Robust H_∞ Control of Uncertain Switched Systems: a Sliding Mode Control Design

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Abstract This paper develops a new method to the robust H_∞ control problem for a class of uncertain switched systems by constructing a single robust H_∞ sliding surface. The method consists of two phases. One is to construct a single sliding surface such that the reduced-order equivalent sliding motion restricted to the sliding surface is robustly stabilizable with H_∞ disturbance attenuation level γ under a hysteresis switching law; the other phase is to design variable structure controllers of subsystems to drive the state of the switched system to reach the single sliding surface in finite time and remain on it thereafter. A numerical example is given to illustrate the effectiveness of the proposed method.

Key words Uncertain switched systems, sliding mode control (SMC), single robust H_∞ sliding surface, hysteresis switching law

Switched systems consist of a family of continuous-time or discrete-time systems and certain rules of logic specifying at each instant of time, by which subsystem is activated along the system trajectory. Switched systems have recently gained a great deal of attention[1−10], mainly because many real-world systems, such as chemical processes and transportation systems, can be modeled as switched systems. In the published works, switched linear systems without uncertainties have been extensively investigated, for instance, see $[5-7]$ and references therein. Since uncertainties are ubiquitous in system models due to the complexity of the system itself, exogenous disturbance, and so on, from a practical point of view, it is much more important to study switched systems with uncertainties.

Among the existing results of switched systems with uncertainties, [8] considered quadratic stabilization of switched systems with norm-bounded time varying uncertainties. In [9], L_2 induced norm of switched systems with external disturbances was considered under the condition of large dwell time. Robust H_∞ control and stabilization of uncertain switched linear systems were addressed in [10] based on the multiple Lyapunov functions approach.

On the other hand, the sliding mode control (SMC) is one of the most important methods in the robust control area, since it possesses various attractive features such as good robustness, fast response, and good transient response[11]. Many results have been reported about SMC[11−13]. However, very few results of SMC applied to switched systems have appeared by now. Reference [14] proposed an SMC method to make a class of switched systems exponentially stable. Reference [15] addressed SMC for planar switched systems under an arbitrary switching sequence. In [16], the sliding motion of switched systems without control input was analyzed and an approach was proposed to estimate the domain, in which the sliding motion might occur. A variable structure controller with sliding mode sector for a hybrid system was presented in [17]. For the robust H_∞ control problem with the SMC technique, to the best of our knowledge, there are no results in the existing literature, which motivates our present study.

In this paper, we investigate the robust H_∞ sliding mode variable structure control problem for a class of uncertain switched systems. This paper is organized as follows. Section 1 presents the problem formulation and the preliminaries. In Section 2, the design method is developed. Section 3 gives a numerical example and simulation results to illustrate the effectiveness of the proposed design, followed by conclusion in Section 4.

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix.

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1 Problem formulation and preliminaries

Consider the following uncertain switched system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (A_\sigma + \Delta A_\sigma)\boldsymbol{x}(t) + B(\boldsymbol{u}_\sigma + \boldsymbol{f}_\sigma(\boldsymbol{x}, t)) + B_1\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) &= C\boldsymbol{x}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the system state, $\sigma : [0,\infty) \to \Xi = \{1, 2, \cdots, l\}$ is the piecewise constant switching signal that may depend on either time t or state $\boldsymbol{x}$, $\boldsymbol{u}_i \in \mathbf{R}^m$ is the control input of the i-th subsystem, $\boldsymbol{z}(t)$ is the controlled output, $\boldsymbol{\omega}(t) \in L_2[0,\infty)$ is the external disturbance input, B, B_1, C, and A_i are constant matrices of appropriate dimensions, $\triangle A_i$ denote the uncertainties, and $\boldsymbol{f}_i(\boldsymbol{x}, t)$ represent nonlinear uncertainties of the system. For convenience, we adopt the following notation from [18]. A switching sequence is expressed by

$$\Psi = \{\boldsymbol{x}_0; (i_0, t_0), (i_1, t_1), \cdots, (i_j, t_j), \cdots \mid i_j \in \Xi\ , j \in \mathbf{N}\} \tag{2}$$

where t_0 is the initial time, $\boldsymbol{x}_0$ is the initial state, and (i_k, t_k) means that the i_k-th subsystem is activated for $[t_k, t_{k+1})$. Therefore, when $t \in [t_k, t_{k+1})$, the trajectory of the switched system (1) is produced by the i_k-th subsystem.

The following assumptions are introduced.

Assumption 1. The uncertainties can be represented and emulated as

$$\Delta A_i = E\Sigma_i(t)F, \quad i \in \Xi$$

where E and F are known constant matrices of appropriate dimensions, and $\Sigma_i(t)$ are unknown time-varying uncertainties satisfying $\Sigma_i^{\mathrm{T}}(t)\Sigma_i(t) \le I$.

Assumption 2. There exist known nonnegative scalar-valued functions $\phi_i(\boldsymbol{x}, t), i \in \Xi$ such that $\|\boldsymbol{f}_i(\boldsymbol{x}, t)\| \le \phi_i(\boldsymbol{x}, t)$ for all t.

