Static Output Feedback Control for Discrete-time Piecewise Linear Systems: an LMI Approach

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Abstract This paper investigates the problem of static output feedback (SOF) control for discrete-time piecewise linear systems. Based on piecewise quadratic Lyapunov functions, new sufficient LMI conditions for the synthesis of SOF stabilization controllers are presented. Meanwhile, by using Finsler′s lemma, a set of slack variables with special structure are introduced to reduce design conservatism. Compared to the existing methods, the proposed method has a good performance and can work successfully in situations where the existing methods fail. An extension of this method is also given in order to incorporate H_∞ performance. Three examples are given to illustrate the effectiveness of our method.

Key words Piecewise linear systems, piecewise quadratic Lyapunov function, Finsler′s lemma, static output feedback (SOF), linear matrix inequality (LMI)

Piecewise linear systems have been receiving much attention in control and system and circuit communities because a large class of nonlinear systems, such as systems with relay, saturation, or dead-zone, can be modeled as piecewise linear systems[1]. In fact, piecewise linear systems are a broad modeling class in the sense that they have been shown to be equivalent to many other classes of systems, such as mixed logic dynamical systems[2] and extended linear complementary systems[3].

Since [4] presented a pioneering work on the analysis of discrete-time piecewise linear systems in the early 1980 s, numerous results[1−17] have been obtained on analysis and synthesis of piecewise linear systems. For example, [6−7] presented results on stability and optimal performance analysis for continuous-time piecewise linear systems based on a piecewise quadratic Lyapunov function. Reference [16] extended the stability analysis method of [6] to discrete-time piecewise linear systems. Meanwhile, controller design for piecewise linear systems arose, such as [8−11] for continuous-time systems and [15, 17] for discrete-time systems. For stability analysis and control synthesis of piecewise linear systems, there are two major differences between the continuous-time and discrete-time case[12]. First, in the former, only continuous Lyapunov functions can be used, whereas in the latter, discontinuous Lyapunov functions are also allowed. Second, in the discrete-time case, switching can also occur between non-adjacent regions and this fact must be properly handled in analysis and synthesis algorithms.

On the other hand, static output feedback (SOF) is one of the most important open problems in control theory and practice. It represents the simplest closed-loop control system, which can be easily implemented with low cost. Therefore, the problem has been extensively studied for the past decades. To deal with the SOF control problem of linear systems, there are various approaches; see [18−21] and references therein.

This paper studies the problem of SOF control for discrete-time piecewise linear systems. In [22−23], the problem of SOF control for discrete-time switched linear systems was investigated and sufficient LMI conditions were given to obtain controller gains. These two methods are also applicable to discrete-time piecewise linear systems. In this paper, based on piecewise quadratic Lyapunov functions, new sufficient LMI conditions for the synthesis of SOF stabilization controllers are given. Meanwhile, by using Finsler′s lemma, a set of slack variables with special structure are introduced to reduce design conservatism. Compared to the methods in [22−23], our method proves to have a good performance and can work successfully in situations where the methods in [22−23] do not. In addition, an extension of this method is also given in order to incorporate H_∞ performance.

The rest of the paper is organized as follows. Section 1 gives the problem statement. Section 2 gives new sufficient LMI-based conditions for SOF stabilization of discrete-time piecewise linear systems. Section 3 extends the method to H_∞ SOF control. Section 4 gives three examples to illustrate the effectiveness of the proposed methods. Finally, we conclude the paper in Section 5.

Notations. We use standard notations throughout this paper. $M^{\rm T}$ is the transpose of matrix M and $M^{-{\rm T}}$ means $(M^{-1})^{\rm T}$. $M > 0$ (< 0) means that M is positive (negative) definite. The symbol "$*$" is used in some matrix expressions to induce a symmetric structure. L^2 is the Lebesgue space consisting of all discrete-time vector-valued functions that are square-summable over $[0, 1, 2, \cdots, \infty)$.

1 Problem statement

Consider the following discrete-time piecewise linear system

$$\begin{cases} \boldsymbol{x}(k+1) = A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k) \\ \boldsymbol{y}(k) = C_i\boldsymbol{x}(k) \end{cases}, \quad \text{for } \boldsymbol{x} \in X_i, i \in I_l \quad (1)$$

where $\boldsymbol{x}(k) \in \mathbf{R}^n$ is the state, $\boldsymbol{u}(k) \in \mathbf{R}^m$ is the control input, and $\boldsymbol{y}(k) \in \mathbf{R}^p$ is the measured output. $\{X_i\}_{i\in I_l} \subseteq \mathbf{R}^n$ denotes a partition of the state space X into a number of closed polyhedral subspaces, i.e., $I_l = \{1, 2, \cdots, l\}$ is the index set of subspaces. We refer to each X_i as a cell. Let S be the set of all ordered pairs (i, j) of indices, denoting the possible switches from cell i to cell j

$$S = \{(i, j) : i, j \in I_l \text{ such that } \boldsymbol{x}(k) \in X_i \text{ and } \boldsymbol{x}(k+1) \in X_j\} \quad (2)$$

The set S can be determined via reachability analysis for mixed logic dynamical (MLD) systems[2].

In this paper, we investigate the SOF stabilization problem, i.e., the problem of designing a static output feedback

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control law

$$\boldsymbol{u}(k) = K_i\boldsymbol{y}(k), \quad i \in I_l \tag{3}$$

such that the closed-loop piecewise linear system

$$\boldsymbol{x}(k+1) = A_{cli}\boldsymbol{x}(k) \tag{4}$$

with

$$A_{cli} = A_i + B_iK_iC_i \tag{5}$$

is exponentially stable.

Without loss of generality, it is assumed that B_i (or C_i), $i = 1, 2, \cdots, l$ are of full column (or row) rank. Then, there exist nonsingular transformation matrices T_{bi} (or T_{ci}), $i = 1, 2, \cdots, l$ such that

$$T_{bi}B_i = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{6}$$

$$C_iT_{ci} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{7}$$

Note that for any given B_i (or C_i), the corresponding T_{bi} (or T_{ci}) are generally not unique. Special T_{bi} and T_{ci} can be obtained by

$$T_{bi} = \begin{bmatrix} (B_i^{\mathrm{T}}B_i)^{-1}B_i^{\mathrm{T}} \\ B_i^{\mathrm{T}\perp\mathrm{T}} \end{bmatrix} \tag{8}$$

$$T_{ci} = \begin{bmatrix} C_i^{\mathrm{T}}(C_iC_i^{\mathrm{T}})^{-1} & C_i^{\perp} \end{bmatrix} \tag{9}$$

where $B_i^{\mathrm{T}\perp\mathrm{T}}$ denotes the transpose of an orthogonal basis for the null space of B_i^{T}, and $C_i^{\perp}$ denotes an orthogonal basis for the null space of C_i.

