H_∞ Measurement-feedback Control for Systems with Input and Measurement Delays

WANG Hong-Xia[1] ZHANG Huan-Shui[2]

Abstract The paper concentrates on finding the H_∞ measurement-feedback control-law for systems with not only an input-delay but also a measurement-delay. Krein space, together with pseudo-measurements, is introduced so that the reorganizing technique can be utilized after we convert the original problem into a minimizing one. Finally, we get the desired controller and see that the separation principle is also applicable to delay systems to some extent.

Key words Krein space, H_∞ control, measurement-feedback, reorganizing innovation

Because of the presence of uncertain exogenous disturbances and model uncertainties, the H_∞ control has intrigued many researchers for decades. The study can be traced back to 1981, when it was originally proposed by Zames[1]. Research related to the H_∞ control stepped into the new epoch as the state-space idea was introduced by Doyle[2]. There are abundant results related to the H_∞ control in time domain[3−6] and frequency domain[1, 7−11]. As a late comer, the time-domain approach gave an impetus to research the H_∞ control problems for time-varying systems, nonlinear systems[12], delay systems[6, 10, 13−16], stochastic systems and so on.

However, it seems hard to find results related to the H_∞ measurement-feedback control for systems with I/O delays except a few works[4, 7, 11, 13]. Mirkin[13] and Meinsma[7] first investigated systems with single I/O delay. Several years later, they generalized nontrivially the single-delay study to the multiple I/O delay case[11]. Transfer functions are the key medium[7, 11, 13] to solve the H_∞ control problem. Unfortunately, they are "aliens" for the time-varying systems. Liu′s idea[4] is applicable to the time-varying systems, but it is only allowed for systems with the delay-measurement.

The paper considers the systems with measurement-delay and input-delay. It needs no transfer function and can be generalized to deal with time-varying or/and multi-delay systems.

We shall organize the paper as follows. The underlying system and problem are presented in Section 1. Section 2 is devoted to achieving an expected quadratic form, which makes a critical preparation for solving the problem. Section 3 works out the problem by means of the indefinite quadratic form and the reorganizing technique. Section 4 provides a numerical experiment and Section 5 arrives at some conclusions.

Notations. Throughout the paper, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ denotes the inner product of vector $\boldsymbol{x}, \boldsymbol{y}$; $col\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ denotes the column vector formed by stacking all the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$; and 0_s denotes the zero matrix of $s \times s$.

1 Problem statement

Consider a linear system described by the following discrete-time model

$$\boldsymbol{x}(t+1) = \Phi\boldsymbol{x}(t) + B_0\boldsymbol{u}_0(t) + B_1\boldsymbol{u}_1(t_d) + G\boldsymbol{w}(t) \tag{1}$$

$$\boldsymbol{y}_0(t) = H_0\boldsymbol{x}(t) + \boldsymbol{v}_0(t) \tag{2}$$

$$\boldsymbol{y}_1(t) = H_1\boldsymbol{x}(t_d) + \boldsymbol{v}_1(t) \tag{3}$$

$$\boldsymbol{z}(t) = \begin{bmatrix} C\boldsymbol{x}(t) \\ D_0\boldsymbol{u}_0(t) \\ D_1\boldsymbol{u}_1(t_d) \end{bmatrix} \tag{4}$$

where $t_d = t - d$. In the following, similar denotation will take the same meaning in the rest paper. $\boldsymbol{x}(t) \in \mathbf{R}^n$, $\boldsymbol{w}(t) \in \mathbf{R}^p$, $\boldsymbol{u}_0(t), \boldsymbol{u}_1(t) \in \mathbf{R}^m$, $\boldsymbol{v}_0(t), \boldsymbol{v}_1(t) \in \mathbf{R}^s$, $\boldsymbol{y}_0(t), \boldsymbol{y}_1(t) \in \mathbf{R}^s$, and $\boldsymbol{z}(t) \in \mathbf{R}^r$ represent the state, input noise, control inputs, measurement noise, measurement outputs, and the signal to be regulated, respectively.