Assumption 3. There exists a known nonnegative constant ϖ such that $\|\boldsymbol{\omega}(t)\| \le \varpi$ for all t.

Assumption 4. The input matrix B has full rank m and $m < n$.

Remark 1. Assumptions $1\sim4$ are standard assumptions in the study of variable structure control.

In order to develop the main design method, we need the following lemmas.

Lemma 1[19]**.** Given real matrices R_1 and R_2 of appropriate dimensions and an unknown matrix $\Sigma(t)$ with $\Sigma(t)^{\mathrm{T}}\Sigma(t) \le I$, we have

$$R_1\Sigma(t)R_2 + R_1^{\mathrm{T}}\Sigma^{\mathrm{T}}(t)R_2^{\mathrm{T}} \le \beta R_1R_1^{\mathrm{T}} + \beta^{-1}R_2^{\mathrm{T}}R_2 \tag{3}$$

where $\beta > 0$.

Now, we introduce a convex combination of system (1) without the matched uncertainties $\boldsymbol{f}_i(\boldsymbol{x}, t)$ as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\bar{A} + \Delta\bar{A})\boldsymbol{x}(t) + B\boldsymbol{u} + B_1\boldsymbol{\omega}(t)\\ \boldsymbol{z}(t) &= C\boldsymbol{x}(t)\end{aligned} \tag{4}$$

where $\bar{A} = \sum_{i=1}^{l}\alpha_iA_i$, $\Delta\bar{A} = \sum_{i=1}^{l}\alpha_i\Delta A_i$, $\alpha_i \ge 0$ with $\sum_{i=1}^{l}\alpha_i = 1$.

Lemma 2. Given a constant $\gamma > 0$, if there exist matrix $P > 0$, state feedback gain K, constant $\lambda > 0$, and scalars $\alpha_i > 0$ with $\sum_{i=1}^{l}\alpha_i = 1$ satisfying

$$\begin{aligned}&(\bar{A} - BK)^{\mathrm{T}}P + P(\bar{A} - BK) + P(\lambda^2EE^{\mathrm{T}} +\\ &\gamma^{-2}B_1B_1^{\mathrm{T}})P + \frac{1}{\lambda^2}F^{\mathrm{T}}F + C^{\mathrm{T}}C < 0\end{aligned} \tag{5}$$

then system (4) is robustly stabilizable with H_∞ disturbance attenuation level γ.

Proof. Let

$$\begin{aligned}Q = (\bar{A} + \Delta\bar{A} - BK)^{\mathrm{T}}P + P(\bar{A} + \Delta\bar{A} - BK) + \gamma^{-2}P\times\\ B_1B_1^{\mathrm{T}}P + C^{\mathrm{T}}C = (\bar{A} - BK)^{\mathrm{T}}P + P(\bar{A} - BK)+\\ \gamma^{-2}PB_1B_1^{\mathrm{T}}P + C^{\mathrm{T}}C + \Delta\bar{A}^{\mathrm{T}}P + P\Delta\bar{A}\end{aligned}$$

Using Lemma 1, one obtains

$$\begin{aligned}\Delta\bar{A}^{\mathrm{T}}P + P\Delta\bar{A} = (\sum_{i=1}^{l}\alpha_i\Delta A_i)^{\mathrm{T}}P + P(\sum_{i=1}^{l}\alpha_i\Delta A_i) =\\ [E(\sum_{i=1}^{l}\alpha_i\Sigma_i(t))F]^{\mathrm{T}}P + P[E(\sum_{i=1}^{l}\alpha_i\Sigma_i(t))F] \le\\ \lambda^2PEE^{\mathrm{T}}P + \lambda^{-2}F^{\mathrm{T}}F\end{aligned}$$

Hence, we have

$$\begin{aligned}Q \le (\bar{A} - BK)^{\mathrm{T}}P + P(\bar{A} - BK) + P(\lambda^2EE^{\mathrm{T}} +\\ \gamma^{-2}B_1B_1^{\mathrm{T}})P + \frac{1}{\lambda^2}F^{\mathrm{T}}F + C^{\mathrm{T}}C < 0\end{aligned}$$

which implies that system (4) is robustly stabilizable with H_∞ disturbance attenuation level γ. □

Remark 2. The inequality (5) can be converted into a linear matrix inequality (LMI) by Schur complement and the change of variable $\hat{K} = KP^{-1}$. Hence, the feasible solutions can be globally found by the LMI method[13].