The following lemmas are useful throughout this paper.

Lemma 1 (Finsler′s lemma). Let $\boldsymbol{\xi} \in \mathbf{R}^n$, $P = P^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, and $H \in \mathbf{R}^{m\times n}$ such that $\mathrm{rank}(H) = r < n$; then the following statements are equivalent:

1) $\boldsymbol{\xi}^{\mathrm{T}}\mathcal{P}\boldsymbol{\xi} < 0$, for all $\boldsymbol{\xi} \neq 0$, $H\boldsymbol{\xi} = 0$;

2) $\exists X \in \mathbf{R}^{n\times m}$ such that $P + XH + H^{\mathrm{T}}X^{\mathrm{T}} < 0$.

Lemma 2[13, 15]**.** If there exist matrices $P_i = P_i^{\mathrm{T}} > 0$, $\forall i \in I_l$ such that the positive definite function $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P_i\boldsymbol{x}$, $\forall \boldsymbol{x} \in X_i$ satisfies $V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) < 0$, then the closed-loop piecewise linear system (4) is exponentially stable.

The piecewise quadratic Lyapunov function appearing in Lemma 2 can be computed as[13, 15]

$$\begin{aligned} A_{cli}^{\mathrm{T}}P_jA_{cli} - P_i < 0, \quad \forall (i,j) \in S \\ P_i = P_i^{\mathrm{T}} > 0, \quad \forall i \in I_l \end{aligned}$$

where S is given in (2).

2 SOF stabilization

In this section, based on a piecewise quadratic Lyapunov function and Finsler′s lemma, new sufficient LMI conditions are deduced to obtain the SOF control gains K_i.

Theorem 1. Assume that B_i, $i = 1, 2, \cdots, l$ are of full column rank. If there exist symmetric matrices $\bar{P}_i$, $\bar{P}_j \in \mathbf{R}^{n\times n}$, and matrices $G_i \in \mathbf{R}^{n\times n}$, $L_i \in \mathbf{R}^{n\times p}$ with the following structure

$$G_i = \begin{bmatrix} G_{i11} & G_{i12} \\ 0 & G_{i22} \end{bmatrix}, \quad L_i = \begin{bmatrix} L_{i1} \\ 0 \end{bmatrix} \tag{10}$$

satisfying the inequalities

$$\begin{bmatrix} \bar{P}_j - G_i - G_i^{\mathrm{T}} & G_i\bar{A}_i + L_i\bar{C}_i \\ * & -\bar{P}_i \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{11}$$

where

$$\bar{A}_i = T_{bi}A_iT_{bi}^{-1}, \quad \bar{C}_i = C_iT_{bi}^{-1} \tag{12}$$

and T_{bi} are given by (8), then the closed-loop piecewise linear system (4) is exponentially stable and the control gain K_i can be obtained by

$$K_i = G_{i11}^{-1}L_{i1}, \quad i \in I_l \tag{13}$$

Proof. Assume that LMIs (11) are feasible. From the structure of L_i and G_i, and from (6) and (13), we can obtain

$$\begin{aligned} L_i = \begin{bmatrix} L_{i1} \\ 0 \end{bmatrix} = \begin{bmatrix} G_{i11}K_i \\ 0 \end{bmatrix} = \\ \begin{bmatrix} G_{i11} & G_{i12} \\ 0 & G_{i22} \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix}K_i = G_iT_{bi}B_iK_i \end{aligned} \tag{14}$$

Define $P_j = T_{bi}^{\mathrm{T}}\bar{P}_jT_{bi}$, $P_i = T_{bi}^{\mathrm{T}}\bar{P}_iT_{bi}$, then

$$\bar{P}_j = T_{bi}^{-\mathrm{T}}P_jT_{bi}^{-1}, \quad \bar{P}_i = T_{bi}^{-\mathrm{T}}P_iT_{bi}^{-1} \tag{15}$$

Substituting (12), (14), and (15) into (11), we have

$$\begin{bmatrix} T_{bi}^{-\mathrm{T}}P_jT_{bi}^{-1} - G_i - G_i^{\mathrm{T}} & \Xi_i \\ * & -T_{bi}^{-\mathrm{T}}P_iT_{bi}^{-1} \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{16}$$

where $\Xi_i = G_iT_{bi}A_iT_{bi}^{-1} + G_iT_{bi}B_iK_iC_iT_{bi}^{-1}$.

Pre- and post-multiplying (16) by $\begin{bmatrix} T_{bi}^{\mathrm{T}} & 0 \\ 0 & T_{bi}^{\mathrm{T}} \end{bmatrix}$ and its transpose, we have

$$\begin{bmatrix} P_j - T_{bi}^{\mathrm{T}}G_iT_{bi} - T_{bi}^{\mathrm{T}}G_i^{\mathrm{T}}T_{bi} & T_{bi}^{\mathrm{T}}G_iT_{bi}A_{cli} \\ * & -P_i \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{17}$$

Inequalities (17) can be written in the form

$$P + XH + H^{\mathrm{T}}X^{\mathrm{T}} < 0, \quad \forall (i,j) \in S \tag{18}$$

where

$$\begin{aligned} P = \begin{bmatrix} P_j & 0 \\ 0 & -P_i \end{bmatrix}, \quad X = \begin{bmatrix} T_{bi}^{\mathrm{T}}G_iT_{bi} \\ 0 \end{bmatrix} \\ H = \begin{bmatrix} -I & A_{cli} \end{bmatrix} \end{aligned} \tag{19}$$

Define $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{x}(k) \end{bmatrix}$, then the closed-loop system (4) can be written in the form

$$H\boldsymbol{\xi} = 0 \tag{20}$$

It follows from Finsler′s lemma that (18) is equivalent to

$$\boldsymbol{\xi}^{\mathrm{T}}P\boldsymbol{\xi} < 0, \quad \forall (i,j) \in S \tag{21}$$