For convenience, denote

$$\underline{\tau}_j = \tau_d - j,\ \overline{\tau}_j = \tau_d + j,\ \underline{\tau}_i = \tau_d - i,\ \overline{\tau}_i = \tau_d + i \tag{5}$$

$$\underline{N}_j = N_d - j,\ \overline{N}_j = N_d + j,\ \underline{N}_i = N_d - i,\ \overline{N}_i = N_d + i \tag{6}$$

$$\mathcal{K}_j^{41} = \begin{bmatrix} K_0^j \\ K_1^j \\ H_0 \\ H_1 \end{bmatrix},\ \mathcal{K}_j^{42} = \begin{bmatrix} K_0^j & 0 \\ K_1^j & 0 \\ H_0 & 0 \\ 0 & H_1 \end{bmatrix},\ \mathcal{K}_j^{21} = \begin{bmatrix} K_0^j \\ H_0 \end{bmatrix} \tag{7}$$

$$\mathcal{K}_j^{31} = \begin{bmatrix} K_0^j \\ K_1^j \\ H_0 \end{bmatrix},\ \mathcal{H}_j^{31} = \begin{bmatrix} K_0^j \\ H_0 \\ H_1 \end{bmatrix},\ \mathcal{H}_j^{32} = \begin{bmatrix} K_0^j & 0 \\ H_0 & 0 \\ 0 & H_1 \end{bmatrix} \tag{8}$$

The H_∞ control problem under investigation is stated as: given a scalar $\gamma > 0$ and observations $\{\boldsymbol{y}(j)\}_{j=0}^t$, find a finite-horizon measurement-feedback control strategy for (1) ∼ (4)

$$\boldsymbol{u}_i(t) = \boldsymbol{F}_i(\boldsymbol{y}(j)\ |_{0\le j\le t}),\quad i = 0, 1$$

such that

$$\sup_{\mathcal{S}} J(\boldsymbol{x}(0), \boldsymbol{w}(t), \boldsymbol{u}_j(t), \boldsymbol{v}_j(t)) < \gamma^2 \tag{9}$$

where

$$J(\boldsymbol{x}(0), \boldsymbol{w}(t), \boldsymbol{u}_j(t), \boldsymbol{v}_j(t)) = \frac{\|\ \boldsymbol{z}\ \|_{[0,N]}^2}{\boldsymbol{x}^{\mathrm{T}}(0)\Pi_0^{-1}\boldsymbol{x}(0) + \|\ \boldsymbol{w}\ \|_{[0,N]}^2 + \|\ \boldsymbol{v}\ \|_{[0,N]}^2} \tag{10}$$

$$\mathcal{S} = \{\boldsymbol{x}(0), \boldsymbol{w}(t), \boldsymbol{v}_0(t), \boldsymbol{v}_1(s)\ |\ 0 \le t \le N, d \le s \le N\}$$

$\boldsymbol{y}(j) = \boldsymbol{y}_0(j)$ for $0 \le j < d$, $\boldsymbol{y}(j) = col\{\boldsymbol{y}_0(j), \boldsymbol{y}_1(j)\}$ for $j \ge d$, $\boldsymbol{v}(j)$ is totally similar to $\boldsymbol{y}(j)$, and Π_0 is a given positive definite matrix, which reflects the uncertainty of the initial state relative to the energy of the exogenous input.

Let us start by categorizing the problem into three cases with the relationship between the delay d and the terminal time N.

Received on May 25, 2007; in revised form April 3, 2008
Supported by National Natural Science Foundation of China (60574016)
1. Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, P. R. China 2. Control Science and Engineering School, Shandong University, Jinan 250061, P. R. China
DOI: 10.3724/SP.J.1004.2008.01417

Case 1: $N < d$;
Case 2: $d \leq N < 2d$;
Case 3: $N \geq 2d$.

With the disappearance of $\boldsymbol{u}_1$ and $\boldsymbol{y}_1$, the problem in Case 1 is simplified to be a standard H_∞ measurement-feedback control problem for delay-free systems, and can be solved readily[17]; both Case 2 and Case 3 can be analyzed by the similar line of arguments, Krein space, and the reorganizing technique. Therefore, we only need to handle one of them. Here, let Case 3 be the one to be discussed in the remainder of the paper.

2 Construction of ideal quadratic form

In this section, we shall make a crucial preparation for solving the H_∞ measurement-feedback control problem on the basis of the results[16].