To have a regular form of system (1), we define a nonsingular matrix

$$T = \begin{bmatrix}\tilde{B}^{\mathrm{T}}\\ B^{\mathrm{T}}\end{bmatrix} \tag{6}$$

where $\tilde{B}$ is an orthogonal complement of matrix B, and a vector

$$\boldsymbol{\xi}(t) = \begin{bmatrix}\boldsymbol{\xi}_1(t)\\ \boldsymbol{\xi}_2(t)\end{bmatrix} = T\boldsymbol{x}(t) = \begin{bmatrix}\tilde{B}^{\mathrm{T}}\\ B^{\mathrm{T}}\end{bmatrix}\boldsymbol{x}(t) \tag{7}$$

with $\boldsymbol{\xi}_1(t) \in \mathbf{R}^{n-m}$ and $\boldsymbol{\xi}_2(t) \in \mathbf{R}^m$. We can easily show

$$T^{-1} = \begin{bmatrix}\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1} & B(B^{\mathrm{T}}B)^{-1}\end{bmatrix} \tag{8}$$

By means of the state transformation $\boldsymbol{\xi}(t) = T\boldsymbol{x}(t)$, system (1) is transformed into the following regular form

$$\begin{aligned}&\dot{\boldsymbol{\xi}}(t) = (\widehat{A}_\sigma + \Delta\widehat{A}_\sigma)\boldsymbol{\xi}(t) + \widehat{B}(\boldsymbol{u}_\sigma + \boldsymbol{f}_\sigma(\boldsymbol{x}, t)) + \widehat{B}_1\boldsymbol{\omega}(t)\\ &\boldsymbol{z}(t) = \widehat{C}\boldsymbol{\xi}(t)\end{aligned} \tag{9}$$

where $\widehat{A}_\sigma = TA_\sigma T^{-1}$, $\Delta\widehat{A}_\sigma = T\Delta A_\sigma T^{-1}$, $\widehat{B} = TB$, $\widehat{B}_1 = TB_1$, and $\widehat{C} = CT^{-1}$. System (9) is equivalent to the following form

$$\begin{aligned}&\begin{bmatrix}\dot{\boldsymbol{\xi}}_1(t)\\ \dot{\boldsymbol{\xi}}_2(t)\end{bmatrix} = \begin{bmatrix}\widehat{A}_{\sigma11} & \widehat{A}_{\sigma12}\\ \widehat{A}_{\sigma21} & \widehat{A}_{\sigma22}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1(t)\\ \boldsymbol{\xi}_2(t)\end{bmatrix} + \begin{bmatrix}0\\ B^{\mathrm{T}}B\end{bmatrix}\times\\ &\qquad(\boldsymbol{u}_\sigma + \boldsymbol{f}_\sigma(\boldsymbol{x}, t)) + \begin{bmatrix}\tilde{B}^{\mathrm{T}}B_1\\ B^{\mathrm{T}}B_1\end{bmatrix}\boldsymbol{\omega}(t)\\ &\boldsymbol{z}(t) = C\begin{bmatrix}\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1} & B(B^{\mathrm{T}}B)^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1(t)\\ \boldsymbol{\xi}_2(t)\end{bmatrix}\end{aligned} \tag{10}$$

where $\widehat{A}_{\sigma11} = \tilde{B}^{\mathrm{T}}A_\sigma\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1} + \tilde{B}^{\mathrm{T}}E\Sigma_\sigma(t)F\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1}$, $\widehat{A}_{\sigma12} = \tilde{B}^{\mathrm{T}}A_\sigma B(B^{\mathrm{T}}B)^{-1} + \tilde{B}^{\mathrm{T}}E\Sigma_\sigma(t)FB(B^{\mathrm{T}}B)^{-1}$, $\widehat{A}_{\sigma21} = B^{\mathrm{T}}A_\sigma\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1} + B^{\mathrm{T}}E\Sigma_\sigma(t)F\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1}$, and $\widehat{A}_{\sigma22} = B^{\mathrm{T}}A_\sigma B(B^{\mathrm{T}}B)^{-1} + B^{\mathrm{T}}E\Sigma_\sigma(t)FB(B^{\mathrm{T}}B)^{-1}$.

Without loss of generality, we assume that the single robust H_∞ sliding surface is given by

$$\boldsymbol{\zeta}(t) = M\boldsymbol{\xi}_1(t) + \boldsymbol{\xi}_2(t) = \mathbf{0} \tag{11}$$

where $M \in \mathbf{R}^{m\times(n-m)}$ is a matrix to be chosen. Then, it follows that $\boldsymbol{\zeta}(t) = S\boldsymbol{x}(t) = (M\tilde{B}^{\mathrm{T}} + B^{\mathrm{T}})\boldsymbol{x}(t)$. Substituting $\boldsymbol{\xi}_2(t) = -M\boldsymbol{\xi}_1(t)$ into (10) yields the sliding motion

$$\begin{aligned}\dot{\boldsymbol{\xi}}_1(t) &= (\widehat{A}_{\sigma11} - \widehat{A}_{\sigma12}M)\boldsymbol{\xi}_1(t) + \tilde{B}^{\mathrm{T}}B_1\boldsymbol{\omega}(t)\\ \boldsymbol{z}(t) &= C\tilde{B}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1}\boldsymbol{\xi}_1(t) - CB(B^{\mathrm{T}}B)^{-1}M\boldsymbol{\xi}_1(t)\end{aligned} \tag{12}$$

Definition 1. Given a constant $\gamma > 0$, the sliding motion (12) is said to be robustly stabilizable with H_∞ disturbance attenuation level γ via switching if there exists a Lyapunov function $V(\boldsymbol{x})$ and a switching law $\sigma(t)$ such that:

1) Derivative of V along the trajectory of system (12) with $\omega(t) = 0$ satisfies

$$L(t) = \dot{V}(t) < 0$$

for all $t \in \mathbf{R}^+$;

2) With zero-initial condition $\boldsymbol{x}(t) = \mathbf{0}$, $\|\boldsymbol{z}(t)\|_2 < \gamma\|\boldsymbol{\omega}(t)\|_2$ holds for all nonzero $\boldsymbol{\omega}(t) \in L_2[0, \infty)$.

