Synthesis for Spatially Interconnected Systems with Distributed Output Feedback Controllers

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Abstract This paper considers the design of distributed control architecture for spatially interconnected systems that are composed of several similar interconnected sub-units. Each sub-unit is a linear continuous time system and directly interacts with its nearest neighbors. This class of systems exists in several applications such as automated highway systems, power systems, and computer networks. Hybrid Lyapunov criterion and the hybrid real bounded lemma are derived to determine the stability and H_∞ performance of the overall system. In order to stabilize this class of systems, distributed dynamic output feedback controllers are considered, and tractable linear matrix inequality (LMI)-based algorithms for the derivation of distributed controllers are presented. The change variable approach is introduced in the LMI-based algorithms due to its higher efficiency and numerically stable implementation than the elimination algorithm as introduced in previous works. It is shown through a numerical example that the distributed H_∞ controllers developed in this article are superior to decentralized controllers in several aspects.

Key words Spatially interconnected systems, distributed control, dynamic output feedback, linear matrix inequality (LMI)

Spatially interconnected systems (SISs) are a special set of multidimensional (n-D) systems characterized by several variables, which are indexed by spatial coordinates in addition to time. The SISs consist of nonhomogeneous or homogeneous interacting components and each component is a continuous time subsystem. SISs arise from many engineering applications, such as power systems, computer networks, web-transport systems, and automated highway systems. One common feature of such physical systems is that their components are geographically isolated. To deal with this kind of large-scale systems, up till now, most published works have been focused on decentralized control strategy[1−3] and only in recent years some scholars began to explore distributed control strategy over SISs[4−9]. In [6], a frequency domain criterion for optimal distributed controllers was established; in [4 − 5], a distributed control architecture was implemented in robot formation control; and in [7−8], a systemic approach of distributed controllers design for SISs was developed via linear matrix inequality (LMI) theory[10].

As we all know, decentralized control schemes present a practical method for controller design. The decentralized controllers utilize only the states of the local subsystems without any information exchange with other subsystems. The no–information–exchange restriction of decentralized controllers makes it flexible and simple during controller design and implementation, and thus decentralized control has been studied widely and deeply over the last decade.

However, there is a negative side of this seemingly advantageous decentralized control: the information injected into the controllers is completely localized and thus is limited. It is natural that the more information we get from the local subsystem, the better control effects the controller will have. In order to overcome this shortcoming of the decentralized schemes, distributed control problem for the distributed parameter systems was originally proposed by Bamieh[9]. Later, D′Andrea[7] developed a more generated method for distributed control synthesis for the SISs, which was successfully applied to a team of aerial vehicles to perform close formation flight. With the information exchange between the neighboring subsystems, the distributed controllers can behave more optimistically both in response time and system gain in comparison with decentralized controllers. One thing to be aware of is that the distributed controllers we are going to design are only relevant to its related subsystems, and thus when other subsystems′ models or the number of subsystems changes, the invariant subsystems′ controllers still work.

In this paper, we present a more in-depth and comprehensive stability analysis of SISs and we adopt dynamic output feedback H_∞ control scheme, which is similar to [7], to generate a distributed close-loop interconnected system. In this distributed system, every subsystem is treated as a hybrid Roesser model (RM) whose stability criterion can be recast into linear matrix inequalities[11].

Furthermore, based on real bounded lemma, we extend this criterion matrix to the form which embraces the H_∞ property of the subsystems. This extended criterion can be considered as a hybrid real bounded lemma.

It is well-known that the real bounded lemma, which is theoretical foundation of the H_∞ dynamic output design, provides a LMI to determine the stability and the H_∞ gain property of a system. However, for the H_∞ dynamic output design, the inequality in the bounded real lemma becomes a bilinear matrix inequality (BMI), which is an NP hard problem[1, 12−15]. In [13], the Lyapunov matrix in the BMIs was set as block diagonal form, and the BMIs were rewritten as LMIs. Reference [15] used the idea of the homotopy method to solve the BMIs. However, the more generally used approach dealing with this BMIs is called elimination algorithm, which was originally introduced by Gahinet[14] and Iwasaki[16]. In [14, 16], necessary and sufficient conditions of the existence of $\gamma-$suboptimal controller was given based on LMIs. Moreover, this algorithm was applied to distributed output feedback controller design of spatially interconnected systems[7]. In contrast to [7], we applied another algorithm called change variable algorithm during distributed controllers design, which was originally proposed by Gahinet[12]. Based on the controller development method of [7], some improvements are made for the design approach, which is more efficient and numerically stably implemented. To verify the effectiveness of the proposed algorithm, simulation results are included for applications on a spatially interconnected system.

This paper is organized as follows. In Section 1, we establish the state-space description of the SISs. In Section 2, we define the performance requirements and the hybrid criterions in the form of LMIs for the SISs. In Section 3, we present the controller design process based on change vari-

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able algorithm. Section 4 shows how distributed control architecture works in a spring-mass system. The performance of the overall system under distributed controllers is compared with the performance under decentralized controllers at the end of the paper.

1 Problem statement

Throughout this paper, we consider spatially invariant continuous systems and restrict ourselves to infinite spatial extent system for the reason that if the infinite extent system is well-posed, stable, and contractive, these properties are inherited by all periodic interconnections[7].

In parallel to [7], we consider vector valued signals, which are functions of $L+1$ independent variables, $u = u(t, s_1, s_2, \cdots, s_L)$, where t is a time-based variable that belongs to non-negative real value, i.e., $t \in \{\mathbf{R}^+, 0\}$, s_i is an integer, i.e., $s_i \in \mathbf{Z}$ or $s_i \in Z_i = \{1, 2, \cdots, N_i\}$, which represents periodicity of period N_i in spatial dimension i, and L is the dimension of the spatial coordinates. For simplification, the L-tuple $(s_1, s_2, \cdots, s_L)$ is denoted by $\boldsymbol{s}$. Particularly, when $L = 1$, we get the one spatial dimension interconnected systems.

We consider signals $u(t, s)$, where t denotes the temporal variable and $\boldsymbol{s}$ denotes the spatial variables both in continuous time domain and discrete spatial domain. The norm definitions of these two domains are

$$\|u\|_{l_2}^2 = \langle u, u\rangle_{l_2} = \sum_{s_1\in Z_1} \cdots \sum_{s_L \in Z_L} |u(\boldsymbol{s})|^2 \tag{1}$$

$$\|u\|_{L_2}^2 = \langle u, u\rangle_{L_2} = \int_{\infty}^{0} |u(t)|^2 \mathrm{d}t \tag{2}$$

Comparing with the Laplace operator which acts on time domain, we define the spatial shift operator s_i [7], acting on signals in l_2, as follows

$$\begin{aligned}(\mathbf{S}_i u(t))(\boldsymbol{s}) &= u(t, s_1, \cdots, s_i + 1, \cdots, s_L), i = 1, 2, \cdots, L\\ (\mathbf{S}_i^{-1} u(t))(\boldsymbol{s}) &= u(t, s_1, \cdots, s_i - 1, \cdots, s_L), i = 1, 2, \cdots, L\end{aligned} \tag{3}$$

where $\mathbf{S}_i$ is a shift operator, which acts on i-th spatial dimension.

