Improved Results on Delay-dependent H_∞ Control for Singular Time-delay Systems

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Abstract The problem of delay-dependent H_∞ control for singular systems with state delay is discussed. In terms of linear matrix inequality (LMI) approach, a delay-dependent bounded real lemma (BRL) is presented to ensure the system to be regular, impulse free, and stable with H_∞ performance condition via an augmented Lyapunov functional. Based on the BRL obtained, the delay-dependent condition for the existence of H_∞ state feedback controller is presented via strict LMI. An explicit expression for the desired state feedback controller is also given. Numerical examples are presented to illustrate the significant improvement on the conservativeness of some reported results in the literature.

Key words Singular systems, time-delay systems, delay-dependent, H_∞ control, linear matrix inequality (LMI)

In the past few years, singular time-delay systems have been extensively studied by many researchers from mathematics and control communities because they can better describe and analyze physical systems than the state-space time-delay ones[1]. The singular time-delay system contains three kinds of modes, that is, finite dynamic modes, infinite dynamic modes, which generate the undesired impulsive behavior of the system, and nondynamic modes, while the latter two do not appear in the state-space time-delay system[2]. Therefore, the study for such systems is much more complicated than that for state-space systems.

Many problems for singular time-delay systems have been tackled, and a great number of fundamental concepts and results based on the theory of state-space time-delay systems have been successfully extended to singular time-delay systems. For instance, several delay-independent bounded real lemmas (BRLs) for singular time-delay systems were provided, and the delay-independent H_∞ control problem was solved via state feedback controller in [2−3]. Recent research effort is focused more on the study for delay-dependent H_∞ control of singular time-delay systems, because it has been shown that delay-dependent results are less conservative than delay-independent ones, especially in the case where time delays are small. The problem of delay-dependent H_∞ control for singular time-delay systems was solved in [4−6] in terms of LMI approach. Several delay-dependent BRLs were obtained and the design algorithms of desired controllers, including state feedback controllers, filters, and output feedback controllers, were also given. It should be pointed out that in [4−5] decomposition and transformation of the original system coefficient matrices are required, which makes the analysis and design procedures complex and unreliable. The delay-dependent robust H_∞ control problem for singular time-delay systems was discussed in [7−9] when norm-bounded parameter uncertainties arose, and some delay-dependent BRLs and sufficient conditions for the solvability of this problem were also obtained. Recently, a free-weighting matrix method, which is based on Leibniz-Newton formula, has been proposed to efficiently improve the delay-dependent results for state-space time-delay systems[10−11], in which the bounding techniques on some cross product terms are no longer involved. The free-weighting matrix method was used to deal with the delay-dependent H_∞ control problem for singular time-delay systems in [12−13], and the obtained results have improved the conservativeness of the results of [4−9] to a certain extent. But it is should be pointed out that the proposed results in [6−9, 12−13] are all formulated in terms of non-strict LMIs whose solutions are difficult to calculate since equality constraints are often fragile and usually do not met perfectly.

In this paper, an augmented Lyapunov functional is proposed to discuss the delay-dependent H_∞ control problem for singular time-delay systems. Owing to the augmented Lyapunov functional, an improved delay-dependent BRL is derived which guarantees the singular time-delay systems to be regular, impulse free, and delay-dependent stable while satisfying a prescribed H_∞ performance level. Based on the BRL obtained, a strict LMI-based method is proposed to solve the delay-dependent H_∞ control problem and the desired state feedback controllers can be constructed by solving a set of strict LMIs. Numerical examples show that the proposed methods are much less conservative than the existing corresponding ones in the literature.

Notations. $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, $\mathbf{R}^{m\times n}$ is the set of all $m\times n$ real matrices. $C_{n,d} = C([-d,0],\mathbf{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into $\mathbf{R}^n$. $\mathcal{L}_2[0,\infty)$ stands for the space of square integrable functions on $[0,\infty)$. $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm and $\|\phi(t)\|_d = \sup_{-d\le t\le 0}\|\phi(t)\|$ stands for the norm of a function $\phi(t)\in C_{n,d}$. $\rho(M)$ denotes the spectral radius of the matrix M. The superscript "T" represents the transpose and "$*$" denotes the term that is induced by symmetry.