The objective of this paper is to determine the matrix M, the switching law $\sigma(t)$, and the variable structure controllers $\boldsymbol{u}_i, i \in \Xi$ such that:

1) The sliding motion (12) restricted to the single sliding surface (11) is robustly stabilizable with H_∞ disturbance attenuation level γ under the switching law $\sigma(t)$;

2) The state of system (1) can reach the single sliding surface (11) in finite time and subsequently remains on it.

Remark 3. The single sliding surface $\boldsymbol{\zeta}(t) = S\boldsymbol{x}(t) = \mathbf{0}$ is designed such that the switched system (1) is robustly stabilizable with H_∞ disturbance attenuation level γ based on the single Lyapunov function approach in the sliding surface. The purpose of designing the single sliding surface for the switched system is to reduce the reaching phase in which systems are sensitive to uncertainties and perturbations, and improve the transient performance and robustness.

Remark 4. We can see that the matched uncertainties $\boldsymbol{f}_i(\boldsymbol{x},t)$ disappear in the sliding motion (12) and the order of the switched system (1) is reduced in the sliding surface (11). Therefore, we only need to study the robust H_∞ control problem of the $n-m$ dimensional switched system (12).

2 Main results

In this section, we introduce the variable structure control technique. In general, the design comprises two steps. Firstly, construct the sliding surface so that the controlled system yields the desired dynamic performance. Secondly, design the variable structure controllers such that the trajectory of system (1) reaches the sliding surface and remains on it for all subsequent time.

The following theorem shows that system (1) in the sliding surface (11) is robustly stabilizable with H_∞ disturbance attenuation level γ via switching.

Theorem 1. Suppose that (5) is solvable. Then, the sliding motion (12) with $M = [(B^{\rm T}B)^{-1}B^{\rm T}PB(B^{\rm T}B)^{-1}]^{-1}(B^{\rm T}B)^{-1}B^{\rm T}P\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}$ is robustly stabilizable with H_∞ disturbance attenuation level γ via switching. In this case, the single robust H_∞ sliding surface is

$$\begin{aligned}\boldsymbol{\zeta}(t) = S\boldsymbol{x}(t) = \\ &\{[(B^{\rm T}B)^{-1}B^{\rm T}PB(B^{\rm T}B)^{-1}]^{-1}(B^{\rm T}B)^{-1}\times \\ &B^{\rm T}P\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}\tilde{B}^{\rm T}+B^{\rm T}\}\boldsymbol{x}(t)=\boldsymbol{0}\end{aligned} \tag{13}$$

where P satisfies (5) in Lemma 2.

Proof. Since (5) is solvable, by Lemma 2, system (4) is robustly stabilizable with H_∞ disturbance attenuation level γ. The sliding motion (12) can be rewritten equivalently as

$$\begin{aligned}\dot{\boldsymbol{\xi}}_1(t) &= (\hat{A}_{\sigma 11}-\hat{A}_{\sigma 12}M+\hat{E}\Sigma_\sigma(t)\hat{F})\boldsymbol{\xi}_1(t)+\hat{B}_1\boldsymbol{\omega}(t)\\ \boldsymbol{z}(t) &= C_s\boldsymbol{\xi}_1(t)\end{aligned} \tag{14}$$

where $\hat{A}_{\sigma11}=\tilde{B}^{\rm T}A_\sigma\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}$, $\hat{A}_{\sigma12}=\tilde{B}^{\rm T}A_\sigma B\ (B^{\rm T}B)^{-1}$, $\hat{E}=\tilde{B}^{\rm T}E$, $\hat{F}=F\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}-FB(B^{\rm T}B)^{-1}M$, $\hat{B}_1=\tilde{B}^{\rm T}B_1$, and $C_s=C\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}-CB(B^{\rm T}B)^{-1}M$.

Denote

$$\bar{A}_c = T(\bar{A}-BK)T^{-1} = \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21}-B^{\rm T}BK\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1} & \bar{A}_{22}-B^{\rm T}BKB(B^{\rm T}B)^{-1}\end{bmatrix} \tag{15}$$

with $\bar{A}_{11}=\tilde{B}^{\rm T}\bar{A}\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}$, $\bar{A}_{12}=\tilde{B}^{\rm T}\bar{A}B(B^{\rm T}B)^{-1}$, and calculate

$$\bar{P}=T^{-\rm T}PT^{-1}= \begin{bmatrix}(\tilde{B}^{\rm T}\tilde{B})^{-1}\tilde{B}^{\rm T}P\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1} & (\tilde{B}^{\rm T}\tilde{B})^{-1}\tilde{B}^{\rm T}PB(B^{\rm T}B)^{-1}\\ (B^{\rm T}B)^{-1}B^{\rm T}P\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1} & (B^{\rm T}B)^{-1}B^{\rm T}PB(B^{\rm T}B)^{-1}\end{bmatrix} = \begin{bmatrix}\bar{P}_{11} & \bar{P}_{12}\\ \bar{P}_{12}^{\rm T} & \bar{P}_{22}\end{bmatrix} \tag{16}$$