Then, we have

$$\begin{bmatrix} \boldsymbol{x}^{\mathrm{T}}(k+1) & \boldsymbol{x}^{\mathrm{T}}(k) \end{bmatrix}\begin{bmatrix} P_j & 0 \\ 0 & -P_i \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{x}(k) \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{22}$$

that is,

$$\boldsymbol{x}^{\mathrm{T}}(k+1)P_j\boldsymbol{x}(k+1) - \boldsymbol{x}^{\mathrm{T}}(k)P_i\boldsymbol{x}(k) < 0, \quad \forall (i,j) \in S \tag{23}$$

Substituting (4) into (23) obtains

$$A_{cli}^{\mathrm{T}}P_jA_{cli} - P_i < 0, \quad \forall (i,j) \in S$$

It follows from (11) and (15) that

$$P_i = P_i^{\mathrm{T}} > 0, \quad \forall i \in I_l$$

Based on Lemma 2, the closed-loop piecewise linear system (4) is exponentially stable. □

Note that the invertibility of G_{i11} can be assured by condition (11) in Theorem 1. See the following lemma.

Lemma 3. If LMIs (11) are feasible, then G_{i11} are invertible.

Proof. It follows from (11) that

$$\bar{P}_j - G_i - G_i^{\mathrm{T}} < 0, \quad \forall (i,j) \in S$$
$$-\bar{P}_i < 0, \quad \forall i \in I_l$$

then, we have $G_i + G_i^{\mathrm{T}} > \bar{P}_j > 0$. This implies G_i are invertible[24]. Therefore, G_{i11}, the block $(1,1)$ of G_i, are invertible. □

In the same way, the invertibility of G_{i11} can be assured by the conditions of Theorems $2\sim4$ below.

Theorem 2. Assume that C_i, $i \in I_l$ are of full row rank. If there exist symmetric matrices $\bar{P}_i$ and $\bar{P}_j \in \mathbf{R}^{n\times n}$, and matrices $G_i \in \mathbf{R}^{n\times n}$ and $L_i \in \mathbf{R}^{m\times n}$ with the following structure

$$G_i = \begin{bmatrix} G_{i11} & 0 \\ G_{i21} & G_{i22} \end{bmatrix}, \quad L_i = \begin{bmatrix} L_{i1} & 0 \end{bmatrix} \tag{24}$$

satisfying the inequalities

$$\begin{bmatrix} \bar{P}_j - G_i - G_i^{\mathrm{T}} & * \\ \bar{A}_iG_i + \bar{B}_iL_i & -\bar{P}_i \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{25}$$

where

$$\bar{A}_i = T_{ci}^{-1}A_iT_{ci}, \quad \bar{B}_i = T_{ci}^{-1}B_i \tag{26}$$

and T_{ci} are given by (9), then the closed-loop piecewise linear system (4) is exponentially stable and the control gain K_i can be obtained by

$$K_i = L_{i1}G_{i11}^{-1}, \quad i \in I_l \tag{27}$$

Proof. From the structure of L_i and G_i, and from (7) and (27), we can obtain

$$L_i = \begin{bmatrix} L_{i1} & 0 \end{bmatrix} = \begin{bmatrix} K_iG_{i11} & 0 \end{bmatrix} =$$
$$K_i \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} G_{i11} & 0 \\ G_{i21} & G_{i22} \end{bmatrix} = K_iC_iT_{ci}G_i \tag{28}$$

Define $P_j = T_{ci}\bar{P}_jT_{ci}^{\mathrm{T}}$, $P_i = T_{ci}\bar{P}_iT_{ci}^{\mathrm{T}}$, then

$$\bar{P}_j = T_{ci}^{-1}P_jT_{ci}^{-\mathrm{T}}, \quad \bar{P}_i = T_{ci}^{-1}P_iT_{ci}^{-\mathrm{T}} \tag{29}$$

Substituting (26), (28), and (29) into (25), we have

$$\begin{bmatrix} T_{ci}^{-1}P_jT_{ci}^{-\mathrm{T}} - G_i - G_i^{\mathrm{T}} & * \\ T_{ci}^{-1}A_iT_{ci}G_i + T_{ci}^{-1}B_iK_iC_iT_{ci}G_i & -T_{ci}^{-1}P_iT_{ci}^{-\mathrm{T}} \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{30}$$

Pre- and post-multiplying (30) by $\begin{bmatrix} T_{ci} & 0 \\ 0 & T_{ci} \end{bmatrix}$ and its transpose, we have

$$\begin{bmatrix} P_j - T_{ci}G_iT_{ci}^{\mathrm{T}} - T_{ci}G_i^{\mathrm{T}}T_{ci}^{\mathrm{T}} & * \\ A_{cli}T_{ci}G_iT_{ci}^{\mathrm{T}} & -P_i \end{bmatrix} < 0, \quad \forall (i,j) \in \mathcal{S} \tag{31}$$

Inequalities (31) can be rewritten in the following form

$$P + XH + H^{\mathrm{T}}X^{\mathrm{T}} < 0, \quad \forall (i,j) \in S \tag{32}$$

where

$$P = \begin{bmatrix} P_j & 0 \\ 0 & -P_i \end{bmatrix}, \quad X = \begin{bmatrix} T_{ci}G_iT_{ci}^{\mathrm{T}} \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} -I & A_{cli}^{\mathrm{T}} \end{bmatrix}$$

Consider the dual system of (4)

$$\boldsymbol{x}'(k+1) = A_{cli}^{\mathrm{T}}\boldsymbol{x}'(k) \tag{33}$$

Define $\boldsymbol{\xi}' = \begin{bmatrix} \boldsymbol{x}'(k+1) \\ \boldsymbol{x}'(k) \end{bmatrix}$. Then, (33) can be rewritten in the form

$$H\boldsymbol{\xi}' = 0 \tag{34}$$

From Finsler′s lemma, we know that (32) is equivalent to

$$\boldsymbol{\xi}'^{\mathrm{T}}P\boldsymbol{\xi}' < 0 \tag{35}$$

then, we have

$$\begin{bmatrix} \boldsymbol{x}'^{\mathrm{T}}(k+1) & \boldsymbol{x}'^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} P_j & 0 \\ 0 & -P_i \end{bmatrix} \begin{bmatrix} \boldsymbol{x}'(k+1) \\ \boldsymbol{x}'(k) \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{36}$$

that is,

$$\boldsymbol{x}'^{\mathrm{T}}(k+1)P_j\boldsymbol{x}'(k+1) - \boldsymbol{x}'^{\mathrm{T}}(k)P_i\boldsymbol{x}'(k) < 0, \quad \forall (i,j) \in S \tag{37}$$