1 Problem formulation

Consider uncertain singular time-delay systems:

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d) + B\boldsymbol{u}(t) + B_\omega\boldsymbol{\omega}(t)\\ \boldsymbol{z}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t)\\ \boldsymbol{x}(t) &= \boldsymbol{\phi}(t),\ t\in[-\bar{d},0] \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbf{R}^n$ is the state, $\boldsymbol{u}(t)\in\mathbf{R}^m$ is the control input, $\boldsymbol{\omega}(t)\in\mathbf{R}^p$ is the disturbance input that belongs to $\mathcal{L}_2[0,\infty)$, $\boldsymbol{z}(t)\in\mathbf{R}^s$ is the controlled output, d is an unknown but constant delay satisfying $0\le d\le\bar{d}$, and $\boldsymbol{\phi}(t)\in C_{n,\bar{d}}$ is a compatible vector valued initial function. The matrix $E\in\mathbf{R}^{n\times n}$ may be singular and it is assumed that rank $E = r\le n$. A, A_d, B, B_ω, C, and D are known real constant matrices with appropriate dimensions.

Received April 16, 2008; in revised form August 28, 2008
Supported by National High Technology Research and Development Program of China (863 Program) (2006AA04Z182), National Natural Science Foundation of China (60736021), and National Creative Research Groups Science Foundation of China (60721062)
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DOI: 10.3724/SP.J.1004.2009.01101

Definition 1[13]**.** For a given scalar $\bar{d} > 0$, the singular time-delay system

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d) \\ \boldsymbol{x}(t) &= \boldsymbol{\phi}(t),\ t \in [-\bar{d}, 0] \end{aligned} \tag{2}$$

is said to be regular and impulse free for any constant time delay d satisfying $0 \le d \le \bar{d}$, if the pairs (E, A) and $(E, A + A_d)$ are regular and impulse free.

We are interested in designing a state feedback controller

$$\boldsymbol{u}(t) = K\boldsymbol{x}(t) \tag{3}$$

where $K \in \mathbf{R}^{m\times n}$ is a matrix to be determined. Then, the aim of this paper is for prescribed scalars $\bar{d} > 0$ and $\gamma > 0$, to develop a state feedback controller (3) such that for any constant time-delay d satisfying $0 \le d \le \bar{d}$, the following requirements are satisfied:

1) The closed-loop system with $\boldsymbol{\omega}(t) \equiv 0$ is regular, impulse free, and stable;

2) The closed-loop system possesses H_∞ performance γ, that is, under the zero initial condition, the closed-loop system satisfies

$$J_{z\omega} = \int_0^\infty \left(\boldsymbol{z}^{\rm T}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\rm T}(t)\boldsymbol{\omega}(t)\right) {\rm d}t < 0 \tag{4}$$

for any nonzero $\boldsymbol{\omega}(t) \in \mathcal{L}_2[0, \infty)$.

To end this section, we propose the following lemma which will be used in the proof of our main results.

Lemma 1. For any symmetric positive-definite constant matrices Z_1, $Z_3 \in \mathbf{R}^{n\times n}$, any constant matrices Z_2, $E \in \mathbf{R}^{n\times n}$, $\begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} > 0$, and a scalar $d > 0$, if there exists a vector function $\boldsymbol{x}(\alpha) : [0, \gamma] \to \mathbf{R}^n$ such that the integrations concerned are well defined, then

$$-d\int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\rm T} \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} {\rm d}\alpha \le$$
$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^t \boldsymbol{x}(\alpha){\rm d}\alpha \end{bmatrix}^{\rm T} \begin{bmatrix} -E^{\rm T}Z_3E & E^{\rm T}Z_3E & -E^{\rm T}Z_2^{\rm T} \\ E^{\rm T}Z_3E & -E^{\rm T}Z_3E & E^{\rm T}Z_2^{\rm T} \\ -Z_2E & Z_2E & -Z_1 \end{bmatrix} \times$$
$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^t \boldsymbol{x}(\alpha){\rm d}\alpha \end{bmatrix}$$