With those definitions shown above, we are ready to introduce the state-space models of SISs, which stem from the Roesser model. The Roesser model was originally proposed for image processing and was used to depict n-D discrete systems[17]. If we denote $x_c(\cdot)$ and $x_d(\cdot)$ as the continuous states and the discrete states, respectively, the hybrid version of Roesser model takes the following form[18]

$$\begin{bmatrix} \frac{\partial}{\partial t} x_c(\boldsymbol{t}, \boldsymbol{j}) \\ \mathbf{S} x_d(\boldsymbol{t}, \boldsymbol{j}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_c(\boldsymbol{t}, \boldsymbol{j}) \\ x_d(\boldsymbol{t}, \boldsymbol{j}) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(\boldsymbol{t}, \boldsymbol{j}) \tag{4}$$

$$y(\boldsymbol{t}, \boldsymbol{j}) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_c(\boldsymbol{t}, \boldsymbol{j}) \\ x_d(\boldsymbol{t}, \boldsymbol{j}) \end{bmatrix} + D u(\boldsymbol{t}, \boldsymbol{j}) \tag{5}$$

where

$$\boldsymbol{t} = (t_1, \cdots, t_r), \quad \boldsymbol{j} = (j_{r+1}, \cdots, j_k)$$

$$x_c(\boldsymbol{t}, \boldsymbol{j}) = \begin{bmatrix} x^1(t_1, \cdots, t_r, j_{r+1}, \cdots, j_k) \\ \vdots \\ x^r(t_1, \cdots, t_r, j_{r+1}, \cdots, j_k) \end{bmatrix}$$

$$x_d(\boldsymbol{t}, \boldsymbol{j}) = \begin{bmatrix} x^{r+1}(t_1, \cdots, t_r, j_{r+1} + 1, \cdots, j_k) \\ \vdots \\ x^k(t_1, \cdots, t_r, j_{r+1}, \cdots, j_k + 1) \end{bmatrix}$$

$$\mathbf{S} x_d(\boldsymbol{t}, \boldsymbol{j}) = \begin{bmatrix} x^{r+1}(t_1, \cdots, t_r, j_{r+1} + 1, \cdots, j_k) \\ \vdots \\ x^k(t_1, \cdots, t_r, j_{r+1}, \cdots, j_k + 1) \end{bmatrix}$$

$$\mathbf{S} = \mathrm{diag}\{\mathbf{S}_{r+1}, \mathbf{S}_{r+2}, \cdots, \mathbf{S}_k\}$$

and $\mathbf{S}_i$ is defined in (3).

The feature of this model is that the state vector is partitioned into horizontal and vertical components, and the systems represented by the above model have continuous dynamics along r dimensions and discrete dynamics along $(k-r)$ dimensions.

We adopt the hybrid Roesser model (4) and (5) to describe the subsystems of the SISs considered in this paper. The structure of one spatial dimension SISs is shown in Fig. 1. By considering r in the Roesser model as the dimension of the subsystem′s states and $(k-r)$ as the dimension of the interconnected signals, it is straightforward to rewrite the hybrid Roesser model into the form that represents the i-th subsystem of the SISs as

$$\begin{bmatrix} \dot{x}(t, \boldsymbol{s}) \\ w(t, \boldsymbol{s}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t, \boldsymbol{s}) \\ v(t, \boldsymbol{s}) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} x(t, \boldsymbol{s}) \\ v(t, \boldsymbol{s}) \end{bmatrix} \tag{6}$$

$$z(t, \boldsymbol{s}) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x(t, \boldsymbol{s}) \\ v(t, \boldsymbol{s}) \end{bmatrix} + D d(t, \boldsymbol{s}) \tag{7}$$

where $x(t, \boldsymbol{s})$ is the continuous time state, $v(t, \boldsymbol{s})$ and $w(t, \boldsymbol{s})$ are the interconnection variables expressed as

$$v(t, \boldsymbol{s}) = \begin{bmatrix} v_+(t, \boldsymbol{s}) \\ v_-(t, \boldsymbol{s}) \end{bmatrix}, \quad w(t, \boldsymbol{s}) = \begin{bmatrix} w_+(t, \boldsymbol{s}) \\ w_-(t, \boldsymbol{s}) \end{bmatrix}$$

$d(t, \boldsymbol{s})$ denotes the disturbance acting on the i-th subsystem and $z(t, \boldsymbol{s})$ is its consequent output.

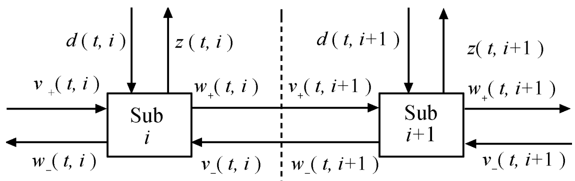

Fig. 1 Basic building blocks (one spatial dimension)

Particularly, for one spatial dimension SISs as shown in Fig. 1, vector $\boldsymbol{s}$ is simplified into one tuple vector, that is, a scale s, $s \in \mathbf{Z}$. s represents the index of a subsystems. The information between two neighboring subsystems propagates along one independent direction and is a two way signal marked by "+" and "−". Thus, $(s+1)$ represents the next subsystem of s-th subsystem along the "+" direction and $(s-1)$ the next subsystem along the "−" direction. Fig. 1 depicts the relations of the interconnected signals as: $v_+(t, s+1) = w_+(t, s)$, $v_-(t, s-1) = w_-(t, s)$.

The relationship between $w(t, \boldsymbol{s})$ is

$$w(t, \boldsymbol{s}) = \begin{bmatrix} w_+(t, \boldsymbol{s}) \\ w_-(t, \boldsymbol{s}) \end{bmatrix} = \begin{bmatrix} \mathbf{S} I_{m+} & 0 \\ 0 & \mathbf{S}^{-1} I_{m-} \end{bmatrix} \begin{bmatrix} v_+(t, \boldsymbol{s}) \\ v_-(t, \boldsymbol{s}) \end{bmatrix} \tag{8}$$

$$\triangleq \Delta_S v(t, \boldsymbol{s}) \tag{9}$$

where m_+ and m_- are the sizes of interconnection variables $v_+(t, \boldsymbol{s})$ and $v_-(t, \boldsymbol{s})$, respectively.