Substituting (33) into (37) yields

$$(A_{cli}^{\mathrm{T}})^{\mathrm{T}}P_jA_{cli}^{\mathrm{T}} - P_i < 0, \quad \forall (i,j) \in S$$

It follows from (25) and (29) that

$$P_i = P_i^{\mathrm{T}} > 0, \quad \forall i \in I_l$$

Based on Lemma 2, the closed-loop piecewise linear system (4) is exponentially stable. □

Remark 1. Theorems 1 and 2 present new sufficient LMI-based conditions for SOF stabilization control for discrete-time piecewise linear systems. These conditions are convex and numerically well tractable with commercially available software[25−26]. Free slack variables G_i with special structure are introduced to reduce design conservatism.

Remark 2. For discrete-time piecewise linear systems, the state may switch among non-adjacent regions of the state-space partition. In this paper, we define the set S that contains all the ordered pairs of indices denoting the possible switches and can be computed via reachability analysis for MLD systems[2]. However, when designing an SOF controller, the set of all possible switches is seldom known in advance, and it may be necessary to consider all pairs of indices in $S_{all} = I_l \times I_l$. Therefore, our synthesis approach in Theorems 1 and 2 can be used to design SOF control for discrete-time switched linear systems with arbitrary switching.

Remark 3. The proposed method is different from that in [22], where equality constraint was imposed on Lyapunov matrics P_i or slack variables G_i. It also differs from the method in [23], which works well only when matrix $A_i^{(22)}$ (the block $(2,2)$ of A_i) are Schur stable. Numerical examples (in Section 4) will show that our method can work successfully in situations where [22−23] do not.

3 Extension to H_∞ SOF control

Consider the following discrete-time piecewise linear system

$$\begin{cases} \boldsymbol{x}(k+1) = A_i\boldsymbol{x}(k) + B_{i1}\boldsymbol{u}(k) + B_{i2}\boldsymbol{w}(k) \\ \boldsymbol{z}(k) = C_{i1}\boldsymbol{x}(k) + D_{i11}\boldsymbol{u}(k) + D_{i12}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) = C_{i2}\boldsymbol{x}(k) + D_{i21}\boldsymbol{w}(k) \end{cases} \tag{38}$$

for $\boldsymbol{x}(k) \in X_i$, $i \in I_l$, where $\boldsymbol{x}(k) \in \mathbf{R}^n$ is the system state, $\boldsymbol{u}(k) \in \mathbf{R}^m$ is the control input, $\boldsymbol{w}(k) \in \mathbf{R}^r$ is disturbance input, $\boldsymbol{z}(k) \in \mathbf{R}^q$ is the controlled output, and $\boldsymbol{y}(k) \in \mathbf{R}^p$ is the measured output. Let S be the set of all ordered pairs (i, j) of indices, denoting the possible switches from cell i to cell j

$$S = \{(i,j) : i,j \in I_l \text{ such that } \boldsymbol{x}(k) \in X_i \text{ and } \boldsymbol{x}(k+1) \in X_j\} \tag{39}$$

In this section, we will design SOF control for the discrete-time piecewise linear system (38) in the H_∞ framework: Given a real number $\gamma > 0$, the exogenous signal $\boldsymbol{w}$ is attenuated by γ if, assuming $\boldsymbol{x}(0) = \boldsymbol{0}$, for each integer $N \geq 0$ and for every $\boldsymbol{w} \in L_2([0, N], \mathbf{R}^r)$

$$\sum_{k=0}^{N} \|\boldsymbol{z}(k)\|^2 < \gamma^2 \sum_{k=0}^{N} \|\boldsymbol{w}(k)\|^2 \tag{40}$$

With the controller (3), the closed-loop piecewise linear system becomes

$$\begin{cases} \boldsymbol{x}(k+1) = A_{cli}\boldsymbol{x}(k) + B_{cli}\boldsymbol{w}(k) \\ \boldsymbol{z}(k) = C_{cli}\boldsymbol{x}(k) + D_{cli}\boldsymbol{w}(k) \end{cases}, \quad \text{for } i \in I_l \tag{41}$$

where

$$\begin{aligned} A_{cli} &= A_i + B_{i1}K_iC_{i2}, & B_{cli} &= B_{i2} + B_{i1}K_iD_{i21} \\ C_{cli} &= C_{i1} + D_{i11}K_iC_{i2}, & D_{cli} &= D_{i12} + D_{i11}K_iD_{i21} \end{aligned} \tag{42}$$

Without loss of generality, we assume that B_{i1} (or C_{i2}), $i = 1, \cdots, l$ are of full column (or row) rank. Then, there exist nonsingular transformation matrices T_{bi} (or T_{ci}), $i = 1, \cdots, l$ such that

$$T_{bi}B_{i1} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad C_{i2}T_{ci} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{43}$$

Note that for any given B_{i1} (or C_{i2}), the corresponding T_{bi} (or T_{ci}) are generally not unique. Special T_{bi} and T_{ci} can be given as

$$T_{bi} = \begin{bmatrix} (B_{i1}^{\mathrm{T}}B_{i1})^{-1}B_{i1}^{\mathrm{T}} \\ B_{i1}^{\mathrm{T}\perp\mathrm{T}} \end{bmatrix}, \quad T_{ci} = \begin{bmatrix} C_{i2}^{\mathrm{T}}(C_{i2}C_{i2}^{\mathrm{T}})^{-1} & C_{i2}^{\perp} \end{bmatrix} \tag{44}$$

The following lemma is useful in this section.

Lemma 4[13, 15]**.** Consider the piecewise linear system (38) with zero initial condition $\boldsymbol{x}(0) = \boldsymbol{0}$. If there exists a function $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P_i\boldsymbol{x}$, $\forall \boldsymbol{x} \in X_i$ with $P_i = P_i^{\mathrm{T}} > 0$ satisfying the following inequality

$$V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) < \gamma^2\|\boldsymbol{w}(k)\|^2 - \|\boldsymbol{z}(k)\|^2, \quad \forall k \tag{45}$$

then the H_∞ performance condition (40) is satisfied. Furthermore, the closed-loop piecewise linear system (41) is exponentially stable.

