_1^{\mathrm{T}}P_{1k}^{-1}$; $F_{2k} = P_kM_2^{\mathrm{T}}P_{2k}^{-1}$; $T_{1k} = P_kM_1^{\mathrm{T}}P_{1,k|k-1}^{-1}\Phi_{1;k,k-1}$; $T_{2k} = P_kM_2^{\mathrm{T}}P_{2,k|k-1}^{-1}\Phi_{2;k,k-1}$.

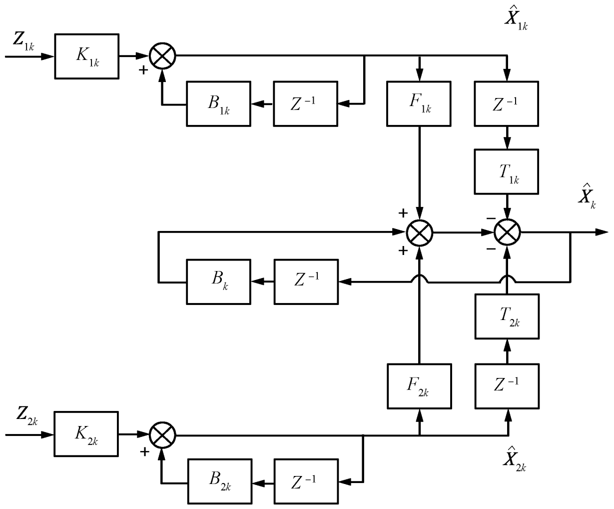


Fig. 1 Signal flow diagram of decentralized filtering

1.3 Federated filtering

1.3.1 $M_i = I$

A special situation is $\Phi_{i;k,k-1} = \Phi_{k,k-1}$ and $M_i = I$. According to the information sharing principle of $\sum_{i=1}^{N,m}\beta_i = 1$ and the state reset of the federated filter, we obtain

$$\begin{cases} P_{i,k-1}^{-1} = \beta_iP_{k-1}^{-1},\ Q_{i,k-1}^{-1} = \beta_iQ_{k-1}^{-1} \\ \beta_m = 1 - \sum\limits_{i=1}^{N}\beta_i,\ \hat{\boldsymbol{X}}_{i,k-1} = \hat{\boldsymbol{X}}_{k-1} \end{cases} \tag{13}$$

where the information-sharing coefficients can be selected by several different strategies according to applications[6, 10].

The formulas of the federated filter are given as

$$\begin{cases} P_{k|k-1}^{-1} = \left[\Phi_{k,k-1}P_{k-1}\Phi_{k,k-1}^{\mathrm{T}} + Q_{k-1}\right]^{-1} \\ \hat{\boldsymbol{X}}_{k|k-1} = \Phi_{k,k-1}\hat{\boldsymbol{X}}_{k-1} \\ P_{i,k|k-1}^{-1} = \beta_iP_{k|k-1}^{-1},\ P_{m,k|k-1}^{-1} = \beta_mP_{k|k-1}^{-1} \\ \hat{\boldsymbol{X}}_{i,k|k-1} = \hat{\boldsymbol{X}}_{k|k-1},\ \hat{\boldsymbol{X}}_{m,k|k-1} = \hat{\boldsymbol{X}}_{k|k-1} \end{cases} \tag{14}$$

for time update,

$$\begin{cases} P_{ik}^{-1} = P_{i,k|k-1}^{-1} + H_{ik}^{\mathrm{T}}R_{ik}^{-1}H_{ik} \\ P_{mk}^{-1} = P_{m,k|k-1}^{-1} \\ \hat{\boldsymbol{X}}_{ik} = P_{ik}P_{i,k|k-1}^{-1}\hat{\boldsymbol{X}}_{i,k|k-1} + P_{ik}H_{ik}^{\mathrm{T}}R_{ik}^{-1}\boldsymbol{Z}_{ik} \\ \hat{\boldsymbol{X}}_{mk} = \hat{\boldsymbol{X}}_{m,k|k-1} \end{cases} \tag{15}$$

for measurement update, and

$$\hat{\boldsymbol{X}}_k = P_kP_{mk}^{-1}\hat{\boldsymbol{X}}_{mk} + \sum_{i=1}^{N} P_kP_{ik}^{-1}\hat{\boldsymbol{X}}_{ik} \tag{16}$$

$$P_k^{-1} = P_{mk}^{-1} + \sum_{i=1}^{N} P_{ik}^{-1} \tag{17}$$

for fusion.