Proof. According to Jensen integral inequality[14], we have

$$-d\int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\rm T} \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} {\rm d}\alpha \le$$
$$-\int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\rm T} {\rm d}\alpha \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} {\rm d}\alpha =$$
$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^t \boldsymbol{x}(\alpha){\rm d}\alpha \end{bmatrix}^{\rm T} \begin{bmatrix} -E^{\rm T}Z_3E & E^{\rm T}Z_3E & -E^{\rm T}Z_2^{\rm T} \\ E^{\rm T}Z_3E & -E^{\rm T}Z_3E & E^{\rm T}Z_2^{\rm T} \\ -Z_2E & Z_2E & -Z_1 \end{bmatrix} \times$$
$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^t \boldsymbol{x}(\alpha){\rm d}\alpha \end{bmatrix}$$

□

Remark 1. Lemma 1 will play a key role in the derivation of a less conservative delay-dependent BRL, and the additional matrix Z_2 will give a potential relaxation.

2 Main results

In this section, we will solve the delay-dependent H_∞ control problem for the singular time-delay systems (1) in terms of LMI approach. Initially, a delay-dependent BRL is given which guarantees the singular time-delay systems (1) with $\boldsymbol{u}(t) \equiv 0$ to be regular, impulse free, and delay-dependent stable while satisfying a prescribed H_∞ performance γ.

Theorem 1. For prescribed scalars $\bar{d} > 0$ and $\gamma > 0$, the singular time-delay system (1) with $\boldsymbol{u}(t) \equiv 0$ is regular, impulse free, and stable with H_∞ performance γ for any constant time delay d satisfying $0 \le d \le \bar{d}$, if there exist symmetric positive-definite matrices Q, $\begin{bmatrix} P_1 & P_2 \\ P_2^{\rm T} & P_3 \end{bmatrix}$, $\begin{bmatrix} Z_1 & Z_2 \\ Z_2^{\rm T} & Z_3 \end{bmatrix}$, and matrices S, T_1, T_2 such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & T_1^{\rm T}B_\omega & C^{\rm T} \\ * & \Xi_{22} & T_2^{\rm T}A_d & P_2 & T_2^{\rm T}B_\omega & 0 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 \\ * & * & * & -Z_1 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{5}$$

where $R \in \mathbf{R}^{n\times(n-r)}$ is any matrix with full column and satisfies $E^TR = 0$ and

$$\begin{aligned}
\Xi_{11} &= T_1^{\rm T}A + A^{\rm T}T_1 + E^{\rm T}P_2 + P_2^{\rm T}E + Q - E^{\rm T}Z_3E + \bar{d}^2 Z_1 \\
\Xi_{12} &= E^{\rm T}P_1 + SR^{\rm T} - T_1^{\rm T} + A^{\rm T}T_2 + \bar{d}^2 Z_2 \\
\Xi_{13} &= T_1^{\rm T}A_d + E^{\rm T}Z_3E - E^{\rm T}P_2 \\
\Xi_{14} &= P_3 - E^{\rm T}Z_2^{\rm T} \\
\Xi_{22} &= -T_2 - T_2^{\rm T} + \bar{d}^2 Z_3 \\
\Xi_{33} &= -Q - E^{\rm T}Z_3E \\
\Xi_{34} &= -P_3 + E^{\rm T}Z_2^{\rm T}
\end{aligned}$$

Proof. First, we prove the singular time-delay system (1) with $\boldsymbol{u}(t) \equiv 0$ is regular, impulse free, and stable. To this end, we consider system (2). It follows from (5) that

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & T_2^{\rm T}A_d \\ * & * & \Xi_{33} \end{bmatrix} - \bar{d}^2 \begin{bmatrix} Z_1 & Z_2 & 0 \\ * & Z_3 & 0 \\ * & * & 0 \end{bmatrix} < 0 \tag{6}$$