The piecewise quadratic Lyapunov function appearing in Lemma 3 can be computed as[13, 15]

$$\boldsymbol{x}^{\mathrm{T}}(k+1)P_j\boldsymbol{x}(k+1) - \boldsymbol{x}^{\mathrm{T}}(k)P_i\boldsymbol{x}(k) < \gamma^2\boldsymbol{w}^{\mathrm{T}}(k)\boldsymbol{w}(k) - \boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k), \quad \forall (i,j) \in S$$

and

$$P_i = P_i^{\mathrm{T}} > 0, \quad \forall i \in I_l$$

Now, we give sufficient LMI conditions to obtain H_∞ SOF control gains K_i.

Theorem 3. Assume that D_{i11} are null matrices and B_{i1}, $i = 1, \cdots, l$ are full column rank matrices. If there exist symmetric matrices $\bar{P}_i$ and $\bar{P}_j \in \mathbf{R}^{n\times n}$ and matrices $G_i \in \mathbf{R}^{n\times n}$ and $L_i \in \mathbf{R}^{n\times p}$ with the following structure

$$G_i = \begin{bmatrix} G_{i11} & G_{i12} \\ 0 & G_{i22} \end{bmatrix}, \quad L_i = \begin{bmatrix} L_{i1} \\ 0 \end{bmatrix} \tag{46}$$

satisfying the inequalities

$$\begin{bmatrix} \Pi_{ij} & 0 & G_i\bar{A}_i + L_i\bar{C}_{i2} & G_i\bar{B}_{i2} + L_iD_{i21} \\ * & -I & \bar{C}_{i1} & D_{i12} \\ * & * & -\bar{P}_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{47}$$

where $\Pi_{ij} = \bar{P}_j - G_i - G_i^{\mathrm{T}}$ and

$$\begin{aligned} \bar{A}_i &= T_{bi}A_iT_{bi}^{-1}, & \bar{B}_{i2} &= T_{bi}B_{i2} \\ \bar{C}_{i1} &= C_{i1}T_{bi}^{-1}, & \bar{C}_{i2} &= C_{i2}T_{bi}^{-1} \end{aligned} \tag{48}$$

and T_{bi} are given by (44), then the piecewise linear system (38) is stabilized by the SOF controller (3) and the H_∞-norm of the closed-loop system (41) is smaller than γ, i.e., $\sum_{k=0}^{N} \|\boldsymbol{z}(k)\|^2 < \gamma^2 \sum_{k=0}^{N} \|\boldsymbol{w}(k)\|^2$. The control gains K_i can be obtained by

$$K_i = G_{i11}^{-1}L_{i1}, \quad i \in I_l \tag{49}$$

Proof. Assume that LMIs (47) are feasible and define $P_j = T_{bi}^{\mathrm{T}}\bar{P}_jT_{bi}$, $P_i = T_{bi}^{\mathrm{T}}\bar{P}_iT_{bi}$. Then,

$$\bar{P}_j = T_{bi}^{-\mathrm{T}}P_jT_{bi}^{-1}, \quad \bar{P}_i = T_{bi}^{-\mathrm{T}}P_iT_{bi}^{-1} \tag{50}$$

Substituting (48) and (50) into (47) leads to

$$\begin{bmatrix} \Upsilon_{ij} & 0 & G_iT_{bi}A_iT_{bi}^{-1} + L_iC_{i2}T_{bi}^{-1} & \Gamma_i \\ * & -I & C_{i1}T_{bi}^{-1} & D_{i12} \\ * & * & -T_{bi}^{-\mathrm{T}}P_iT_{bi}^{-1} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{51}$$

where $\Upsilon_{ij} = T_{bi}^{-\mathrm{T}}P_jT_{bi}^{-1} - G_i - G_i^{\mathrm{T}}$, $\Gamma_i = G_iT_{bi}B_{i2} + L_iD_{i21}$.

Pre- and post-multiplying (51) by

$$\begin{bmatrix} T_{bi}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & T_{bi}^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose yields

$$\begin{bmatrix} \Phi_{ij} & 0 & T_{bi}^{\mathrm{T}}G_iT_{bi}A_i + T_{bi}^{\mathrm{T}}L_iC_{i2} & \Lambda_i \\ * & -I & C_{i1} & D_{i12} \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{52}$$

where

$$\Phi_{ij} = P_j - T_{bi}^{\mathrm{T}}G_iT_{bi} - T_{bi}^{\mathrm{T}}G_i^{\mathrm{T}}T_{bi}$$

$$\Lambda_i = T_{bi}^{\mathrm{T}}G_iT_{bi}B_{i2} + T_{bi}^{\mathrm{T}}L_iD_{i21}$$

From the structure of L_i and G_i and from (43) and (49), we have

$$L_i = \begin{bmatrix} L_{i1} \\ 0 \end{bmatrix} = \begin{bmatrix} G_{i11}K_i \\ 0 \end{bmatrix} = \begin{bmatrix} G_{i11} & G_{i12} \\ 0 & G_{i22} \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix} K_i = G_i T_{bi} B_{i1} K_i \tag{53}$$

By simple algebraic operation, we can obtain

$$\begin{aligned} T_{bi}^{\rm T} G_i T_{bi} A_i + T_{bi}^{\rm T} L_i C_{i2} &= T_{bi}^{\rm T} G_i T_{bi} A_{cli} \\ T_{bi}^{\rm T} G_i T_{bi} B_{i2} + T_{bi}^{\rm T} L_i D_{i21} &= T_{bi}^{\rm T} G_i T_{bi} B_{cli} \end{aligned} \tag{54}$$

Due to the assumption that $D_{i11} = 0$, we have

$$C_{i1} = C_{cli}, \quad D_{i12} = D_{cli} \tag{55}$$

By substituting (54) and (55) into (52), it follows that

$$\begin{bmatrix} \Phi_{ij} & 0 & T_{bi}^{\rm T} G_i T_{bi} A_{cli} & T_{bi}^{\rm T} G_i T_{bi} B_{cli} \\ * & -I & C_{cli} & D_{cli} \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{56}$$