Equations (16) and (17) describe the relation between the global filtering and the local filtering of the federated filter, whose global filtering is optimal while the local filtering is not. If $N = 2$, the signal flow diagram is shown in Fig. 2, where $B_k = \beta_mP_kP_{k|k-1}^{-1}\Phi_{k,k-1}$; $B_{1k} = \beta_1P_{1k}P_{k|k-1}^{-1}\Phi_{k,k-1}$; $B_{2k} = \beta_2P_{2k}P_{k|k-1}^{-1}\Phi_{k,k-1}$; $K_{1k} = P_{1k}H_{1k}^{\mathrm{T}}R_{1k}^{-1}$; $K_{2k} = P_{2k}H_{2k}^{\mathrm{T}}R_{2k}^{-1}$; $F_{1k} = P_kP_{1k}^{-1}$; and $F_{2k} = P_kP_{2k}^{-1}$. Compared with the decentralized filtering in Fig. 1, the federated filter has a simpler structure and fewer computations.

The federated filtering is equivalent to the centralized Kalman filtering, whose global filtering is optimal.

Usually the state vector of the local filters contains common state variables and bias variables, and the state vector of the master filter is composed of common state variables and all bias variables. Thus, we need expand the dimensions of the local filters to meet the condition of $M_i = I$. In this case, the computation amount for local filters will be increased.

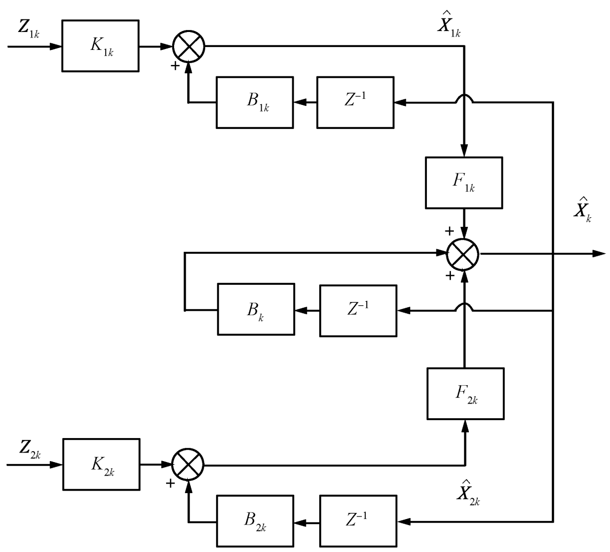


Fig. 2 Signal flow diagram of federated filtering

1.3.2 $\boldsymbol{M_i \neq I}$

In the practical application of federated filters, not only the dimensions of the master filter and each local filter are different, but also the influence of the non-common state on the common state cannot be ignored. In such a situation, the federated filtering is not globally optimal.

If $M_i \neq I, i = 1, 2, \cdots, N$, the dimensions of the master filter and the local filters are different. Two situations will be discussed. One is that the dimension of the master filter is lower than that of the local filter ($n_m < n_i$), and the other is that the dimension of the master filter is higher ($n_m > n_i$).

When $n_m < n_i$, the state of the master filter is the common state $\hat{\boldsymbol{X}}_c$, and the state of the local filters is $\hat{\boldsymbol{X}}_i = \left[\hat{\boldsymbol{X}}_{ic}^{\rm T}, \hat{\boldsymbol{X}}_{ib}^{\rm T}\right]^{\rm T}$, where $\hat{\boldsymbol{X}}_{ic}$ and $\hat{\boldsymbol{X}}_{ib}$ are the common state and the bias state, respectively. The master filter only fuses the common state, and resets the common state and its covariance matrix to each local filter. Therefore, only the common state is considered in time update and measurement update of the local filtering. This situation can only be applied when the bias state has little influence on the common state. Neither the global filtering nor the local filtering is optimal. Considering the influence of the common state on the bias state[5], the common state $\hat{\boldsymbol{X}}_{mc}^+$ by fusion is used to reset the common state of the local filter, and $\hat{\boldsymbol{X}}_{ic}^+ = \hat{\boldsymbol{X}}_{mc}^+$ is used as the measurement value to update the bias state of the local filter $\hat{\boldsymbol{X}}_{ib}$. We deduce that

$$\hat{\boldsymbol{X}}_i^+ = \left[\begin{array}{c} \hat{\boldsymbol{X}}_{mc}^+ \\ \hat{\boldsymbol{X}}_{ib} + S_{ibc}S_{icc}^{-1}\left(\hat{\boldsymbol{X}}_{ic}^+ - \hat{\boldsymbol{X}}_{ic}\right) \end{array}\right]$$
$$S_i^+ = \left[\begin{array}{cc} S_{mcc}^+ r_i^{1/2} & 0 \\ S_{ibc}S_{icc}^{-1}S_{icc}^+ & S_{ibb} \end{array}\right]$$

where the subscripts c and ic, ib denote the common state, the common state, and the bias state of the sensor i, respectively; $\hat{\boldsymbol{X}}_{ic}$ and $\hat{\boldsymbol{X}}_{ib}$ denote the state estimation of the one-step prediction, while S_{icc}, S_{ibc}, and S_{ibb} denote their lower triangular metrics of the covariance matrix decomposition; $\hat{\boldsymbol{X}}_i^+$, $\hat{\boldsymbol{X}}_{mc}^+$, and S_i^+ denote the state estimation and its lower triangular matrix of the covariance matrix decomposition after feedback correction; and $r_i = \beta_i^{-1}$.