Letting

$$V = \begin{bmatrix} I & A^{\rm T} & 0 \\ 0 & A_d^{\rm T} & I \end{bmatrix}$$

and pre- and post-multiplying (6) by V and $V^{\rm T}$, respectively, we get

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ * & \Xi_{33} \end{bmatrix} < 0 \tag{7}$$

where

$$\begin{aligned}
\Upsilon_{11} &= E^{\rm T}P_2 + P_2^{\rm T}E + Q - E^{\rm T}Z_3E + E^{\rm T}P_1A + \\
&\quad SR^{\rm T}A + A^{\rm T}P_1E + A^{\rm T}RS^{\rm T} \\
\Upsilon_{12} &= -E^{\rm T}P_2 + E^{\rm T}Z_3E + E^{\rm T}P_1A_d + SR^{\rm T}A_d
\end{aligned}$$

Choose two nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

Noting that $E^{\mathrm{T}}R = 0$ and $\operatorname{rank} R = n - r$, we can get

$$M^{-\mathrm{T}}R = \begin{bmatrix} 0 \\ H \end{bmatrix} \tag{9}$$

where $H \in \mathbf{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix. Write

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad N^{\mathrm{T}}S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \tag{10}$$

Pre- and post-multiplying $\Upsilon_{11} < 0$ by N^{T} and N, respectively, and then using the expressions in (8) $\sim$ (10), we have

$$S_2H^{\mathrm{T}}A_{22} + A_{22}^{\mathrm{T}}HS_2^{\mathrm{T}} < 0$$

which implies A_{22} is nonsingular. Therefore, the pair (E, A) is regular and impulse free. Pre-and post-multiplying (7) by $\begin{bmatrix} I & I \end{bmatrix}$ and $\begin{bmatrix} I & I \end{bmatrix}^{\mathrm{T}}$, respectively, we get

$$(E^{\mathrm{T}}P_1 + SR^{\mathrm{T}})(A + A_d) + (A + A_d)^{\mathrm{T}}(P_1E + RS^{\mathrm{T}}) < 0 \tag{11}$$

Using the same method, we can find (11) implies the pair $(E, A + A_d)$ is regular and impulse free. Thus, according to Definition 1, the singular time-delay system (2) is regular and impulse free for any constant time delay d satisfying $0 \le d \le \bar{d}$.

Next, we shall show the stability of the singular time-delay system (2). For any $t \ge d$, choose a Lyapunov functional candidate to be

$$V(\boldsymbol{x}_t) = V_1(\boldsymbol{x}_t) + V_2(\boldsymbol{x}_t) + V_3(\boldsymbol{x}_t) \tag{12}$$

where

$$V_1(\boldsymbol{x}_t) = \begin{bmatrix} E\boldsymbol{x}(t) \\ \int_{t-d}^{t} \boldsymbol{x}(\alpha)\mathrm{d}\alpha \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix} \begin{bmatrix} E\boldsymbol{x}(t) \\ \int_{t-d}^{t} \boldsymbol{x}(\alpha)\mathrm{d}\alpha \end{bmatrix}$$

$$V_2(\boldsymbol{x}_t) = \int_{t-d}^{t} \boldsymbol{x}^{\mathrm{T}}(\alpha)Q\boldsymbol{x}(\alpha)\mathrm{d}\alpha$$

$$V_3(\boldsymbol{x}_t) = d\int_{-d}^{0}\int_{t+\beta}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} \mathrm{d}\alpha\mathrm{d}\beta$$

where $\boldsymbol{x}_t = \boldsymbol{x}(t+\theta)$ and $-2\bar{d} \le \theta \le 0$. Then, the time-derivative of $V(\boldsymbol{x}_t)$ along the solution of system (2) gives

$$\dot{V}_1(\boldsymbol{x}_t) = 2\begin{bmatrix} \boldsymbol{x}(t) \\ \int_{t-d}^{t} \boldsymbol{x}(\alpha)\,\mathrm{d}\alpha \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} E^{\mathrm{T}}P_1 + SR^{\mathrm{T}} & E^{\mathrm{T}}P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix} \times \begin{bmatrix} E\dot{\boldsymbol{x}}(t) \\ \boldsymbol{x}(t) - \boldsymbol{x}(t-d) \end{bmatrix}$$

$$\dot{V}_2(\boldsymbol{x}_t) = \boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t-d)Q\boldsymbol{x}(t-d)$$

$$\dot{V}_3(\boldsymbol{x}_t) \le \bar{d}^2\begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix} - \bar{d}\int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} \mathrm{d}\alpha$$