Then, the inequality (5) can be rewritten as

$$\begin{aligned}&\bar{A}_c^{\rm T}\bar{P}+\bar{P}\bar{A}_c+\bar{P}T(\lambda^2EE^{\rm T}+\gamma^{-2}B_1B_1^{\rm T})T^{\rm T}\bar{P}+\\ &T^{-\rm T}(\frac{1}{\lambda^2}F^{\rm T}F+C^{\rm T}C)T^{-1}<0\end{aligned} \tag{17}$$

Pre- and post-multiplying (17) by $[I_{n-m},-\bar{P}_{12}\bar{P}_{22}^{-1}]$ and $[I_{n-m},-\bar{P}_{12}\bar{P}_{22}^{-1}]^{\rm T}$, respectively, we have

$$\begin{aligned}&(\bar{A}_{11}-\bar{A}_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T})^{\rm T}\bar{P}_r+\bar{P}_r(\bar{A}_{11}-\bar{A}_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T})+\\ &\bar{P}_r\tilde{B}^{\rm T}(\lambda^2EE^{\rm T}+\gamma^{-2}B_1B_1^{\rm T})\tilde{B}\bar{P}_r+[\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}-\\ &B(B^{\rm T}B)^{-1}\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T}]^{\rm T}(\frac{1}{\lambda^2}F^{\rm T}F+C^{\rm T}C)\times\\ &[\tilde{B}(\tilde{B}^{\rm T}\tilde{B})^{-1}-B(B^{\rm T}B)^{-1}\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T}]<0\end{aligned} \tag{18}$$

where $\bar{P}_r=\bar{P}_{11}-\bar{P}_{12}\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T}$. Obviously, $\bar{P}_r>0$ since $\bar{P}>0$. Therefore, by setting $M=\bar{P}_{22}^{-1}\bar{P}_{12}^{\rm T}=[(B^{\rm T}B)^{-1}B^{\rm T}PB(B^{\rm T}B)^{-1}]^{-1}(B^{\rm T}\ B)^{-1}\ B^{\rm T}\ P\tilde{B}\ (\tilde{B}^{\rm T}\ \tilde{B})^{-1}$, (18) becomes

$$\begin{aligned}&(\bar{A}_{11}-\bar{A}_{12}M)^{\rm T}\bar{P}_r+\bar{P}_r(\bar{A}_{11}-\bar{A}_{12}M)+\bar{P}_r(\lambda^2\hat{E}\hat{E}^{\rm T}+\\ &\gamma^{-2}\hat{B}_1\hat{B}_1^{\rm T})\bar{P}_r+\frac{1}{\lambda^2}\hat{F}^{\rm T}\hat{F}+C_s^{\rm T}C_s<0\end{aligned} \tag{19}$$

Furthermore, substituting $\bar{A}=\sum_{i=1}^{l}\alpha_iA_i$ into the inequality (19) and denoting

$$\begin{aligned}Q_i=&(\hat{A}_{i11}-\hat{A}_{i12}M)^{\rm T}\bar{P}_r+\bar{P}_r(\hat{A}_{i11}-\hat{A}_{i12}M)+\\ &\bar{P}_r(\lambda^2\hat{E}\hat{E}^{\rm T}+\gamma^{-2}\hat{B}_1\hat{B}_1^{\rm T})\bar{P}_r+\frac{1}{\lambda^2}\hat{F}^{\rm T}\hat{F}+C_s^{\rm T}C_s, i\in\Xi\end{aligned}$$

we have

$$\alpha_1Q_1+\alpha_2Q_2+\cdots+\alpha_lQ_l<0$$

We define the regions

$$\Omega_i=\{\boldsymbol{\xi}_1|\,\boldsymbol{\xi}_1^{\rm T}Q_i\boldsymbol{\xi}_1<0\},\quad i\in\Xi \tag{20}$$

Obviously, $\underset{i\in\Xi}{\cup}\Omega_i=\mathbf{R}^{(n-m)}\backslash\{\boldsymbol{0}\}$.

The hysteresis switching law for the sliding motion (12) is designed as

$$\sigma(0)=\min\arg\{\Omega_i|\,\boldsymbol{\xi}_1(0)\in\Omega_i\}$$

$$\sigma(t)=\begin{cases} i, & \text{if } \boldsymbol{\xi}_1(t)\in\Omega_i \text{ and } \sigma(t^-)=i\\ \min\arg\{\Omega_k|\boldsymbol{\xi}_1(t)\in\Omega_k\}, & \text{if } \boldsymbol{\xi}_1(t)\notin\Omega_i \text{ and } \sigma(t^-)=i\end{cases} \tag{21}$$

We first verify the stabilization of the sliding motion (12) with $\boldsymbol{\omega}(t)=\boldsymbol{0}$. To this end, choose the Lyapunov function candidate as

$$V(t)=\boldsymbol{\xi}_1^{\rm T}(t)\bar{P}_r\boldsymbol{\xi}_1(t) \tag{22}$$

Then, by (19) the derivative of the Lyapunov function (22) along the trajectory of system (14) with $\boldsymbol{\omega}(t)=\boldsymbol{0}$ and under the switching law (21) satisfies

$$\dot{V}<0$$

By the single Lyapunov function method, the sliding motion (12) with $\boldsymbol{\omega}(t)=\boldsymbol{0}$ is robustly stabilizable under the switching law (21).