The above calculation is rather complex, and it is proved[7] that if the state estimation errors of the local filters and the master filter are completely uncorrelated, the algorithm can achieve a globally optimal estimation. However, because the local filters and the master filter of the federated filters belong to the same reference system and the state noise is coupled, only suboptimal solution can be obtained.

When $n_m > n_i$, the state of the master filter includes not only the common state $\hat{\boldsymbol{X}}_c$, but also the bias state $\hat{\boldsymbol{X}}_{ib}$ of partial or entire local filters. For example, in decentralized navigation systems, more sensor-dedicated local filters feed a larger master filter. The master filter fuses local filter outputs and yields global estimates.

In terms of Carlson′s approach, reset

$$P_{i,k-1}^{-1} = \beta_i M_i P_{k-1}^{-1} M_i^{\rm T}, \quad Q_{i,k-1}^{-1} = \beta_i M_i Q_{k-1}^{-1} M_i^{\rm T}$$

and substitute the above into (14). We will not result in $P_{i,k|k-1}^{-1} = \beta_i M_i P_{k|k-1}^{-1} M_i^{\rm T}$, which will induce difficulties in the subsequent treatment. Hence, we modify the reset structure as

$$\begin{cases} P_{i,k|k-1}^{-1} = \beta_i M_i P_{k|k-1}^{-1} M_i^{\rm T}, \\ \hat{\boldsymbol{X}}_{i,k|k-1} = M_i \hat{\boldsymbol{X}}_{k|k-1}, \end{cases} \quad i = 1, 2, \cdots, N, m \tag{18}$$

By substituting (18) into (12), the theoretical value of the state error information matrix of the general system is obtained as

$$\begin{aligned} P_k^{-1^*} = & P_{k|k-1}^{-1} - \sum_{i=1}^{N} M_i^{\rm T} P_{i,k|k-1}^{-1} M_i + \sum_{i=1}^{N} M_i^{\rm T} P_{ik}^{-1} M_i = \\ & P_{k|k-1}^{-1} - \sum_{i=1}^{N} \beta_i M_i^{\rm T} M_i P_{k|k-1}^{-1} M_i^{\rm T} M_i + \sum_{i=1}^{N} M_i^{\rm T} P_{ik}^{-1} M_i \end{aligned} \tag{19}$$

where matrix $M_i^{\rm T} M_i$ has the upper bound unit matrix I, i.e.,

$$M_i^{\rm T} M_i \leq I$$

which indicates that the matrix I is more positive-definite than $M_i^{\rm T} M_i$. Substituting this result in (19) and taking the upper bound, by which the information of the information matrix $P_k^{-1^*}$ is decreased, we can obtain the conservative estimation as

$$\begin{aligned} P_k^{-1} = & P_{k|k-1}^{-1} - \sum_{i=1}^{N} \beta_i P_{k|k-1}^{-1} + \sum_{i=1}^{N} M_i^{\rm T} P_{ik}^{-1} M_i = \\ & \left(1 - \sum_{i=1}^{N} \beta_i\right) P_{k|k-1}^{-1} + \sum_{i=1}^{N} M_i^{\rm T} P_{ik}^{-1} M_i = \\ & \beta_m P_{k|k-1}^{-1} + \sum_{i=1}^{N} M_i^{\rm T} P_{ik}^{-1} M_i \end{aligned}$$

Further, by substituting $M_m^{\rm T} M_m P_{k|k-1}^{-1} M_m^{\rm T} M_m$ for $P_{k|k-1}^{-1}$ into the above equation, a more conservative result can be obtained as

$$P_k^{-1} = \sum_{i=1}^{N,m} M_i^{\rm T} P_{ik}^{-1} M_i \tag{20}$$

Similarly, with the substitution of (18) into (12), we obtain the theoretical value of the state estimation for the general system

$$\begin{aligned}&\hat{\boldsymbol{X}}_k^* = P_k^* P_{k|k-1}^{-1} \hat{\boldsymbol{X}}_{k|k-1} - \\ &\sum_{i=1}^{N} \beta_i P_k^* M_i^{\mathrm{T}} M_i P_{k|k-1}^{-1} M_i^{\mathrm{T}} M_i \hat{\boldsymbol{X}}_{k|k-1} + \\ &\sum_{i=1}^{N} P_k^* M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik}\end{aligned} \tag{21}$$