Inequalities (56) can be written in the form

$$P + XH + H^{\rm T} X^{\rm T} < 0, \quad \forall (i,j) \in S \tag{57}$$

where

$$P = \begin{bmatrix} P_j & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -P_i & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \quad X = \begin{bmatrix} T_{bi}^{\rm T} G_i T_{bi} & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$H = \begin{bmatrix} -I & 0 & A_{cli} & B_{cli} \\ 0 & -I & C_{cli} & D_{cli} \end{bmatrix} \tag{58}$$

In addition, the closed-loop piecewise linear system (41) can be written in the form

$$H\boldsymbol{\xi}(k) = 0 \tag{59}$$

where

$$\boldsymbol{\xi}(k) = \begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \\ \boldsymbol{w}(k) \end{bmatrix} \tag{60}$$

It follows from Finsler′s lemma that (57) is equivalent to

$$\boldsymbol{\xi}^{\rm T}(k) P \boldsymbol{\xi}(k) < 0 \tag{61}$$

Substituting (58) and (60) into (62) yields

$$\boldsymbol{x}^{\rm T}(k+1) P_j \boldsymbol{x}(k+1) - \boldsymbol{x}^{\rm T}(k) P_i \boldsymbol{x}(k) < \gamma^2 \boldsymbol{w}^{\rm T}(k)\boldsymbol{w}(k) - \boldsymbol{z}^{\rm T}(k)\boldsymbol{z}(k), \quad \forall (i,j) \in S \tag{62}$$

It follows from (47) and (50) that

$$P_i = P_i^{\rm T} > 0$$

Based on Lemma 4, the closed-loop piecewise linear system is exponentially stable and the H_∞-norm is smaller than γ. □

Theorem 4. Assume that D_{i21} are null matrices and C_{i2}, $i = 1, \cdots, l$ are full row rank matrices. If there exist symmetric matrices $\bar{P}_i$ and $\bar{P}_j \in \mathbf{R}^{n\times n}$ and matrices $G_i \in \mathbf{R}^{n\times n}$ and $L_i \in \mathbf{R}^{m\times n}$ with the following structure

$$G_i = \begin{bmatrix} G_{i11} & 0 \\ G_{i21} & G_{i22} \end{bmatrix}, \quad L_i = \begin{bmatrix} L_{i1} & 0 \end{bmatrix} \tag{63}$$

satisfying the inequalities

$$\begin{bmatrix} \bar{P}_j - G_i - G_i^{\rm T} & * & * & * \\ 0 & -I & * & * \\ \bar{A}_i G_i + \bar{B}_{i1} L_i & \bar{B}_{i2} & -\bar{P}_i & * \\ \bar{C}_{i1} G_i + D_{i11} L_i & D_{i12} & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{64}$$

where

$$\begin{aligned} \bar{A}_i &= T_{ci}^{-1} A_i T_{ci}, \quad & \bar{B}_{i1} &= T_{ci}^{-1} B_{i1} \\ \bar{B}_{i2} &= T_{ci}^{-1} B_{i2}, \quad & \bar{C}_{i1} &= C_{i1} T_{ci} \end{aligned} \tag{65}$$

and T_{ci} are given in (44), then the piecewise linear system (38) is stabilized by the SOF controller (3) and the H_∞-norm of the closed-loop system (41) is smaller than γ, i.e., $\sum_{k=0}^N \|\boldsymbol{z}(k)\|^2 < \gamma^2 \sum_{k=0}^N \|\boldsymbol{w}(k)\|^2$. The control gains K_i can be obtained by

$$K_i = L_{i1} G_{i11}^{-1}, \quad i \in I_l \tag{66}$$

Proof. Assume that the LMI conditions (64) are feasible. Define $P_j = T_{ci} \bar{P}_j T_{ci}^{\rm T}$ and $P_i = T_{ci} \bar{P}_i T_{ci}^{\rm T}$. Then,

$$\bar{P}_j = T_{ci}^{-1} P_j T_{ci}^{-\rm T}, \quad \bar{P}_i = T_{ci}^{-1} P_i T_{ci}^{-\rm T} \tag{67}$$

Substituting (65) and (67) into (64) yields

$$\begin{bmatrix} T_{ci}^{-1} P_j T_{ci}^{-\rm T} - G_i - G_i^{\rm T} & * & * & * \\ 0 & -I & * & * \\ T_{ci}^{-1} A_i T_{ci} G_i + T_{ci}^{-1} B_{i1} L_i & T_{ci}^{-1} B_{i2} & \Omega_i & * \\ C_{i1} T_{ci} G_i + D_{i11} L_i & D_{i12} & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{68}$$

where $\Omega_i = -T_{ci}^{-1} P_i T_{ci}^{-\rm T}$.

Pre- and post-multiplying (68) by

$$\begin{bmatrix} T_{ci} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & T_{ci} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose leads to

$$\begin{bmatrix} P_j - T_{ci} G_i T_{ci}^{\rm T} - T_{ci} G_i^{\rm T} T_{ci}^{\rm T} & * & * & * \\ 0 & -I & * & * \\ A_i T_{ci} G_i T_{ci}^{\rm T} + B_{i1} L_i T_{ci}^{\rm T} & B_{i2} & -P_i & * \\ C_{i1} T_{ci} G_i + D_{i11} L_i & D_{i12} & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{69}$$

From the structure of L_i and G_i, and from (43) and (66), we have

$$L_i = \begin{bmatrix} L_{i1} & 0 \end{bmatrix} = \begin{bmatrix} K_i G_{i11} & 0 \end{bmatrix} = K_i \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} G_{i11} & 0 \\ G_{i21} & G_{i22} \end{bmatrix} = K_i C_{i2} T_{ci} G_i \tag{70}$$

By simple algebraic operation, we can obtain

$$\begin{aligned} A_iT_{ci}G_iT_{ci}^{\mathrm{T}} + B_{i1}L_iT_{ci}^{\mathrm{T}} &= A_{cli}T_{ci}G_iT_{ci}^{\mathrm{T}} \\ C_{i1}T_{ci}G_i + D_{i11}L_i &= C_{cli}T_{ci}G_i \end{aligned} \tag{71}$$

Due to the assumption that $D_{i21} = 0$, we have

$$B_{i2} = B_{cli}, \quad D_{i12} = D_{cli} \tag{72}$$

By substituting (71) and (72) into (69), it follows that

$$\begin{bmatrix} P_j - T_{ci}G_iT_{ci}^{\mathrm{T}} - T_{ci}G_i^{\mathrm{T}}T_{ci}^{\mathrm{T}} & * & * & * \\ 0 & -I & * & * \\ A_{cli}T_{ci}G_iT_{ci}^{\mathrm{T}} & B_{cli} & -P_i & * \\ C_{cli}T_{ci}G_i & D_{cli} & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i,j) \in S \tag{73}$$