We may thus write the interconnected system in (6) and (7) as follows:

$$\begin{bmatrix} \dot{x}(t,\boldsymbol{s}) \\ (\Delta_S v(t))(\boldsymbol{s}) \\ z(t,\boldsymbol{s}) \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{TS} & B_T \\ A_{ST} & A_{SS} & B_S \\ C_T & C_S & D \end{bmatrix} \begin{bmatrix} x(t,\boldsymbol{s}) \\ v(t,\boldsymbol{s}) \\ d(t,\boldsymbol{s}) \end{bmatrix} \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \chi(t,\boldsymbol{s}) \\ d(t,\boldsymbol{s}) \end{bmatrix}$$
$$x(0,\boldsymbol{s}) = x_0(\boldsymbol{s}) \tag{10}$$

where

$$\chi(t,\boldsymbol{s}) = \begin{bmatrix} x(t,\boldsymbol{s}) \\ v(t,\boldsymbol{s}) \end{bmatrix} \tag{11}$$

and Δ_S is defined in (9).

In this paper, we focus on spatially interconnected system and all of our analysis will be based on the system model (10). We use the notation $\mathcal{M} = \{A, B, C, D\}$ to denote the interconnected systems, which was generated from $\{A, B, C, D\}$.

Expression (10) is similar to the interconnected models described in [7].

We have the following theorem which can be used to test the well-posedness, stability, and H_∞ performance of the SISs.

Theorem 1[7]**.** A system $\mathcal{M} = \{A, B, C, D\}$ is said to be well-posed, stable, and satisfy $\|T_{zd}\|_\infty < \gamma$, where $\gamma \in \mathbf{R}^+$, if there exist scaling matrices $X_T = X_T^{\mathrm{T}} > 0$, and $X_S = X_S^{\mathrm{T}}$ such that

1) $I - A_{SS}$ is invertible;

2) The following inequality is satisfied:

$$\begin{bmatrix} \bar{A}^{\mathrm{T}} X + X\bar{A} & X\bar{B} & \bar{C}^{\mathrm{T}} \\ \bar{B}^{\mathrm{T}} X & -\gamma I & \bar{D}^{\mathrm{T}} \\ \bar{C} & \bar{D} & -\gamma I \end{bmatrix} < 0 \tag{12}$$

where $X = \mathrm{diag}\{X_T, X_S\}$, $\bar{\mathcal{M}} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{\boldsymbol{m}}\} = f_{D2C}(\mathcal{M})$ and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are the modified bilinear transformation as introduced in [7].

Theorem 1 was obtained by treating the spatially invariant systems as interconnection in the robust control/linear fractional transformation (LFT) framework and the LMI (12) was derived based on μ analysis. In this article, we discuss the SISs from the multidimensional point of view. The hybrid version of Lyapunov criterion of the SISs is proposed and the criterion is further extended into the form that includes H_∞ performance of the SISs. The stability of the overall SISs is proved by Lyapunov theory. The extended criterion can be considered as a hybrid version of the real bounded lemma.

2 Performance analysis of SISs

We evaluate the spatially interconnected system on the following three performances[7]: well-posedness, exponential stability, and $\|T_{zd}\|_\infty < \gamma$.

The standard interpretation of well-posedness can be found in [19]: there must exist unique solutions to the system equations when signals are injected anywhere in the loop. For the class of systems considered in this paper, we should guarantee that the interconnection signals $v(t,\boldsymbol{s})$ and $w(t,\boldsymbol{s})$ are bounded.

We can extend this definition to the systems considered in this paper as follows.

Proposition 1. A system is well-posed if and only if $(\Delta_S - A_{SS})$ is invertible on l_2.

Proof. The solution to the system described in (10) is

$$x(t) = \int_0^t \exp(\Phi(t-\tau))\Gamma \mathrm{d}\tau \tag{13}$$

where

$$\Phi = A_{TT} + A_{TS}(\Delta_S - A_{SS})^{-1} A_{ST} \tag{14}$$

and

$$\Gamma = [B_T \ 0] + A_{TS}(\Delta_S - A_{SS})^{-1}[B_S \ I] \tag{15}$$

Thus, the result follows since $\exp(\Phi t)$ is bounded on the compact interval $[0, T]$. Another direction of the proof can be found in the appendix of [7]. □

Definitions of the exponential stability and the H_∞ performance are similar to the standard one, and thus are omitted. The reader can refer to [7] for an in-depth discussion of the problem.

Based on those performance indicators, criterion for the interconnected systems can be derived from the discrete version of the Kalman-Yakubovich-Popov lemma.

2.1 Stability analysis

As shown in (10), the state of each subsystem is composed of two parts: continuous-time part and discrete spatial part. Thus, the stability analysis should be on consideration of both parts. A remainder of the stability conditions for linear continuous and discrete 1-D/n-D systems can be found in [11, 18].

Theorem 2. The subsystem of SISs described in (11) is stable if there exist symmetric matrices $X_t \in \mathbf{R}^{n_t \times n_t}$ and symmetric matrix $X_s \in \mathbf{R}^{n_s \times n_s}$, where n_t is the size of $x(t,\boldsymbol{s})$ and n_s is the size of $v(t,\boldsymbol{s})$, such that the following two conditions are satisfied.

1) $X_t > 0$

2) $$\begin{bmatrix} A_{TT}^{\mathrm{T}} X_t + X_t A_{TT} & X_t A_{TS} & A_{ST}^{\mathrm{T}} X_s \\ * & -X_s & A_{SS}^{\mathrm{T}} X_s \\ * & * & -X_s \end{bmatrix} < 0 \tag{16}$$

Proof. The stability of the subsystem is a combination of two parts: continuous time Lyapunov criterion and discrete Lyapunov criterion. Thus, the hybrid Lyapunov criterion for the subsystems takes the following form:

$$A^{\mathrm{T}} \hat{X}_1 + \hat{X}_1 A + A^{\mathrm{T}} \hat{X}_2 A - \hat{X}_2 < 0 \tag{17}$$

where

$$A = \begin{bmatrix} A_{TT} & A_{TS} \\ A_{ST} & A_{SS} \end{bmatrix}, \quad \hat{X}_1 = \begin{bmatrix} X_t & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{X}_2 = \begin{bmatrix} 0 & 0 \\ 0 & X_s \end{bmatrix} \tag{18}$$

and $X_t > 0$. (17) is an extension of the Lyapunov inequality for the 2-D continuous-discrete linear systems[20]. Based on (18), (17) is equivalent to

$$\begin{bmatrix} A_{TT}^{\mathrm{T}} X_t + X_t A_{TT} & X_t A_{TS} \\ * & 0 \end{bmatrix} + \begin{bmatrix} A_{ST}^{\mathrm{T}} X_s A_{ST} & A_{ST}^{\mathrm{T}} X_s A_{SS} \\ * & A_{SS}^{\mathrm{T}} X_s A_{SS} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -X_s \end{bmatrix} < 0 \tag{19}$$