Substituting $M_i^{\mathrm{T}} M_i \leq I$ and taking the upper bound, we can obtain the conservative estimation as

$$\begin{aligned}&\hat{\boldsymbol{X}}'_k = P_k^* P_{k|k-1}^{-1} \hat{\boldsymbol{X}}_{k|k-1} - \\ &\sum_{i=1}^{N} \beta_i P_k^* P_{k|k-1}^{-1} \hat{\boldsymbol{X}}_{k|k-1} + \sum_{i=1}^{N} P_k^* M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik} = \\ &\left(1 - \sum_{i=1}^{N} \beta_i\right) P_k^* P_{k|k-1}^{-1} \hat{\boldsymbol{X}}_{k|k-1} + \sum_{i=1}^{N} P_k^* M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik} = \\ &\beta_m P_k^* P_{k|k-1}^{-1} \hat{\boldsymbol{X}}_{k|k-1} + \sum_{i=1}^{N} P_k^* M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik}\end{aligned}$$

Further, substitute $M_m^{\mathrm{T}} M_m P_{k|k-1}^{-1} M_m^{\mathrm{T}} M_m$ for $P_{k|k-1}^{-1}$, and P_k for P_k^*, and the conservative estimation of the global filtering corresponding to (20) is

$$\hat{\boldsymbol{X}}_k = \sum_{i=1}^{N,m} P_k M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik} \tag{22}$$

where $P_{mk}^{-1} = \beta_m M_m P_{k|k-1}^{-1} M_m^{\mathrm{T}}$. Here, federated filtering is not equivalent to centralized Kalman filtering and is not globally optimal.

2 Global optimal filtering

The global filtering of the generalized federated filters expressed by (20) and (22) is suboptimal. The reason is that the influence of the non-common state, which is usually the bias state of sensors, is not considered completely, and some information is lost. We propose a novel method to achieve a globally optimal solution.

Firstly, a master filter and local filters with different dimensions are structured. Based on the information sharing principle, the one-step prediction state and its state error information matrix are reset to obtain the suboptimal solution of the global filtering. Then, the suboptimal solution is used as the observation feedback to correct the one-step prediction state. And the optimal solution of the global filtering is obtained as

$$\hat{\boldsymbol{X}}_k^* = \hat{\boldsymbol{X}}_{k|k-1} + K\left(\hat{\boldsymbol{X}}_k - \hat{\boldsymbol{X}}_{k|k-1}\right) \tag{23}$$

where K is the optimal gain matrix of the generalized federated filters.

Define

$$\hat{\boldsymbol{X}}_k^* = \boldsymbol{X}_k + \tilde{\boldsymbol{X}}_k^* \tag{24a}$$

$$\hat{\boldsymbol{X}}_{k|k-1} = \boldsymbol{X}_k + \tilde{\boldsymbol{X}}_{k|k-1} \tag{24b}$$

$$\hat{\boldsymbol{X}}_k = \boldsymbol{X}_k + \tilde{\boldsymbol{X}}_k \tag{24c}$$

where $\boldsymbol{X}_k$ is the real state vector; the vectors with "$\sim$" are the state error vectors relative to $\boldsymbol{X}_k$. With the above substitution into (23), we obtain

$$\tilde{\boldsymbol{X}}_k^* = \tilde{\boldsymbol{X}}_{k|k-1} + K\left(\tilde{\boldsymbol{X}}_k - \tilde{\boldsymbol{X}}_{k|k-1}\right) \tag{25}$$

The covariance matrixes of both sides of the above equation are calculated as follows. Define $\mathrm{E}\left[\tilde{\boldsymbol{X}}_k^* \tilde{\boldsymbol{X}}_k^{*\mathrm{T}}\right] = P_k^*$ and $\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1} \tilde{\boldsymbol{X}}_{k|k-1}^{\mathrm{T}}\right] = P_{k|k-1}$. Then,

$$\begin{aligned}&P_k^* = P_{k|k-1} + \mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1} \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right] K^{\mathrm{T}} - P_{k|k-1} K^{\mathrm{T}} + \\ &K\mathrm{E}\left[\tilde{\boldsymbol{X}}_k \tilde{\boldsymbol{X}}_{k|k-1}^{\mathrm{T}}\right] + K\mathrm{E}\left[\tilde{\boldsymbol{X}}_k \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right] K^{\mathrm{T}} - \\ &K\mathrm{E}\left[\tilde{\boldsymbol{X}}_k \tilde{\boldsymbol{X}}_{k|k-1}^{\mathrm{T}}\right] K^{\mathrm{T}} - K P_{k|k-1} - \\ &K\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1} \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right] K^{\mathrm{T}} + K P_{k|k-1} K^{\mathrm{T}}\end{aligned} \tag{26}$$

where $\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1} \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right] = \left\{\mathrm{E}\left[\tilde{\boldsymbol{X}}_k \tilde{\boldsymbol{X}}_{k|k-1}^{\mathrm{T}}\right]\right\}^{\mathrm{T}}$. And the key to obtain the analytic equation of P_k^* is to calculate $\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1} \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right]$ and $\mathrm{E}\left[\tilde{\boldsymbol{X}}_k \tilde{\boldsymbol{X}}_k^{\mathrm{T}}\right]$.