Inequalities (73) can be rewritten in the form

$$P + XH + H^{\mathrm{T}}X^{\mathrm{T}} < 0, \quad \forall (i,j) \in S \tag{74}$$

where

$$P = \begin{bmatrix} P_j & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -P_i & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \tag{75}$$

$$X = \begin{bmatrix} T_{ci}G_iT_{ci}^{\mathrm{T}} & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -I & 0 & A_{cli}^{\mathrm{T}} & C_{cli}^{\mathrm{T}} \\ 0 & -I & B_{cli}^{\mathrm{T}} & D_{cli}^{\mathrm{T}} \end{bmatrix} \tag{76}$$

Consider the dual system of (41)

$$\begin{aligned} \boldsymbol{x}'(k+1) &= A_{cli}^{\mathrm{T}}\boldsymbol{x}'(k) + C_{cli}^{\mathrm{T}}\boldsymbol{w}'(k) \\ \boldsymbol{z}'(k) &= B_{cli}^{\mathrm{T}}\boldsymbol{x}'(k) + D_{cli}^{\mathrm{T}}\boldsymbol{w}'(k) \end{aligned} \tag{77}$$

which can be written in the form

$$H\boldsymbol{\xi}'(k) = 0$$

where

$$\boldsymbol{\xi}'(k) = \begin{bmatrix} \boldsymbol{x}'(k+1) \\ \boldsymbol{z}'(k) \\ \boldsymbol{x}'(k) \\ \boldsymbol{w}'(k) \end{bmatrix} \tag{78}$$

It follows from Finsler′s lemma that (74) is equivalent to

$$\boldsymbol{\xi}'^{\mathrm{T}}(k)P\boldsymbol{\xi}'(k) < 0 \tag{79}$$

Substituting (76) and (78) into (79) yields

$$\begin{aligned} &\boldsymbol{x}'^{\mathrm{T}}(k+1)P_j\boldsymbol{x}'(k+1) - \boldsymbol{x}'^{\mathrm{T}}(k)P_i\boldsymbol{x}'(k) < \\ &\quad \gamma^2\boldsymbol{w}'^{\mathrm{T}}(k)\boldsymbol{w}'(k) - \boldsymbol{z}'^{\mathrm{T}}(k)\boldsymbol{z}'(k), \quad \forall (i,j) \in S \end{aligned} \tag{80}$$

It follows from (64) and (67) that

$$P_i = P_i^{\mathrm{T}} > 0$$

Based on Lemma 4, the closed-loop piecewise linear system is exponentially stable and the H_∞-norm is smaller than γ. □

Remark 4. Theorems 3 and 4 present sufficient LMI conditions for H_∞ SOF control for discrete-time piecewise linear systems. By using Finsler′s lemma, a set of slack variables G_i with special structure are introduced to improve H_∞ performance and to reduce design conservatism. Note that by letting $C_i = I$ in (4) or $C_{i2} = I$ in (38) and slack variables G_i be general matrices, SOF control in Theorems 1 ∼ 4 reduces to the state-feedback control in [14].

Remark 5. In Theorems 1 and 2, it is assumed that B_i are of full column rank or C_i are of full row rank. If this assumption is not satisfied, i.e., both B_i and C_i are not of full rank, we can introduce non-singular linear transformation to system (1) and obtain a new system model satisfying the assumption[27]. Then, Theorems 1 and 2 can be used for the newly-built model. Therefore, this method can also be used to deal with the situation where both B_{i1} and C_{i2} in system (38) are not of full rank.

4 Examples

In this section, three examples are given to illustrate the effectiveness of our method. Examples 1 and 2 provide a comparison of the proposed method to the methods presented in [22−23]. These two examples show that our synthesis method can work successfully in situations where [22−23] do not, respectively. In Example 3, an H_∞ SOF controller is designed to show the effectiveness of Theorems 3 and 4.

Example 1. Consider the following system borrowed from [23]

$$\boldsymbol{x}(k+1) = A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k), \quad i = 1,2,3,4$$

where

$$A_1 = \begin{bmatrix} 0.7786 & 0.9908 & 0.1270 \\ 0.1616 & 0.8443 & 0.8144 \\ 0.9214 & 0.9747 & 0.7825 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.3894 & 0.3263 & 0.7746 \\ 0.7806 & 0.9886 & 0.1297 \\ 0.8814 & 0.4718 & 0.3110 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0.3049 & 0.4247 & 0.8979 \\ 0.8448 & 0.2485 & 0.6921 \\ 0.7558 & 0.9160 & 0.3636 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0.1194 & 0.3964 & 0.2454 \\ 0.1034 & 0.2515 & 0.4983 \\ 0.6981 & 0.8655 & 0.2403 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 0.2458 & 0.7409 \\ 0.2501 & 0.5257 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2722 & 0.6055 \\ 0.1576 & 0.1580 \\ 0 & 0 \end{bmatrix}$$
$$B_3 = \begin{bmatrix} 0.4945 & 0.3020 \\ 0.9237 & 0.9118 \\ 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0.9894 & 0.7205 \\ 0.1709 & 0.1519 \\ 0 & 0 \end{bmatrix}$$

Note that $A_1 \sim A_4$ are all unstable. The system is allowed to switch arbitrarily between these four modes. Output matrices are selected as[23]

$$C_1 = \begin{bmatrix} 0.3815 & 0.6916 & 0.7183 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0.0591 & 0.8258 & 0.4354 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0.5204 & 0.8010 & 0.9708 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 0.6995 & 0.3081 & 0.8767 \end{bmatrix}$$

Reference [23] has shown that this system cannot be stabilized using the method in [22]. However, it can be stabilized using our method and the control gains are given as

$$K_1 = \begin{bmatrix} -5.1512 \\ 0.6370 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -4.1075 \\ -0.0307 \end{bmatrix}$$
$$K_3 = \begin{bmatrix} -3.2011 \\ 2.4172 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 1.7127 \\ -2.9989 \end{bmatrix}$$