Assume that S_{11}, S_{12}, and S_{22} in Schur complement arguments satisfy

$$S_{11} = \begin{bmatrix} A_{TT}^{\mathrm{T}} X_t + X_t A_{TT} & X_t A_{TS} \\ * & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -X_s \end{bmatrix} \tag{20}$$

and

$$S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}} = -\begin{bmatrix} A_{ST}^{\mathrm{T}}X_sA_{ST} & A_{ST}^{\mathrm{T}}X_sA_{SS} \\ * & A_{SS}^{\mathrm{T}}X_sA_{SS} \end{bmatrix} \tag{21}$$

If we set

$$S_{12} = \begin{bmatrix} 0 & A_{ST}^{\mathrm{T}}X_s \\ 0 & A_{SS}^{\mathrm{T}}X_s \end{bmatrix} \tag{22}$$

then, S_{22} takes the following form:

$$S_{22} = \begin{bmatrix} -X_0 & 0 \\ 0 & -X_s \end{bmatrix} \tag{23}$$

where $X_0 > 0$ is an arbitrary symmetric matrix.

Let us consider (20), (22) and (23) as the matrices in Schur complement arguments; then we get the sufficient condition of (17) as

$$\begin{bmatrix} A_{TT}^{\mathrm{T}}X_t + X_tA_{TT} & X_tA_{TS} & 0 & A_{ST}^{\mathrm{T}}X_s \\ * & -X_s & 0 & A_{SS}^{\mathrm{T}}X_s \\ * & * & -X_0 & 0 \\ * & * & * & -X_s \end{bmatrix} < 0 \tag{24}$$

where "$*$" follows symmetry of matrix. (24) is equivalent to (17). Note that the third row and the third column of (24) can be omitted without affecting the existence of the inequality. This operation results in (16), which is the sufficient stability condition for the subsystems in Theorem 2. □

It is well-known that the stability of every subsystem cannot ensure the overall system′s stability. For the decentralized control synthesis, we consider a Lyapunov function for each subsystem which contains the information of all other interconnected subsystems[21]. By doing this, we can find the decentralized controller for the overall system through n LMIs (where n is the number of subsystems). In this paper, the overall system′s stability is transformed into a discussion of the decoupled subsystem′s stability. Recalling the state-space model of subsystems as defined in (10), we know that the interconnected signals are considered as states of the subsystem. Note that the stability criterion in (16) ensures all states in the subsystem, including the interconnected variables, to converge to zero as $t \to \infty$. Thus, we have the following theorem that can be used to testify the overall system′s stability.

Theorem 3. An SIS that is composed of N subsystems is stable if for all $i \in [1, N] \in \mathbf{Z}$, there exist symmetric block diagonal matrices $X_{TT} = \mathrm{diag}\{X_{1t}, \cdots, X_{Nt}\}$, $X_{it} \in \mathbf{R}^{n_{it}\times n_{it}}$, $X_{SS} = \mathrm{diag}\{X_{1s}, \cdots, X_{Ns}\}$, and $X_{is} \in \mathbf{R}^{n_{is}\times n_{is}}$, where n_{it} and n_{is} are the dimensions of the continuous time state and the spatially discrete stat of the i-th subsystem, respectively, such that

(i) $X_{it} > 0$

(ii) $$\begin{bmatrix} A_{iTT}^{\mathrm{T}}X_{it} + X_{it}A_{iTT} & X_{it}A_{iTS} & A_{iST}^{\mathrm{T}}X_{is} \\ * & -X_{is} & A_{iSS}^{\mathrm{T}}X_{is} \\ * & * & -X_{is} \end{bmatrix} < 0 \tag{25}$$

Proof. The hybrid Lyapunov criterion of the i-th subsystem is defined as

$$f_i = A_i^{\mathrm{T}}\hat{X}_{i1} + \hat{X}_{i1}A_i + A_i^{\mathrm{T}}\hat{X}_{i2}A_i - \hat{X}_{i2} < 0 \tag{26}$$

where A_i, X_{i1}, and X_{i2} are the relevant matrices of the i-th subsystem as defined in (18). Thus, the overall subsystem is stable and the hybrid Lyapunov inequality has solutions:

$$f = \mathrm{diag}\{f_1, \cdots, f_N\} < 0 \tag{27}$$

where the states that go with the overall system′s Lyapunov function are

$$\chi = [\chi_1^{\mathrm{T}}, \cdots, \chi_N^{\mathrm{T}}]^{\mathrm{T}} \tag{28}$$

As proved in Theorem 2, (25) is the sufficient condition of (26). Thus, if (25) is satisfied for all $i \in [1, N]$, then, the Lyapunov inequality of the overall system (27) is satisfied, which means that the stability of every subsystem′s Lyapunov function leads to the stability of the overall system. □

2.2 Criteria for the stability and H_∞ performance

Now, we are in a position to give the criterion with which we can test stability and H_∞ performance of the interconnected systems. The real bounded lemma is a crucial basis for our study during H_∞ performance of the system, and in the following context, we extend the real bounded lemma to the field of multidimensional systems.

Theorem 4. The subsystem of SISs described in (10) is said to be stable and satisfy $\|T_{zd}\|_\infty < \gamma$ where $\gamma \in \mathbf{R}^+$, if there exist scaling matrices $X_t = X_t^{\mathrm{T}} > 0, X_s = X_s^{\mathrm{T}}$ such that

$$\begin{bmatrix} A^{\mathrm{T}}\hat{X}_1 + \hat{X}_1A + A^{\mathrm{T}}\hat{X}_2A - \hat{X}_2 & \hat{X}_1B + A^{\mathrm{T}}\hat{X}_2B & C^{\mathrm{T}} \\ * & B^{\mathrm{T}}\hat{X}_2A - \gamma^2I & D^{\mathrm{T}} \\ * & * & -I \end{bmatrix} < 0 \tag{29}$$

where $\hat{X}_1$and $\hat{X}_2$ are defined in (18).