On the other hand, for any appropriately dimensional matrices T_1 and T_2, the following equation is true:

$$\alpha(t) = 2\left[\boldsymbol{x}^{\mathrm{T}}(t)T_1^{\mathrm{T}} + (E\dot{\boldsymbol{x}}(t))^{\mathrm{T}}T_2^{\mathrm{T}}\right] \times [E\dot{\boldsymbol{x}}(t) + A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d)] \equiv 0$$

Hence, taking Lemma 1 into account, we have that there exists a scalar $\lambda > 0$ such that

$$\dot{V}(\boldsymbol{x}_t) = \dot{V}_1(\boldsymbol{x}_t) + \dot{V}_2(\boldsymbol{x}_t) + \dot{V}_3(\boldsymbol{x}_t) + \alpha(t) \le \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^{t}\boldsymbol{x}(\alpha)\mathrm{d}\alpha \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & T_2^{\mathrm{T}}A_d & P_2 \\ * & * & \Xi_{33} & \Xi_{34} \\ * & * & * & -Z_1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^{t}\boldsymbol{x}(\alpha)\mathrm{d}\alpha \end{bmatrix} < -\lambda\|\boldsymbol{x}(t)\|^2 \tag{13}$$

Note that the regularity and the absence of impulses of the pair (E, A) imply that there always exist two nonsingular matrices $\tilde{M}$ and $\tilde{N}$ such that

$$\tilde{M}E\tilde{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tilde{M}A\tilde{N} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{14}$$

Write

$$\begin{aligned} \tilde{M}A_d\tilde{N} &= \begin{bmatrix} A_{d1} & A_{d2} \\ A_{d3} & A_{d4} \end{bmatrix} \\ \tilde{N}^{\mathrm{T}}Q\tilde{N} &= \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \\ \tilde{N}^{\mathrm{T}}S &= \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix} \\ \tilde{M}^{-\mathrm{T}}R &= \begin{bmatrix} 0 \\ \tilde{H} \end{bmatrix} \end{aligned} \tag{15}$$

where $\tilde{H} \in \mathbf{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix. Pre- and post- multiplying (7) by $\begin{bmatrix} \tilde{N} & 0 \\ 0 & \tilde{N} \end{bmatrix}^{\mathrm{T}}$ and $\begin{bmatrix} \tilde{N} & 0 \\ 0 & \tilde{N} \end{bmatrix}$, respectively, and then using the expressions in (14) and (15), we have

$$\begin{bmatrix} S_{21}\tilde{H}^{\mathrm{T}} + \tilde{H}S_{21}^{\mathrm{T}} + Q_{22} & S_{21}\tilde{H}^{\mathrm{T}}A_{d4} \\ * & -Q_{22} \end{bmatrix} < 0 \tag{16}$$

which implies[15−16]

$$\rho(A_{d4}) < 1 \tag{17}$$

Noting this and noting (12) and (13), and following a line similar to that in the proof of Theorem 1 in [15−16], we can deduce that the singular time-delay system (2) is stable for any constant time delay d satisfying $0 \le d \le \bar{d}$.

In the following, we will establish the H_∞ performance of the singular time-delay systems (1) with $\boldsymbol{u}(t) \equiv 0$. Under the zero initial condition, it can be shown that for any nonzero $\omega(t) \in \mathcal{L}_2[0, \infty)$,

$$J_{z\omega} = \int_0^\infty \left(\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t)\right)\,\mathrm{d}t \le \int_0^\infty \left(\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) + \dot{V}(\boldsymbol{x}_t)\right)\,\mathrm{d}t \le \int_0^\infty \boldsymbol{\zeta}^{\mathrm{T}}(t)\Omega\boldsymbol{\zeta}(t)\,\mathrm{d}t$$

where

$$\boldsymbol{\zeta}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ \boldsymbol{x}(t-d) \\ \int_{t-d}^{t} \boldsymbol{x}(\alpha)\mathrm{d}\alpha \\ \boldsymbol{\omega}(t) \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \Xi_{11} + C^{\mathrm{T}}C & \Xi_{12} & \Xi_{13} & \Xi_{14} & T_1^{\mathrm{T}}B_\omega \\ * & \Xi_{22} & T_2^{\mathrm{T}}A_d & P_2 & T_2^{\mathrm{T}}B_\omega \\ * & * & \Xi_{33} & \Xi_{34} & 0 \\ * & * & * & -Z_1 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