From (22), we have

$$\begin{aligned}&\hat{\boldsymbol{X}}_k = \sum_{i=1}^{N,m} P_k M_i^{\mathrm{T}} P_{ik}^{-1} \hat{\boldsymbol{X}}_{ik} = \\ &\sum_{i=1}^{N,m} P_k \left[\beta_i M_i^{\mathrm{T}} M_i P_{k|k-1}^{-1} M_i^{\mathrm{T}} M_i \hat{\boldsymbol{X}}_{k|k-1} + M_i^{\mathrm{T}} A_{ik}^{\mathrm{T}} R_{ik}^{-1} \boldsymbol{Z}_{ik}\right] = \\ &P_k \left(P_{k|k-1}^{-1} - \Delta P_k^{-1}\right) \hat{\boldsymbol{X}}_{k|k-1} + P_k H_k^{\mathrm{T}} R_k^{-1} \boldsymbol{Z}_k\end{aligned} \tag{27}$$

where

$$\Delta P_k^{-1} = P_{k|k-1}^{-1} - \sum_{i=1}^{N,m} \beta_i M_i^{\mathrm{T}} M_i P_{k|k-1}^{-1} M_i^{\mathrm{T}} M_i \tag{28}$$

From (20), we have

$$\begin{aligned}&P_k^{-1} = \sum_{i=1}^{N,m} M_i^{\mathrm{T}} P_{ik}^{-1} M_i = \\ &\sum_{i=1}^{N,m} \beta_i M_i^{\mathrm{T}} M_i P_{k|k-1}^{-1} M_i^{\mathrm{T}} M_i + \sum_{i=1}^{N} M_i^{\mathrm{T}} A_{ik}^{\mathrm{T}} R_{ik}^{-1} A_{ik} M_i\end{aligned}$$

Adding $P_{k|k-1}^{-1}$ to and subtract $P_{k|k-1}^{-1}$ from the right side of the equation, we have

$$\begin{aligned}&P_{k|k-1}^{-1} + \sum_{i=1}^{N} M_i^{\mathrm{T}} A_{ik}^{\mathrm{T}} R_{ik}^{-1} A_{ik} M_i = \\ &P_{k|k-1}^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k = P_k^{-1^*}\end{aligned}$$

while

$$\sum_{i=1}^{N,m} \beta_i M_i^{\mathrm{T}} M_i P_{k|k-1}^{-1} M_i^{\mathrm{T}} M_i - P_{k|k-1}^{-1} = -\Delta P_k^{-1}$$

Finally,

$$P_k^{-1} = P_k^{-1^*} - \Delta P_k^{-1} \tag{29}$$

Substituting (24) and $\boldsymbol{Z}_k = H_k \boldsymbol{X}_k + \boldsymbol{V}_k$ into (27), we obtain

$$\begin{aligned}&\tilde{\boldsymbol{X}}_k = -\boldsymbol{X}_k + P_k P_{k|k-1}^{-1} \boldsymbol{X}_k + P_k P_{k|k-1}^{-1} \tilde{\boldsymbol{X}}_{k|k-1} - \\ &P_k \Delta P_k^{-1} \boldsymbol{X}_k - P_k \Delta P_k^{-1} \tilde{\boldsymbol{X}}_{k|k-1} + \\ &P_k H_k^{\mathrm{T}} R_k^{-1} H_k \boldsymbol{X}_k + P_k H_k^{\mathrm{T}} R_k^{-1} \boldsymbol{V}_k\end{aligned}$$

The sum of the second and the sixth items in the right side of the above equation is

$$P_k P_{k|k-1}^{-1}\boldsymbol{X}_k + P_k H_k^{\rm T} R_k^{-1} H_k \boldsymbol{X}_k =$$
$$P_k\left(P_{k|k-1}^{-1} + H_k^{\rm T} R_k^{-1} H_k\right)\boldsymbol{X}_k = P_k P_k^{-1^*}\boldsymbol{X}_k =$$
$$\boldsymbol{X}_k + P_k \Delta P_k^{-1}\boldsymbol{X}_k$$

After the above substitution, the equation becomes

$$\tilde{\boldsymbol{X}}_k = P_k\left(P_{k|k-1}^{-1} - \Delta P_k^{-1}\right)\tilde{\boldsymbol{X}}_{k|k-1} + P_k H_k^{\rm T} R_k^{-1}\boldsymbol{V}_k \quad (30)$$

From (30), it follows that

$$\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1}\tilde{\boldsymbol{X}}_k^{\rm T}\right] = \mathrm{E}\Big[\tilde{\boldsymbol{X}}_{k|k-1}\tilde{\boldsymbol{X}}_{k|k-1}^{\rm T}\left(P_{k|k-1}^{-1} - \Delta P_k^{-1}\right)\times P_k + \tilde{\boldsymbol{X}}_{k|k-1}\boldsymbol{V}_k^{\rm T} R_k^{-1} H_k P_k\Big]$$