Proof. If there exist $X_t = X_t^{\mathrm{T}} > 0$ and $X_s = X_s^{\mathrm{T}}$ such that (29), then the inequality in (17) is satisfied. This ensures the stability of the subsystem.

We define the Lyapunov function of subsystem (10) as

$$\begin{aligned} V(\chi) = \ & \chi^{\mathrm{T}}(t, \boldsymbol{s}) \begin{bmatrix} X_t & 0 \\ 0 & X_s \end{bmatrix} \chi(t, \boldsymbol{s}) = \\ & \chi^{\mathrm{T}}(t, \boldsymbol{s})\hat{X}_1\chi(t, \boldsymbol{s}) + \chi^{\mathrm{T}}(t, \boldsymbol{s})\hat{X}_2\chi(t, \boldsymbol{s}) \\ & \triangleq V_T(\chi) + V_S(\chi) \end{aligned} \tag{30}$$

where χ is defined in (11). The derivation of the Lyapunov function $\mathcal{D}V(\chi)$ should be divided into two parts: continuous differential part and discrete differential part.

$$\mathcal{D}V(\chi) = \frac{\mathrm{d}}{\mathrm{d}t}V_T(\chi) + \Delta V_S(\chi) \tag{31}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}V_T(\chi) = \chi^{\mathrm{T}}\hat{X}_1(A\chi + Bd) + \chi^{\mathrm{T}}A^{\mathrm{T}} + d^{\mathrm{T}}B^{\mathrm{T}})\hat{X}_1\chi \tag{32}$$

$$\Delta V_S(\chi) = (A\chi + Bd)^{\mathrm{T}}\hat{X}_2(A\chi + Bd) - \chi^{\mathrm{T}}\hat{X}_2\chi \tag{33}$$

A, B, C, and D are defined in (10).

Consider the following function

$$J_{TS} = \|z(t, \boldsymbol{s})\|^2 - \gamma^2\|d(t, \boldsymbol{s})\|^2 \tag{34}$$

we have

$$J_{TS} = [\|z\|^2 - \gamma^2\|d\|^2 + \mathcal{D}V(\chi)] - \mathcal{D}V(\chi) =$$

$$\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix}C^{\mathrm{T}}\\ D^{\mathrm{T}}\end{bmatrix}[C\ D] + \begin{bmatrix}0 & 0\\ 0 & -\gamma^2 I\end{bmatrix}\right)\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix} +$$

$$\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A^{\mathrm{T}}\hat{X}_1 + \hat{X}_1 A & \hat{X}_1 B\\ * & 0\end{bmatrix}\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix} +$$

$$\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A^{\mathrm{T}}\hat{X}_2 A - \hat{X}_2 & A^{\mathrm{T}}\hat{X}_2 B\\ * & B^{\mathrm{T}}\hat{X}^2 A\end{bmatrix}\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix} -$$

$$\mathcal{D}V(\chi) = \begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix}C^{\mathrm{T}}\\ D^{\mathrm{T}}\end{bmatrix}[C\ D] + \begin{bmatrix}A^{\mathrm{T}}\hat{X}_1 + \hat{X}_1 A + A^{\mathrm{T}}\hat{X}_2 A - \hat{X}_2 & \hat{X}_1 B + A^{\mathrm{T}}\hat{X}_2 B\\ * & B^{\mathrm{T}}\hat{X}^2 A - \gamma^2 I\end{bmatrix}\right)\times$$

$$\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix} - \mathcal{D}V(\chi) \tag{35}$$

Applying Schur complement arguments to (29), we get $J_{TS} < -\mathcal{D}V(\chi)$. Thus,

$$\int_0^T \|z\|^2 \mathrm{d}t - \gamma^2 \int_0^T \|d\|^2 \mathrm{d}t + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} V_T(\chi) + \sum_{S_0}^{S_n} \Delta V_S(\chi) < 0 \tag{36}$$

Let us assume $d \in L_2[0,\infty)$ and the zero initial condition, and then, when $T \to +\infty, n \to +\infty$, (36) is congruent to

$$\|z\|_2^2 - \gamma^2\|d\|_2^2 + V(\chi) < 0$$

So, $\|z\|_2^2 < \gamma^2\|d\|_2^2$, and (29) is the sufficient condition for the H_∞ performance. □

The LMI derived in (29) can be considered as a hybrid real bounded lemma. As our goal is to design distributed output feedback controllers, we should transform the hybrid LMI into standard form so that the existing controller synthesis method can be applied. This work has been solved by D′Andrea[7] through a modified bilinear transformation and the result is given in Theorem 1 in this article. Note that in Theorem 1, well-posedness performance is included.

3 Control implementation

Suppose that the open-loop plant we considered in Fig.2 has the following form

$$\begin{bmatrix}\dot{\chi}^G(t,\boldsymbol{s})\\ z(t,\boldsymbol{s})\\ y(t,\boldsymbol{s})\end{bmatrix} = \begin{bmatrix}A^G & B_d^G & B_u^G\\ C_z^G & D_{zd}^G & D_{zu}^G\\ C_y^G & D_{yd}^G & D_{yu}^G\end{bmatrix}\begin{bmatrix}\chi^G(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\\ u(t,\boldsymbol{s})\end{bmatrix}$$

$$\triangleq \begin{bmatrix}A^G & B^G\\ C^G & D^G\end{bmatrix}\begin{bmatrix}\chi^G(t,\boldsymbol{s})\\ \mathrm{d}(t,\boldsymbol{s})\\ u(t,\boldsymbol{s})\end{bmatrix} \triangleq \mathcal{G} \tag{37}$$

The distributed controller for each open-loop subsystem has the same structure as its related subsystems as depicted in Fig. 2, and we can express the state-space equation for the controllers as

$$\begin{bmatrix}\dot{x}^K(t,\boldsymbol{s})\\ w^K(t,\boldsymbol{s})\\ u(t,\boldsymbol{s})\end{bmatrix} = \begin{bmatrix}A_{TT}^K & A_{TS}^K & B_T^K\\ A_{ST}^K & A_{SS}^K & B_S^K\\ C_T^K & C_S^K & D^K\end{bmatrix}\begin{bmatrix}x^K(t,\boldsymbol{s})\\ v^K(t,\boldsymbol{s})\\ y(t,\boldsymbol{s})\end{bmatrix}$$

$$\triangleq \begin{bmatrix}A^K & B^K\\ C^K & D^K\end{bmatrix}\begin{bmatrix}x^K(t,\boldsymbol{s})\\ v^K(t,\boldsymbol{s})\\ y(t,\boldsymbol{s})\end{bmatrix} \triangleq \mathcal{K} \tag{38}$$