These SOF control gains give the following closed-loop poles for each modes:

$$\text{Mode 1}: \{0.6351, 0.4040 \pm 0.5478\mathrm{i}\}$$
$$\text{Mode 2}: \{0.7029, 0.1902 \pm 0.6019\mathrm{i}\}$$
$$\text{Mode 3}: \{0.4600, -0.2949 \pm 0.3924\mathrm{i}\}$$
$$\text{Mode 4}: \{0.7737, -0.2693 \pm 0.3320\mathrm{i}\}$$

Example 2. Consider system (1) with three modes, which is described by the following matrices:

$$A_1 = \begin{bmatrix} 3 & 0.3 & 2 \\ 1 & 0 & 1 \\ 0.3 & 0.6 & 0.6 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.5871 & -0.8441 & -0.0092 \\ -0.6865 & -0.5090 & -0.8561 \\ 0.0974 & 0.4523 & -0.2280 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0.1089 & 0.2458 & -0.9035 \\ 0.3998 & -0.9213 & -0.4161 \\ 0.6745 & -0.5750 & 0.7138 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1930 & -0.4204 \\ -0.7359 & 0.0346 \\ 0.5073 & -0.9077 \end{bmatrix}$$
$$B_3 = \begin{bmatrix} -0.4164 & 0.0244 \\ 0.8297 & -0.4366 \\ -0.0900 & -0.8416 \end{bmatrix}$$

Note that all the modes are unstable. The output matrices are given as

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

For this system, the method developed in [23] does not allow to compute an SOF controller. However, our method provides the following control gains:

$$K_1 = \begin{bmatrix} -0.9273 \\ 0.0032 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -1.0162 \\ -0.4316 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0.3791 \\ 0.5438 \end{bmatrix}$$

These SOF control gains give the following closed-loop poles for each modes:

$$\text{Mode 1}: \{0.0412, 0.7174 \pm 0.6314\mathrm{i}\}$$
$$\text{Mode 2}: \{0.5364, -0.4631 \pm 0.2830\mathrm{i}\}$$
$$\text{Mode 3}: \{-0.8203, 0.1535 \pm 0.5217\mathrm{i}\}$$

Example 3. Consider system (38) with two modes. System matrices are given as

$$A_1 = \begin{bmatrix} -0.5871 & -0.8441 & -0.0092 \\ -0.6865 & -0.5090 & -0.8561 \\ 0.0974 & 0.4523 & -0.2280 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.1089 & 0.2458 & -0.9035 \\ 0.3998 & -0.9213 & -0.4161 \\ 0.6745 & -0.5750 & 0.7138 \end{bmatrix}$$
$$B_{11} = \begin{bmatrix} 0.1930 & -0.4204 \\ -0.7359 & 0.0346 \\ 0.5073 & -0.9077 \end{bmatrix}$$
$$B_{21} = \begin{bmatrix} -0.4164 & 0.0244 \\ 0.8297 & -0.4366 \\ -0.0900 & -0.8416 \end{bmatrix}$$
$$B_{12} = B_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_{11} = C_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$C_{12} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$
$$D_{111} = D_{211} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_{112} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad D_{212} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

D_{121}, D_{221} are null matrices.

By Theorem 4, the control gains are obtained as

$$K_1 = \begin{bmatrix} -1.0832 \\ -0.5259 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.3563 \\ -0.1241 \end{bmatrix}$$

and the H_∞-norm is 5.6853.

The following two figures are the responses of open-loop and closed-loop states with initial states chosen as $\boldsymbol{x}(0) = [-2 \quad 2 \quad 4]^{\mathrm{T}}$ and disturbances chosen as $w = \begin{cases} 2, & k < 10 \\ 0, & k \geq 10 \end{cases}$. Fig. 1 shows that the open-loop system is unstable and Fig. 2 shows that the closed-loop system is exponentially stable.

5 Conclusion

In this paper, the problem of SOF control for discrete-time piecewise linear systems has been addressed. By the aid of piecewise quadratic Lyapunov functions combined with Finsler′s lemma, new sufficient LMI conditions for the synthesis of SOF stabilization controllers have been given. The proposed method can work successfully where the existing ones do not. Extension to H_∞ control has also been presented. The numerical examples have shown the effectiveness of the proposed methods.

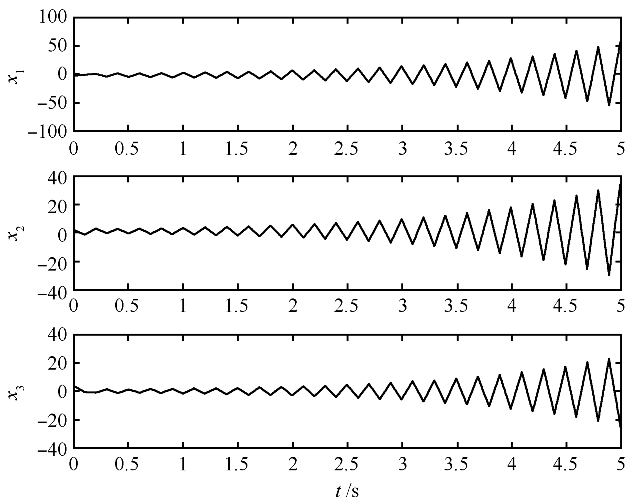

Fig. 1 Responses of open-loop states

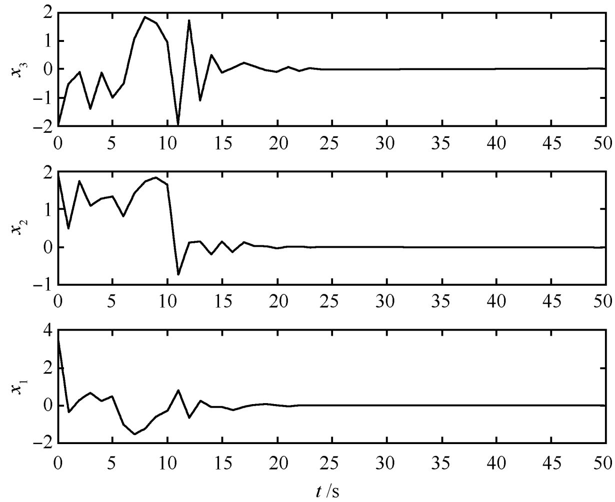

Fig. 2 Responses of closed-loop states

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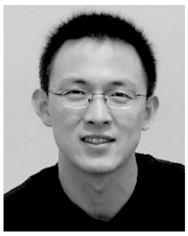

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