In the following, we show that the overall L_2-gain from $\boldsymbol{\omega}$ to $\boldsymbol{z}$ is less than or equal to γ in the single sliding surface

(13). We suppose $\boldsymbol{x}(0) = \mathbf{0}$ and without loss of generality, for $\forall\, T \geq t_0 = 0$, assume $T \in [t_k, t_{k+1})$ for some k.

Now, we introduce

$$J = \int_0^T (\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2)\mathrm{d}t$$

According to the switching sequence (2), when $T \in [t_k, t_{k+1})$, we have

$$J = \sum_{j=0}^{k-1} \left(\int_{t_j}^{t_{j+1}} (\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2 + \dot{V}(t))\mathrm{d}t - (V(t_{j+1}) - V(t_j)) \right) + \int_{t_k}^{T} (\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2 + \dot{V}(t))\mathrm{d}t - (V(T) - V(t_k)) = \sum_{j=0}^{k-1} \left(\int_{t_j}^{t_{j+1}} (\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2 + \dot{V}(t))\mathrm{d}t \right) + \int_{t_k}^{T} (\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2 + \dot{V}(t))\mathrm{d}t - V(T)$$

Note that

$$\|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{\omega}\|^2 + \dot{V}(t) \leq \boldsymbol{\xi}_1^{\mathrm{T}} Q_{i_j} \boldsymbol{\xi}_1 - (\gamma^{-1}\hat{B}_1^{\mathrm{T}}\bar{P}_r\boldsymbol{\xi}_1 - \gamma\boldsymbol{\omega})^{\mathrm{T}}(\gamma^{-1}\hat{B}_1^{\mathrm{T}}\bar{P}_r\boldsymbol{\xi}_1 - \gamma\boldsymbol{\omega}) < 0$$

Therefore, $J < 0$ for $\forall \boldsymbol{\omega} \in L_2[0, \infty)$. That is $\|\boldsymbol{z}(t)\|_2 < \gamma \|\boldsymbol{\omega}(t)\|_2$. □

Remark 5. The classical state-depended switching laws appeared in many references[8−10] may result in sliding motions in subsystems switching surfaces. We referred to [1, 20] to design the hysteresis switching law to avoid Zeno phenomenon. The value of the hysteresis switching signal is not determined by the current value of state alone, but depends also on the previous value of switching signal.

Next, we design controllers for subsystems to reach the sliding surface in finite time.

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and the sliding surface of system (1) is given by (13). Then, under the controllers

$$\boldsymbol{u}_i = -(SB)^{-1}SA_i\boldsymbol{x} - (SB)^{-1}(\|SE\|\,\|F\boldsymbol{x}\| + \|SB\|\,\phi_i(\boldsymbol{x}, t) + \varpi\,\|SB_1\| + \mu)\mathrm{sgn}(\boldsymbol{\zeta}), \quad i \in \Xi \tag{23}$$

the state of system (1) can reach in finite time and subsequently remains on the sliding surface, where μ is a positive scalar to adjust the convergent rate.

Proof. The derivative of the sliding function $\boldsymbol{\zeta}(t) = S\boldsymbol{x}(t)$ along the trajectory of system (1) is

$$\begin{aligned}\dot{\boldsymbol{\zeta}}(t) = S(A_i + \Delta A_i)\boldsymbol{x}(t) + \\ SB\boldsymbol{u}_i + SB\boldsymbol{f}_i(\boldsymbol{x}, t) + SB_1\boldsymbol{\omega}(t)\end{aligned} \tag{24}$$

By Assumptions 1 ∼ 3, substituting the controllers (23) into (24) yields $\boldsymbol{\zeta}^{\mathrm{T}}(t)\dot{\boldsymbol{\zeta}}(t) \leq -\mu\|\boldsymbol{\zeta}(t)\|$, which implies that the state of system (1) reaches the sliding surface (13) in finite time and remains on it thereafter. □

3 Example

In this section, we present a numerical example to demonstrate the effectiveness of the proposed design method.

Consider the following uncertain switched system

$$\begin{aligned}&\dot{\boldsymbol{x}}(t) = (A_\sigma + \Delta A_\sigma)\boldsymbol{x}(t) + \boldsymbol{B}(u_\sigma + f_\sigma(\boldsymbol{x}, t)) + \boldsymbol{B}_1\omega(t) \\ &z(t) = \boldsymbol{C}\boldsymbol{x}(t)\end{aligned} \tag{25}$$

where $\sigma(t) \in \Xi = \{1, 2\}$, $A_1 = \begin{bmatrix} -3 & -0.5 & 1 \\ 1 & -0.5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix}$, $\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ -0.1 \\ 0.1 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, the uncertainties $\Delta A_i = \boldsymbol{E}\Sigma_i(t)\boldsymbol{F}$, with $\boldsymbol{E} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{F} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\Sigma_1 = \eta_1 = -1 \in [-1, 1]$, $\Sigma_2 = \eta_2 = 0.8 \in [-1, 1]$, and $f_1 = f_2 = 0$.