By applying the Schur complement to (5), we have $\Omega < 0$. Therefore, $J_{z\omega} < 0$ for any nonzero $\boldsymbol{\omega}(t) \in \mathcal{L}_2[0, \infty)$. □

Remark 2. Theorem 1 proposes a new version of the bound real lemma (BRL) for the singular time-delay systems (1) with $\boldsymbol{u}(t) \equiv 0$ to be regular, impulse free, and delay-dependently stable with H_∞ performance γ in terms of strict LMI formulated by all the coefficient matrices of the original system, which is contrast to those of [6−9, 12−13], where nonstrict LMI conditions were reported, and is also different from the conditions of [4−5], where decomposition of the given singular system were used. Testing such a strict LMI-based condition can avoid some numerical problems arising from equality constraints and decomposition of the original singular system. Thus, the BRL in this paper is more elegant and has computational advantages from the mathematical point of view.

Remark 3. References [8, 12] used the Lyapunov functional method to deal with delay-dependent H_∞ control problem for singular time-delay systems. However, their methods are based on (12) with $P_2 = P_3 = Z_1 = Z_2 = 0$. This implies that the Lyapunov functional (12) is more generalized and includes more weighting matrices. Therefore, Theorem 1 has less conservative than the results of the aforementioned papers. This will be demonstrated by numerical examples in Section 3.

In the following theorem, we will apply Theorem 1 to design the state feedback controller (3) for the singular time-delay system (1) such that the resultant closed-loop system is regular, impulse free, and delay-dependently stable with H_∞ performance γ.

Theorem 2. For prescribed scalars $\bar{d} > 0$ and $\gamma > 0$, the singular time-delay system (1) controlled by $\boldsymbol{u}(t) = VG^{-1}\boldsymbol{x}(t)$ is regular, impulse free, and stable with H_∞ performance γ for any constant time delay d satisfying $0 \le d \le \bar{d}$, if there exist symmetric positive-definite matrices Q,

$$\begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix}, \begin{bmatrix} Z_1 & Z_2 \\ Z_2^{\mathrm{T}} & Z_3 \end{bmatrix}$$

and matrices S, G, V such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & B_\omega \\ * & \Xi_{22} & G^{\mathrm{T}}A_d^{\mathrm{T}} & P_2 & \Xi_{15} & 0 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 \\ * & * & * & -Z_1 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (18)$$

where $R \in \mathbf{R}^{n\times(n-r)}$ is any matrix with full column and satisfies $ER = 0$ and

$$\begin{aligned} \Xi_{11} &= G^{\mathrm{T}}A^{\mathrm{T}} + V^{\mathrm{T}}B^{\mathrm{T}} + AG + BV + EP_2 + \\ &\quad P_2^{\mathrm{T}}E^{\mathrm{T}} + Q - EZ_3E^{\mathrm{T}} + \bar{d}^2 Z_1 \\ \Xi_{12} &= EP_1 + SR^{\mathrm{T}} - G^{\mathrm{T}} + AG + BV + \bar{d}^2 Z_2 \\ \Xi_{13} &= G^{\mathrm{T}}A_d^{\mathrm{T}} + EZ_3E^{\mathrm{T}} - EP_2 \\ \Xi_{14} &= P_3 - EZ_2^{\mathrm{T}} \\ \Xi_{15} &= G^{\mathrm{T}}C^{\mathrm{T}} + V^{\mathrm{T}}D^{\mathrm{T}} \\ \Xi_{22} &= -G - G^{\mathrm{T}} + \bar{d}^2 Z_3 \\ \Xi_{33} &= -Q - EZ_3E^{\mathrm{T}} \\ \Xi_{34} &= -P_3 + EZ_2^{\mathrm{T}} \end{aligned}$$

Proof. Substituting the state feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ into system (1) yields the following closed-loop system