Since $\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1}\boldsymbol{V}_k^{\rm T}\right] = 0$, we have

$$\mathrm{E}\left[\tilde{\boldsymbol{X}}_{k|k-1}\tilde{\boldsymbol{X}}_k^{\rm T}\right] = P_{k|k-1}\left(P_{k|k-1}^{-1} - \Delta P_k^{-1}\right)P_k = P_k - P_{k|k-1}P_k^{-1^*}P_k + P_{k|k-1}$$

$$\mathrm{E}\left[\tilde{\boldsymbol{X}}_k\tilde{\boldsymbol{X}}_{k|k-1}^{\rm T}\right] = P_k - P_k P_k^{-1^*}P_{k|k-1} + P_{k|k-1}$$

Similarly,

$$\mathrm{E}\left[\tilde{\boldsymbol{X}}_k\tilde{\boldsymbol{X}}_k^{\rm T}\right] = 2P_k - P_k P_k^{-1^*}P_k +$$
$$P_k P_k^{-1^*}P_{k|k-1}P_k^{-1^*}P_k - P_k P_k^{-1^*}P_{k|k-1} -$$
$$P_{k|k-1}P_k^{-1^*}P_k + P_{k|k-1}$$

Substituting it into (26), we obtain

$$P_k^* = P_{k|k-1} + P_k K^{\rm T} + K P_k -$$
$$P_{k|k-1}P_k^{-1^*}P_k K^{\rm T} - K P_k P_k^{-1^*}P_{k|k-1} -$$
$$K P_k P_k^{-1^*}P_k K^{\rm T} + K P_k P_k^{-1^*}P_{k|k-1}P_k^{-1^*}P_k K^{\rm T}$$

Let

$$\tilde{P} = K P_k P_k^{-1^*} \quad (31)$$

Then, $KP_k = KP_kP_k^{-1^*}P_k^* = \tilde{P}P_k^*$. With the above substitution, the result is

$$P_k^* - P_{k|k-1} = \tilde{P}\left(P_k^* - P_{k|k-1}\right) + \left(P_k^* - P_{k|k-1}\right)\tilde{P}^{\rm T} - \tilde{P}\left(P_k^* - P_{k|k-1}\right)\tilde{P}^{\rm T} \quad (32)$$

In order to obtain the optimal gain matrix K, we calculate

$$\frac{\partial\left[\mathrm{tr}\left(P_k^* - P_{k|k-1}\right)\right]}{\partial\tilde{P}} = 0$$

From (32), we have

$$2\left(P_k^* - P_{k|k-1}\right) - 2\tilde{P}\left(P_k^* - P_{k|k-1}\right) = 0$$

Then, we obtain $\tilde{P} = I$. And from (31), we obtain

$$K = P_k^* P_k^{-1} \quad (33)$$

With the substitution of (33) into (23), we prove that it is equivalent to the centralized Kalman filtering:

$$\hat{\boldsymbol{X}}_k^* = \hat{\boldsymbol{X}}_{k|k-1} + P_k^* P_k^{-1}\left(\hat{\boldsymbol{X}}_k - \hat{\boldsymbol{X}}_{k|k-1}\right) =$$
$$\hat{\boldsymbol{X}}_{k|k-1} + P_k^* P_k^{-1}\hat{\boldsymbol{X}}_k - P_k^* P_k^{-1}\hat{\boldsymbol{X}}_{k|k-1}$$

From (27), we have

$$P_k^{-1}\hat{\boldsymbol{X}}_k = \left(P_{k|k-1}^{-1} - \Delta P_k^{-1}\right)\hat{\boldsymbol{X}}_{k|k-1} + H_k^{\rm T} R_k^{-1}\boldsymbol{Z}_k =$$
$$\left(P_{k|k-1}^{-1} - P_k^{-1^*} + P_k^{-1}\right)\hat{\boldsymbol{X}}_{k|k-1} + H_k^{\rm T} R_k^{-1}\boldsymbol{Z}_k$$

With the above substitution, the result is

$$\hat{\boldsymbol{X}}_k^* = \hat{\boldsymbol{X}}_{k|k-1} + P_k^*\left(P_{k|k-1}^{-1} - P_k^{-1^*} + P_k^{-1}\right)\hat{\boldsymbol{X}}_{k|k-1} +$$
$$P_k^* H_k^{\rm T} R_k^{-1}\boldsymbol{Z}_k - P_k^* P_k^{-1}\hat{\boldsymbol{X}}_{k|k-1} =$$
$$\hat{\boldsymbol{X}}_{k|k-1} + P_k^* P_{k|k-1}^{-1}\hat{\boldsymbol{X}}_{k|k-1} - \hat{\boldsymbol{X}}_{k|k-1} + P_k^* P_k^{-1}\hat{\boldsymbol{X}}_{k|k-1} +$$
$$P_k^* H_k^{\rm T} R_k^{-1}\boldsymbol{Z}_k - P_k^* P_k^{-1}\hat{\boldsymbol{X}}_{k|k-1} =$$
$$P_k^* P_{k|k-1}^{-1}\hat{\boldsymbol{X}}_{k|k-1} + P_k^* H_k^{\rm T} R_k^{-1}\boldsymbol{Z}_k$$

This is just the formula of the centralized Kalman filtering.