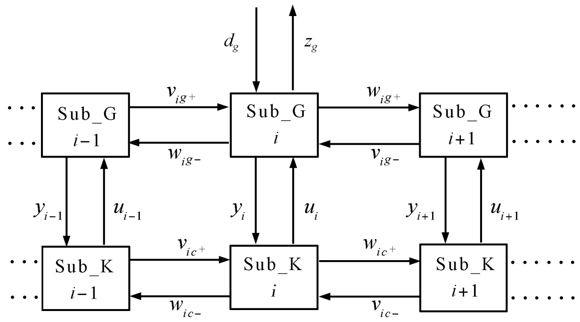

Fig. 2 Basic building block with its controller (one spatial dimension)

By canceling the interconnected signals y and u, the close-loop state space equation can be written as

$$\begin{bmatrix}\begin{bmatrix}\dot{x}^G(t,\boldsymbol{s})\\ \dot{x}^K(t,\boldsymbol{s})\end{bmatrix}\\ \begin{bmatrix}w_+^C(t,\boldsymbol{s})\\ w_-^C(t,\boldsymbol{s})\end{bmatrix}\\ z(t,\boldsymbol{s})\end{bmatrix} = \begin{bmatrix}A_{TT}^C & A_{TS}^C & B_T^C\\ A_{ST}^C & A_{SS}^C & B_S^C\\ C_T^C & C_S^C & D^C\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x^G(t,\boldsymbol{s})\\ x^K(t,\boldsymbol{s})\end{bmatrix}\\ \begin{bmatrix}v_+^C(t,\boldsymbol{s})\\ v_-^C(t,\boldsymbol{s})\end{bmatrix}\\ d(t,\boldsymbol{s})\end{bmatrix} \tag{39}$$

$$\triangleq \begin{bmatrix}A^C & B^C\\ C^C & D^C\end{bmatrix}\begin{bmatrix}\chi(t,\boldsymbol{s})\\ d(t,\boldsymbol{s})\end{bmatrix} \triangleq \mathcal{C}$$

Denote the modified bilinear transformations of $\mathcal{G}$ and $\mathcal{K}$ as $\overline{\mathcal{G}}$ and $\overline{\mathcal{K}}$, and denote the relevant close-loop system as $\overline{\mathcal{C}}$. Thus, we can express the inequality in (12) as

$$\begin{bmatrix}(\bar{A}^C)^{\mathrm{T}}\bar{X} + \bar{X}\bar{A}^C & \bar{X}\bar{B}^C & (\bar{C}^C)^{\mathrm{T}}\\ (\bar{B}^C)^{\mathrm{T}}\bar{X} & -I & (\bar{D}^C)^{\mathrm{T}}\\ \bar{C}^C & \bar{D}^C & -I\end{bmatrix} < 0 \tag{40}$$

where $f_{D2C}(\mathcal{M}^c) = \bar{\mathcal{M}}^C = \{\bar{A}^C, \bar{B}^C, \bar{C}^C, \bar{D}^C\}$. Note that $\bar{A}^C, \bar{B}^C, \bar{C}^C$, and $\bar{D}^C$ are affine transforms of $\overline{\mathcal{K}}$. Hence, the matrix inequality (40) is a biaffine matrix inequality on the variables $\overline{\mathcal{K}}$ and $\bar{X}$. The biaffine matrix inequality problem is non-convex and known to be NP-hard to solve[22].

To solve this problem, we have two methods known as elimination algorithm[7, 14] and change variable algorithm[12]. In the following, we discuss the change variable algorithm, by which we can translate the nonlinear inequality into linear inequality and thus get the $H_\infty\gamma$-suboptimal distributed controller for each subsystem.

Theorem 5. For a given γ, a close-loop subsystem $f_{D2C}(\mathcal{M}^c) = \bar{\mathcal{M}}^C = \{\bar{A}^C, \bar{B}^C, \bar{C}^C, \bar{D}^C\}$ is said to be well-posed, stable, and satisfying $\|T_{zd}\|_\infty < \gamma$ if there exist $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$ with suitable dimensions and scaling matrixes $\hat{X}$ and $\hat{Y}$ such that the following two conditions are satisfied

1)

$$\begin{bmatrix}\Psi_{11} & \Psi_{21}^{\mathrm{T}}\\ \Psi_{21} & \Psi_{22}\end{bmatrix} < 0 \tag{41}$$

with the shorthand notation

$$\Psi_{11}(:,1) = \begin{bmatrix}\bar{A}^G X^G + X^G(\bar{A}^G)^{\mathrm{T}} + \bar{B}_u^G\hat{C} + (\bar{B}_u^G\bar{C})^{\mathrm{T}}\\ (\hat{A}^{\mathrm{T}} + (\bar{A}^G + \bar{B}_u^G\hat{D}\bar{C}_y^G))^{\mathrm{T}}\end{bmatrix}$$

$$\Psi_{11}(:,2) = \begin{bmatrix} \hat{A}^{\mathrm{T}} + (\bar{A}^G + \bar{B}_u^G \hat{D} \bar{C}_y^G) \\ (\bar{A}^G)^{\mathrm{T}} Y^G + Y^G \bar{A}^G + \hat{B} \bar{C}_y^G + (\hat{B} \bar{C}_y^G)^{\mathrm{T}} \end{bmatrix}$$

$$\Psi_{21} = \begin{bmatrix} (\bar{B}_d^G + \bar{B}_u^G \hat{D} \bar{D}_{yd}^G)^{\mathrm{T}} & (Y^G \bar{B}_d + \hat{B} \bar{D}_{yd}^G)^{\mathrm{T}} \\ \bar{C}_z^G X^G + \bar{D}_{zu}^G \hat{C} & \bar{C}_z^G + \bar{D}_{zu}^G \hat{D} \bar{C}_y \end{bmatrix}$$

$$\Psi_{22} = \begin{bmatrix} -\gamma I & (\bar{D}_{zd}^G + \bar{D}_{zd}^G \hat{D} \bar{D}_{yd})^{\mathrm{T}} \\ * & -\gamma I \end{bmatrix}$$

where "$*$" follows the symmetry of the matrix.