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= (A + BK)\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d) + B_\omega\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) &= (C + DK)\boldsymbol{x}(t) \end{aligned} \quad (19)$$

Since $\det(sE-(A+BK)) = \det(sE^{\mathrm{T}}-(A+BK)^{\mathrm{T}})$, the pair $(E, (A+BK))$ is regular and impulse free if and only if the pair $(E^{\mathrm{T}}, (A+BK)^{\mathrm{T}})$ is regular and impulse free. Moreover, since the solution of $\det(sE-(A+BK)-\mathrm{e}^{-ds}A_d) = 0$ is the same as that of $\det(sE^{\mathrm{T}}-(A+BK)^{\mathrm{T}}-\mathrm{e}^{-ds}A_d^{\mathrm{T}}) = 0$ and the $\det(sE^{\mathrm{T}} - (A+BK)^{\mathrm{T}} - \mathrm{e}^{-ds}A_d^{\mathrm{T}}) = 0$ and the

$$\|G(s)\|_\infty = \sup_{\omega\in[0,\infty)} \sigma_{\max}\{(C+DK)(\mathrm{j}\omega E - (A+BK) - \mathrm{e}^{-d\mathrm{j}\omega}A_d)^{-1}B_\omega\}$$

is equal to

$$\|H(s)\|_\infty = \sup_{\omega\in[0,\infty)} \sigma_{\max}\{B_\omega^{\mathrm{T}}(\mathrm{j}\omega E^{\mathrm{T}}-(A+BK)^{\mathrm{T}}-\mathrm{e}^{-d\mathrm{j}\omega}A_d^{\mathrm{T}})^{-1}(C+DK)^{\mathrm{T}}\}$$

as long as the regularity, absence of impulses, and stability with H_∞ performance are the only concern, system (19) is equivalent to the system

$$\begin{aligned} E^{\mathrm{T}}\dot{\boldsymbol{x}}(t) &= (A+BK)^{\mathrm{T}}\boldsymbol{x}(t) + A_d^{\mathrm{T}}\boldsymbol{x}(t-d) + \\ &\quad (C+DK)^{\mathrm{T}}\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) &= B_\omega^{\mathrm{T}}\boldsymbol{x}(t) \end{aligned} \quad (20)$$

Hence, applying Theorem 1 to the above system and setting $T_1 = T_2 = G$ and $V = KG$ yields (18) straightforwardly. □

Remark 4. Theorem 2 provides a sufficient condition for the solvability of delay-dependent H_∞ control problem for the singular time-delay system (1). The desired state feedback controller can be obtained by solving the strict LMI (18), without any parameter tuning and decomposition or transformation of the original system, and can be solved numerically very efficiently by using LMI toolbox of Matlab. While the decomposition or transformation of the system matrices is needed in [4−5], the equality constraints appear in the state feedback controller design processes of [7−9, 12−13]. Thus, Theorem 2 is much more general and elegant. Moreover, if (18) is feasible, it follows from $\Xi_{22} = -G - G^{\mathrm{T}} + \bar{d}^2 Z_3 < 0$ that G is nonsingular and thus the desired state feedback gain K can be readily obtained. It is worth pointing out that such a strict

LMI-based condition on the delay-dependent H_∞ control in the context of singular time-delay systems has not been reported in the literature.

Remark 5. Note that using the methods we derived, the problems of finding the largest $\bar{d}$ for a given γ, or the smallest γ for a given $\bar{d}$ can be easily solved by solving a quasi-convex optimization problem without the need of explicitly tuning any parameters.

3 Numerical examples

In this section, some examples are used to demonstrate that the methods presented in this paper are effective and are an improvement over the existing methods.

Example 1. Consider the following singular time-delay system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0.6341 & 0.5413 \\ -0.6121 & -1.1210 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -0.4500 & 0 \\ 0 & -0.1210 \end{bmatrix} \boldsymbol{x}(t-d)$$

By comparing the stability criterion of Theorem 1 with those of [4−7, 16−17] for the above system, we have Table 1. Hence, for this example, the stability criterion we derived is less conservative than those reported in the above-mentioned papers.