We choose the convex combination coefficients $\alpha_1 = \alpha_2 = 0.5$ and the constant $\lambda = 1/\sqrt{2}$. The disturbance attenuation level is given by $\gamma = 1/\sqrt{2}$.

Let $\boldsymbol{K} = \boldsymbol{B}^{\mathrm{T}}P$. Solving inequality (5) leads to the solution

$$P = \begin{bmatrix} 9.2006 & 7.8894 & 3.7381 \\ 7.8894 & 8.1363 & 3.6193 \\ 3.7381 & 3.6193 & 3.2078 \end{bmatrix}$$

Then, we obtain $\boldsymbol{M} = [0.0831, 0.2195]$. The single robust H_∞ sliding function is given as

$$\zeta(t) = \boldsymbol{S}\boldsymbol{x}(t) = [-0.1591, -0.3458, 1.0771]\boldsymbol{x}(t) \tag{26}$$

Taking $\mu = 1$, according to (23), one has the controllers for subsystems given as

$$\begin{aligned} u_1 &= -0.1054x_1 - 1.0637x_2 + 2.1273x_3 - \\ &\quad 0.8(0.1867\,\|x_1 + x_2\| + 1)\mathrm{sgn}(\zeta) \\ u_2 &= -0.4358x_1 + 0.1493x_2 + 1.5741x_3 - \\ &\quad 0.8(0.1867\,\|x_1 + x_2\| + 1)\mathrm{sgn}(\zeta) \end{aligned} \tag{27}$$

The state responses of the two subsystems with initial state $\boldsymbol{x}_0 = [1, 2, -1]^{\mathrm{T}}$ are shown in Figs. 1 and 2, respectively. We can easily see that both subsystems are unstable.

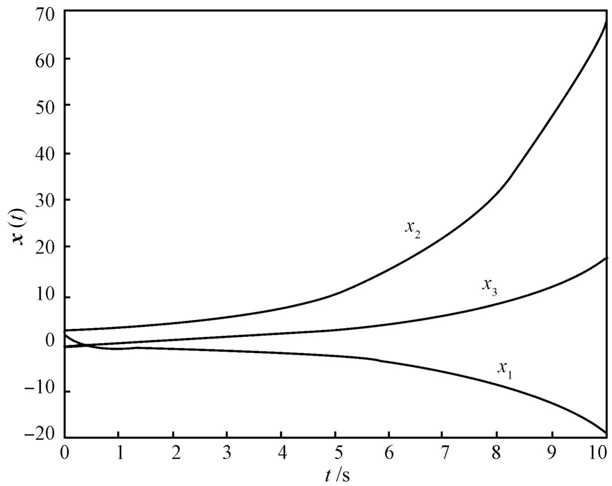

Fig. 1 The state responses of subsystem 1

It is easy to verify that the conditions of Theorems 1 and 2 are satisfied. According to the hysteresis switching law (21), for system (25), we design the switching law as

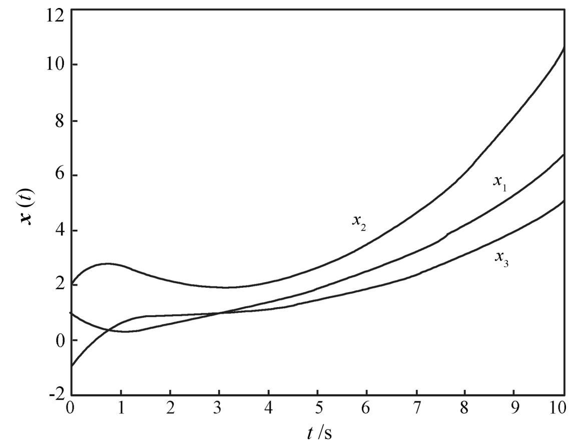

Fig. 2 The state responses of subsystem 2

$$
\sigma(t)=\begin{cases}1, \text{ if}(\boldsymbol{x}(0)\in\Omega_1) \text{ or } (\boldsymbol{x}(t)\in\Omega_1 \text{ and } \sigma(t^-)=1)\\ \quad \text{or } (\boldsymbol{x}(t)\notin\Omega_2 \text{ and } \sigma(t^-)=2)\\ 2, \text{ if}(\boldsymbol{x}(0)\notin\Omega_1) \text{ or } (\boldsymbol{x}(t)\in\Omega_2 \text{ and } \sigma(t^-)=2)\\ \quad \text{or } (\boldsymbol{x}(t)\notin\Omega_1 \text{ and } \sigma(t^-)=1)\end{cases} \tag{28}
$$

where $\Omega_1 = \{\boldsymbol{x}|\boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}-32.5006, -13.8465, -6.9232\\ -13.8465, -0.5939, -0.2969\\ -6.9232, -0.2969, -0.1485\end{bmatrix}\times$ $\boldsymbol{x}<0\}$, $\Omega_2=\{\boldsymbol{x}|\boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}23.2917, 9.9988, 4.9994\\ 9.9988, -2.3998, -1.1999\\ 4.9994, -1.1999, -0.5999\end{bmatrix}\boldsymbol{x}<0\}$

The simulation results are depicted in Figs 3 $\sim$ 6.