As mentioned above, before the optimization, the structure of the generalized federated filtering is the same as the federated filtering. The signal flow diagrams show that it has fewer computations compared with the decentralized filtering. To gain the global optimality for generalized federated filters, further computing is to calculate the optimal gain matrix $K = P_k^* P_k^{-1}$. Since P_k^{-1} is obtained before, P_k^* can be obtained by (29). It just takes fewer computations in the master filter by using efficient square root algorithms. Besides, in generalized federated filtering the dimensions of local filters are much fewer than the dimension of the master filter. This design achieves major improvement in speed for the local filters.

3 Simulation

Simulations are carried out for a dual-SINS/GPS integrated navigation system. SINS and GPS are the abbreviations of strap-down inertial navigation system and global positioning system, respectively. The system structure is shown in Fig. 3, where the symbol "(-)" denotes the one-step prediction estimation.

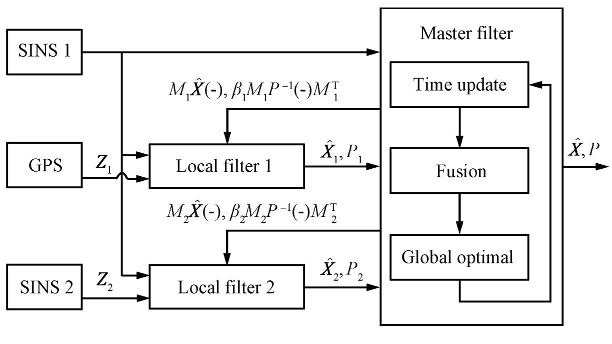


Fig. 3 The structure of the dual-SINS/GPS integrated navigation system

Both the error dimensions of SINS 1 and SISN 2 are 15, i.e.,

$$\boldsymbol{X} = [\varepsilon_{\rm E}\ \ \varepsilon_{\rm N}\ \ \varepsilon_{\rm H}\ \ \delta\varphi\ \ \delta\lambda\ \ \delta h\ \ \delta\dot{\varphi}\ \ \delta\dot{\lambda}\ \ \delta\dot{h}\ \ \Delta d_{\rm E}\ \ \Delta d_{\rm N}\ \ \Delta d_{\rm H}\ \ \Delta a_{\rm N}\ \ \Delta a_{\rm E}\ \ \Delta a_{\rm H}]^{\rm T}$$

where $\varepsilon_{\rm E}$, $\varepsilon_{\rm N}$, and $\varepsilon_{\rm H}$ denote the error angles between the calculating coordinates and the geographic coordinates; $\delta\varphi$, $\delta\lambda$, and δh denote the error of geographic latitude, longitude, and altitude; $\delta\dot{\varphi}$, $\delta\dot{\lambda}$ and $\delta\dot{h}$ are the corresponding velocity errors; Δd_{E}, Δd_{N}, Δd_{H} and Δa_{E}, Δa_{N}, Δa_{H} are the dynamic bias of gyros and accelerators in the east, north, and up coordinate frame.

Suppose SINS 1 is the reference system. The position and velocity differences between GPS and SINS 1 are taken as the observation vector to constitute local filter 1. The position and velocity differences between SINS 1 and SINS 2 are taken as the observation vector to constitute local filter 2. The precision of the two strap-down inertial navigation systems is set as follows. For SINS 1, the drift bias of the gyros is $0.02\,^{\circ}/\mathrm{h}$ and the random drift is $0.01\,^{\circ}/\mathrm{h}$; the zero bias of the accelerators is $10^{-4}g$ and the output noise is $5\times10^{-5}g$. SINS 2 is taken as measurement update, whose precision is one order of magnitude higher than that of SINS 1. For GPS, the position error is 15 m (1σ)and the velocity error is 0.05 m/s (1σ).

Measurement baseline is determined to provide the reference position before simulation. The baseline shape is of an "L" pattern. The carrier runs eastward for 0.5 h and then northward for 0.5 h. It stops for 1 min every 9 min. The circulative movement characteristic is "accelerating-uniform velocity-decelerating-stationary". The acceleration is $\pm3\,\mathrm{m/s^2}$. The accelerating time is 3 s, the uniform velocity movement lasts 543 s, and the decelerating time is also 3 s. The whole distance is about 30 km.