Table 1 Comparison of maximum allowed time-delays $\bar{d}$'s

[6]	[17]	[4-5]	[7, 16]	Theorem 1
–	2.1328	2.1372	2.4841	2.4865

Example 2. To compare the delay-dependent BRL in Theorem 1 with the existing ones, we consider the singular time-delay system (1) with $\boldsymbol{u}(t) \equiv 0$ and

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.6 & 0.5 \\ -0.6 & -1 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -0.7 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad \boldsymbol{B}_\omega = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$$
$$\boldsymbol{C} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

For a given $\gamma > 0$, the maximum allowed time-delay $\bar{d}$ satisfying the LMI in (5) can be calculated by solving a quasi-convex optimization problem. Similarly, for a given $\bar{d} > 0$, the minimum allowed γ satisfying the LMI in (5) can also be calculated by solving a quasi-convex optimization problem. Tables 2 and 3 provide the comparison results on the maximum allowed time-delay $\bar{d}$ for given $\gamma > 0$ and the minimum allowed γ for given $\bar{d} > 0$, respectively, via the methods in [4−5, 7−8, 12−13] and Theorem 1 in this paper. In addition, the result of [6] cannot deal with the above system. Thus, the BRL in Theorem 1 of this paper is less conservative than those in [4−8, 12−13].

Example 3. To show the reduced conservatism of the H_∞ control result in Theorem 2 in this paper, we now consider singular system with time delay[4]:

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A_d\boldsymbol{x}(t-d) + \boldsymbol{B}u(t) + \boldsymbol{B}_\omega\omega(t) \\ \boldsymbol{z}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + Du(t) \end{aligned}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$
$$\boldsymbol{B} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad \boldsymbol{B}_\omega = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\boldsymbol{C} = \begin{bmatrix} 1 & 0.2 \end{bmatrix}, \quad D = 0.1$$

For a given time-delay $d = 1.2$, the minimum γ's are 21 and 15.0268, which can be obtained using the methods of [4, 12], respectively. However, by resorting to Theorem 2 in this paper, for the same time-delay, the minimum $\gamma = 9.6754$ by solving the strict LMI (18), which is 53.93% and 35.61% larger than those in [4, 12], respectively. Furthermore, the state feedback controller achieving the minimum $\gamma = 9.6754$ can be obtained as

$$u(t) = \begin{bmatrix} 0.4834 & -2.3868 \end{bmatrix} \boldsymbol{x}(t)$$

while the result of [6] can not deal with the H_∞ control problem for the above system. Therefore, Theorem 2 in this paper is less conservative than those in [4, 6, 12].

Table 2 Comparison of maximum allowed time-delays $\bar{d}$'s for Example 2

γ	2.0	2.2	2.4	2.6	2.8	3.0	3.2
[7−8]	0.3121	0.4109	0.4760	0.5237	0.5607	0.5906	0.6156
[4−5]	0.6711	0.9213	1.0533	1.1334	1.1864	1.2237	1.2512
[12−13]	0.6745	0.9381	1.1102	1.2261	1.3061	1.3626	1.4034
Theorem 1	0.9508	1.1681	1.2865	1.3559	1.3973	1.4272	1.4525

Table 3 Comparison of minimum γ's for Example 2

$\bar{d}$	0.90	0.95	1.00	1.05	1.10	1.15	1.20
[7−8]	5.8270	6.0969	6.3722	6.6540	6.9438	7.2440	7.5590
[4−5]	2.1763	2.2348	2.3056	2.3933	2.5054	2.6547	2.8653
[12−13]	2.1650	2.2115	2.2630	2.3205	2.3857	2.4604	2.5479
Theorem 1	1.9688	1.9995	2.0346	2.0751	2.1221	2.1774	2.2438

4 Conclusion

The problem of delay-dependent H_∞ control for singular systems with state delay has been solved in terms of LMI approach and an augmented Lyapunov functional. A new version of delay-dependent BRL and the design method of the desired state feedback controller are established. The obtained results are all formulated by strict LMIs involving no decomposition of the system matrices, which can be tested easily by the LMI control toolbox and make the analysis and design relatively simple and reliable. Numerical examples are given to demonstrate the reduced conservatism of the obtained stability, BRL as well as H_∞ control results in this paper.

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