2)

$$\begin{bmatrix} X_T^G & I \\ I & Y_T^G \end{bmatrix} \geq 0 \tag{42}$$

where

$$X^G = (X^G)^{\mathrm{T}} = \mathrm{diag}\ \{X_T^G,\ X_X^G\}$$

$$Y^G = (Y^G)^{\mathrm{T}} = \mathrm{diag}\ \{Y_T^G,\ Y_{\boldsymbol{s}}^G\}$$

$$\hat{X} = \begin{bmatrix} Y^G & N \\ N^{\mathrm{T}} & W \end{bmatrix}, \quad \hat{X}^{-1} = \begin{bmatrix} X^G & M \\ M^{\mathrm{T}} & Z \end{bmatrix} \tag{43}$$

and

$$\hat{A} = Y^G(\bar{A}^G + \bar{B}_u^G \bar{D}^K \bar{C}_y^G) X^G + N \bar{B}^K \bar{C}_y X^G + Y^G \bar{B}_u^G \bar{C}^K M^{\mathrm{T}} + N \bar{A}^K M^{\mathrm{T}} \tag{44}$$

$$\hat{B} = Y^G \bar{B}_u^G \bar{D}^K + N \bar{B}^K \tag{45}$$

$$\hat{C} = \bar{D}^K \bar{C}_y X^G + \bar{C}^K M^{\mathrm{T}} \tag{46}$$

$$\hat{D} = \bar{D}^K \tag{47}$$

And the controller for each subsystem can be obtained by inverse substitution of (44) $\sim$ (47).

Proof. From $\hat{X} \times \hat{X}^{-1} = I$ and (43), we have

$$\hat{X} \begin{bmatrix} X^G & I \\ M^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} I & Y^G \\ 0 & N^{\mathrm{T}} \end{bmatrix} \tag{48}$$

If we define $F_1 = \begin{bmatrix} X^G & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}$ and $F_2 = \begin{bmatrix} I & Y^G \\ 0 & N^{\mathrm{T}} \end{bmatrix}$, we can calculate the following equalities

$$F_1^{\mathrm{T}} \hat{X} \bar{A}^C F_1 = F_2^{\mathrm{T}} \bar{A}^C F_1 = \begin{bmatrix} \Theta_1 & \Theta_2 \end{bmatrix} \tag{49}$$

where

$$\Theta_1 = \begin{bmatrix} \bar{A}^G X + \bar{B}_u^G (\bar{D}^K \bar{C}_y^G X^G + \bar{C}^K M^{\mathrm{T}}) \\ Y^G(\bar{A}^G + \bar{B}_u^G \bar{D}^K \bar{C}_y^G) X^G + N \bar{B}^K \bar{C}_y X^G + \\ Y^G \bar{B}_u^G \bar{C}^K M^{\mathrm{T}} + N \bar{A}^K M^{\mathrm{T}} \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} \bar{A}^G + \bar{B}_u^G \bar{D}^K \bar{C}_y^G \\ Y^G \bar{A}^G + (Y^G \bar{B}_u^G \bar{D}^K + N \bar{B}^K) \bar{C}_y^G \end{bmatrix}$$

and

$$F_1^{\mathrm{T}} \bar{X} \bar{B}^C = \begin{bmatrix} \bar{B}_u^G \bar{D}^K \bar{D}_{yd}^G + \bar{B}_d^G \\ (Y^G \bar{B}_u^G \bar{D}^K + N \bar{B}^K) \bar{D}_{yd}^G + Y^G \bar{B}_d^G \end{bmatrix} \tag{50}$$

$$\bar{C}^C F_1(1,1) = \bar{C}_z X^G + \bar{D}_{zu}^G (\bar{D}^K \bar{C}_y X^G + \bar{C}^K M^{\mathrm{T}}) \tag{51}$$

$$\bar{C}^C F_1(1,2) = \bar{C}_z + \bar{D}_{zu}^G \bar{D}^K \bar{C}_y \tag{52}$$

$$F_1^{\mathrm{T}} \bar{X} F_1 = F_2^{\mathrm{T}} F_1 = \begin{bmatrix} X^G & I \\ I & Y^G \end{bmatrix} \tag{53}$$

Note that for given X^G, Y^G and their associated full rank matrixes M and N, A^K, B^K, C^K, and D^K can uniquely be determined by $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$.

If we pre- and post-multiply (40) pre- and post-by diag $\{F_1^{\mathrm{T}}, I, I\}$ and diag$\{F_1, I, I\}$, respectively, then, we can get the result in (41) through equalities (44) $\sim$ (48).□

Furthermore, we can get the H_∞ optimal controller by minimizing γ in (41). This can easily be implemented by the LMI-toolbox[23].

The elimination algorithm introduced in [7] and [14] requires to discuss feasibility of three LMIs firstly and one LMI secondly, which have numerical shortcomings and may sometimes lead to infeasible results (An example given in Section 4 demonstrates this property). In contrast to the elimination algorithm, the change variable algorithm contains only one LMI, and this LMI gives a sufficient and necessary condition for the existence of distributed output feedback controllers. Furthermore, the controllers can be obtained directly from the feasible solutions of the LMI. Another advantage of this simple and convenient method is that more restrictions can be combined together to deal with multi-objects design[12].

4 Numerical example

In this section, we apply the distributed control design method to a spring-mass system using the change variable algorithm. The framework of the system is depicted in Fig. 3.

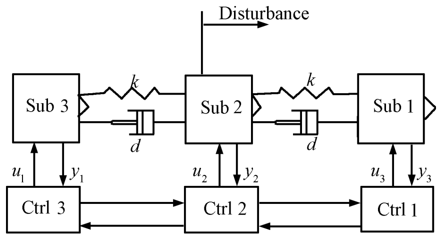

Fig. 3 Distributed control architecture of the spring-mass system with pulse disturbance on block 2

The kinematic equation of the i-th block is

$$M\ddot{x}_i = \sum_{j=1, j\neq i}^{3} \delta_{ij}[K(x_j - x_i) + D(\dot{x}_j - \dot{x}_i)]$$

$$\delta_{ij} = \begin{cases} 1, & |i-j| = 1 \\ 0, & \text{else} \end{cases}, \quad i, j = 1, 2, 3 \tag{54}$$

where x_i is the bias position of the i-th block, u_i is the control signal, K and D are the coefficients of the spring and the damper, respectively, and M is the mass of each block. Let us set the state variable in the time domain as $r_{i1} = x_i, r_{i2} = \dot{x}_i$, and $A_{TT} \in \mathbf{R}^{2\times 2}$, $A_{SS} \in \mathbf{R}^{2\times 2}$. We then get the three blocks' models (55) $\sim$ (57).