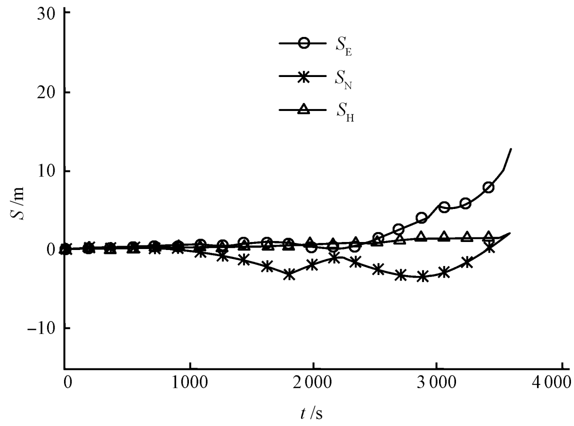
Firstly, the simulation is designed as the centralized Kalman filter. The simulation result is shown in Fig. 4 (a), where S_{E}, S_{N}, and S_{H} indicate the position errors in the eastern, northern, and upper directions. Secondly, the simulation is designed as the generalized federated filter. The dimensions of the local filter 1 are 15 and that of the local filter 2, and the master filter are 30. The information sharing coefficients are $\beta_1 = 0.5$, $\beta_2 = 0.3$, and $\beta_m = 0.2$. The simulation result is shown in Fig. 4 (b). Finally, the simulation is designed by the globally optimal filtering approach. The output of the generalized federated filter is used as the observation feedback to correct the one-step prediction state so that the optimal solution of the global filtering can be achieved. The simulation result is shown in Fig. 4 (c).

The navigation deviations of the three filters are listed in Table 1.

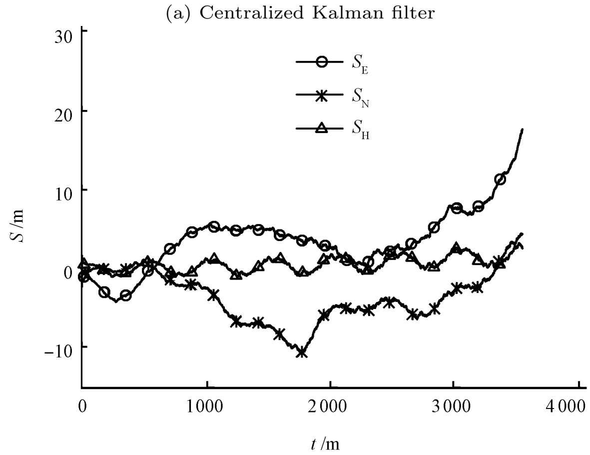
The simulation results indicate that if the dimension of local filter 1 is 15 and the dimensions of local filter 2 and the master filter are 30, i.e., $M_i \neq I$, only suboptimal solution can be obtained by the generalized federated filter. When the navigation time is shorter (the simulation time is 1 h), rather high estimation precision can be still achieved. If the feedback correction filtering algorithm is used in the generalized federated filter, the globally optimal solution can be obtained, which is completely equivalent to the centralized Kalman filter.

Table 1 Navigation deviations with different filters

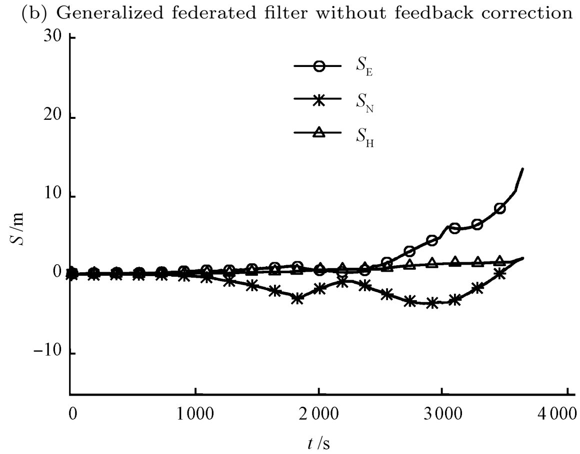
Filter	Navigation deviation (m)		
	East	North	Height
Centralized Kalman filter	3.27	1.82	0.80
Generalized federated filter (without feedback correction)	5.68	4.63	1.30
Generalized federated filter (with feedback correction)	3.28	1.83	0.79



(a) Centralized Kalman filter



(b) Generalized federated filter without feedback correction



(c) Generalized federated filter with feedback correction

Fig. 4 Navigation position errors with different filters

4 Conclusion

Based on the matrix theory and the information sharing principle, the analytic relation among centralized Kalman filtering, decentralized filtering, and federated filtering is delivered. It is proved that the global filtering of federated filters is optimal only when the dimensions of the master filter and the local filters are totally equal. Compared with decentralized filtering, federated filtering has the simpler structure and fewer computations, which is illustrated by signal flow diagram visually and clearly. If the dimensions of the master filter and the local filters are different, only suboptimal solution can be obtained. Hereby, the feedback correction approach by which the generalized federated filter may realize global optimality is presented. The optimal feedback gain matrix is mathematically derived, so the filtering result is theoretically proved to be equivalent to the centralized Kalman filtering. The result of the simulation experiments with a dual-SINS/GPS integrated navigation system demonstrates the validity of the algorithm.

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