The simulation results for the system state responses in the closed-loop with the same initial state $\boldsymbol{x}_0=[1,2,-1]^{\mathrm{T}}$ are shown in Fig. 3. It is clearly seen that the closed-loop system of the switched system (25) with the designed controllers (27) and the switching law (28) is asymptotically stable. Fig. 4 gives the input signal of the switched system (25). The trajectory of the sliding function (26) is shown in Fig. 5. The switching signal is given in Fig. 6.

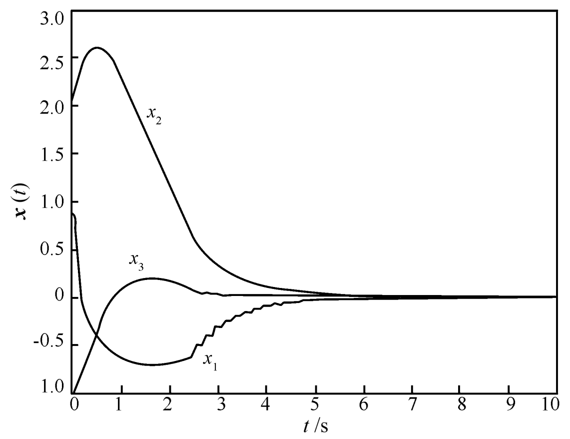

Fig. 3 The system state responses of the switched system (25)

4 Conclusion

This paper has developed a new approach to the robust H_∞ control problem for a class of uncertain switched systems by constructing the single robust H_∞ sliding surface.

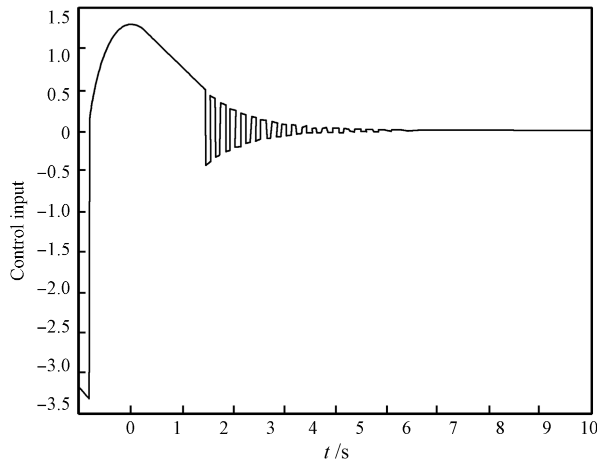

Fig. 4 The input signal of the switched system (25)

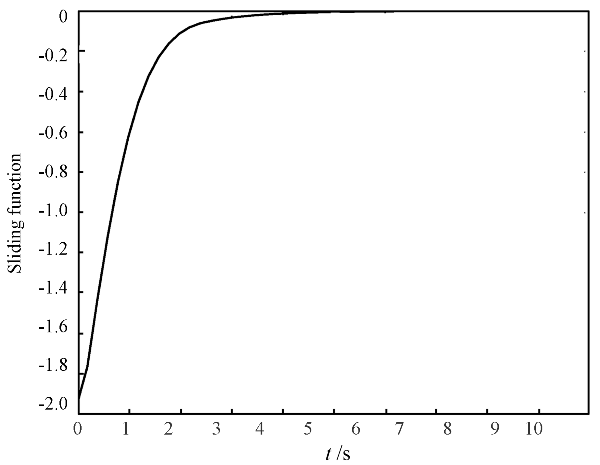

Fig. 5 The trajectory of the sliding function (26)

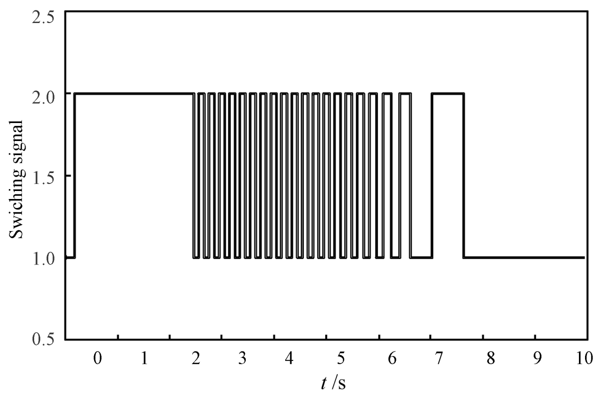

Fig. 6 The switching signal (28)

The sufficient condition for the existence of the single robust H_∞ sliding surface has been derived in terms of Riccati inequality associated with the convex combination of the switched system. The switching law has been constructed such that the $n-m$ dimensional sliding motion is robustly stabilizable with H_∞ disturbance attenuation level γ. Variable structure controllers have been designed to drive the state of the switched system to reach the single robust H_∞ sliding surface in a finite time.

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