$$\text{Sub 1:} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ -\dfrac{k}{m} & -\dfrac{d}{m} & 0 & 1 & 0 & \dfrac{1}{m} \\ \dfrac{k}{m} & \dfrac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$\text{Sub 2: } \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ -\dfrac{2k}{m} & -\dfrac{2d}{m} & 1 & 1 & 0 & \dfrac{1}{m} \\ \dfrac{k}{m} & \dfrac{d}{m} & 0 & 0 & 0 & 0 \\ \dfrac{k}{m} & \dfrac{d}{m} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

$$\text{Sub 3: } \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ -\dfrac{k}{m} & -\dfrac{d}{m} & 1 & 0 & 0 & \dfrac{1}{m} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{k}{m} & \dfrac{d}{m} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{57}$$

The equilibrium state of this system is when the distances between Sub 1 and Sub 2, Sub 2 and Sub 3 are equal to the natural length of the spring. But this equilibrium can easily be destroyed with a tiny disturbance. Thus, we design both decentralized controller and distributed controller for every subsystem so that the overall system could be stabilized under disturbance. Set $K = 2$, $D = 1$, and $M = 4$. The decentralized control strategy is

$$\begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} = C_i \begin{bmatrix} r_{i1} \\ r_{i2} \end{bmatrix}, \quad i = 1, 2, 3$$

where C_i are the controllers that are designed using the methods introduced in [21]. The controller realization results are shown in (58) $\sim$ (60). The distributed controllers are designed using the algorithm we presented in Section 3 and the results are shown in (61) $\sim$ (63).

$$C_1 : [-1.997 \quad -0.157] \tag{58}$$

$$C_2 : [-3.958 \quad -0.714] \tag{59}$$

$$C_3 : [-1.997 \quad -0.157] \tag{60}$$

$$D_1: \begin{bmatrix} -3.182 & -6.277 & -0.023 & 0.044 & 0.042 \\ 14.543 & -13.167 & 0.057 & 0.033 & 9.446 \\ -2.384 & -0.245 & -0.003 & -0.891 & -0.787 \\ -8.844 & 2.436 & -1.256 & -0.007 & -3.088 \\ -91.033 & -119.178 & -0.503 & 0.647 & -25.672 \end{bmatrix} \tag{61}$$

$$D_2: \begin{bmatrix} -0.673 & -5.347 & 0.004 & -0.004 & -0.010 \\ 10.115 & -10.147 & 0.002 & -0.002 & 6.704 \\ 8.791 & -0.239 & 0.001 & 0.844 & 3.109 \\ -8.794 & 0.239 & 0.845 & 0.001 & -3.110 \\ -12.948 & -135.715 & -0.108 & 0.117 & -0.215 \end{bmatrix} \tag{62}$$

$$D_3: \begin{bmatrix} -1.972 & -3.553 & 0.012 & -0.011 & -1.303 \\ 12.695 & -16.71 & 0.02 & -0.503 & 11.011 \\ -0.288 & 0.533 & 0.009 & 0.385 & 4.085 \\ -1.984 & -0.993 & -0.187 & 0.046 & -1.166 \\ -14.344 & -43.523 & -0.834 & -0.344 & -9.335 \end{bmatrix} \tag{63}$$

The control effects are shown in Figs. 4 and 5, where x-axis stands for the time sequence and y-axis the equilibrium displacement. From Figs. 4 and 5, we can see clearly that with the distributed controllers, the second block can be driven back to the equivalence point at 20 s with an error 0.00017 m, in comparison with the decentralized controller which made the block still oscillate at time 50 s. Actually, it takes approximately 75 s for the decentralized system to get to the equivalence point with an error 0.00066 m.

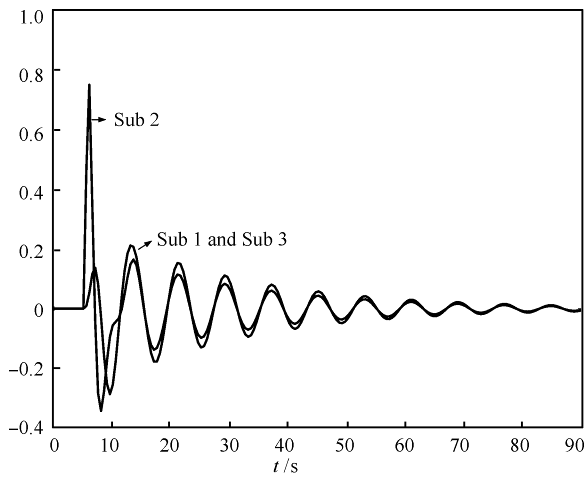

Fig. 4 Three blocks responses under pulse disturbance (amplitude=1, width=1 s) on Sub 2 (decentralized control)

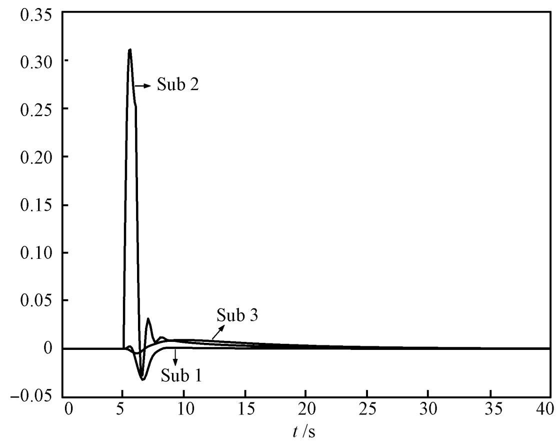

Fig. 5 Three blocks responses under pulse disturbance (amplitude=1, width=1 s) on Sub 2 (distributed control)

With distributed controllers, the maximum equilibrium only amounts to 0.31 m in response to a pulse amplitude of 1 m, in comparison with 0.79 m under decentralized control.

From the experimental results shown above, we can see that compared with decentralized controllers, the distributed control architecture performs more optimistically both in response time and system gain.

One thing should be pointed out is that during distributed controller synthesis, the elimination algorithm will lead to an infeasible solution in LMI-toolbox, while the change variable algorithm introduced in this paper can successfully solve this problem.

5 Concluding remarks

This paper has discussed the distributed control design problem and applied the control implementation algorithm to a spring-mass system. Hybrid version of Lyapunov equation and the real bounded lemma for the SISs were derived, and the stability of the overall system was proved by Lyapunov criterion. It has been shown through the spring-mass system that the distributed control architecture performs more optimistically both in response time and systems gain than decentralized controllers. The introduction of the change variable algorithm successfully solved the distributed controllers design problem, while the elimination algorithm introduced in [7] leads to an infeasible solution.

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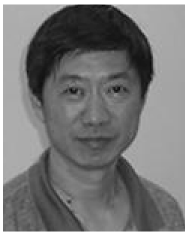

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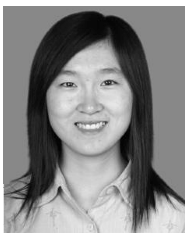

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