For convenience, we first introduce a couple of Riccati equations[16] as

$$P_j = \Phi^{\mathrm{T}} P_{j+1}\Phi + Q - \Phi^{\mathrm{T}} P_{j+1}\Gamma_j (R_j + \Gamma_j^{\mathrm{T}} P_{j+1}\Gamma_j)^{-1}\Gamma_j^{\mathrm{T}} P_{j+1}\Phi$$
$$j = N, N-1, \cdots, 0,\ P_{N+1} = 0 \tag{11}$$

$$P_j^\tau = \Phi^{\mathrm{T}} P_{j+1}^\tau\Phi + Q - \Phi^{\mathrm{T}} P_{j+1}^\tau\Gamma_j (R_j + \Gamma_j^{\mathrm{T}} P_{j+1}^\tau\Gamma_j)^{-1}\Gamma_j^{\mathrm{T}} P_{j+1}^\tau\Phi$$
$$j = \min\{d, N_d + 1\}, \cdots, 0,\ P_{N_\tau+1}^\tau = 0, \text{ or } P_d^\tau = P_{\tau+d} \tag{12}$$

Obviously, $P_j^\tau = P_{j+\tau}$ as $j \geq d$, where

$$Q = C^{\mathrm{T}} C,\ \Gamma_j = \begin{cases} [B_0, G], & 0 \leq j < d \\ [B_0, G, B_1], & j \geq d \end{cases}$$

$$R_j = \begin{cases} \mathrm{diag}\{D_0^{\mathrm{T}} D_0, -\gamma^2 I_p\}, & 0 \leq j < d \\ \mathrm{diag}\{D_0^{\mathrm{T}} D_0, -\gamma^2 I_p, D_1^{\mathrm{T}} D_1\}, & j \geq d \end{cases}$$

Assume that the above Riccati equations have bounded solutions, and denote

$$M_j^\tau = R_j + \Gamma_j^{\mathrm{T}} P_{j+1}^\tau \Gamma_j,\ K_j^\tau = \Phi^{\mathrm{T}} P_{j+1}^\tau \Gamma_j (M_j^\tau)^{-1} \tag{13}$$

throughout the paper.

Remark 1. It should be noted that

1) Despite the same structures of the above two Riccati equations, they still have different solutions because of different initial values and the range of j.

2) The two equations have the same structures for linear time invariant (LTI) systems. Nevertheless, for linear time varying (LTV) systems, they just have similar structure of the one in [16] but not the same one.

Considering the performance index (9), we define

$$J_N^\infty = \boldsymbol{x}^{\mathrm{T}}(0)\Pi_0^{-1}\boldsymbol{x}(0) + \|\ \boldsymbol{v}\ \|_{[0,N]}^2 - \gamma^{-2} J_N \tag{14}$$

where

$$J_N = \|\ \boldsymbol{z}\ \|_{[0,N]}^2 - \gamma^2 \|\ \boldsymbol{w}\ \|_{[0,N]}^2 \tag{15}$$

It is clear that the H_∞ controller $\boldsymbol{u}_i(t)$ satisfies (9) if and only if it renders that J_N^∞ in (14) is positive for all non-zero $\{\boldsymbol{x}(0);\ \boldsymbol{w}(t),\ \boldsymbol{v}_0(t),\ 0 \leq t \leq N;\ \boldsymbol{v}_1(t),\ d \leq t \leq N\}$.

It is not hard to see that (14) is almost the same as (9) in [16], but the additional term $\|\ \boldsymbol{v}\ \|_{[0,N]}^2$ is only involved by the measurement equations; what is more important is that (6) in [16] and (15) are totally identical. Therefore, in view of Lemma 7 in [16], (14) is rewritten as

$$J_N^\infty = \boldsymbol{x}^{\mathrm{T}}(0)(\Pi_0^{-1} - \gamma^{-2} P_0)\boldsymbol{x}(0) + \|\ \boldsymbol{v}\ \|_{[0,N]}^2 -$$
$$\gamma^{-2}\sum_{\tau=0}^{N} [\boldsymbol{v}_r(\tau) - \check{\boldsymbol{v}}_r(\tau)]^{\mathrm{T}} \tilde{M}_\tau [\boldsymbol{v}_r(\tau) - \check{\boldsymbol{v}}_r(\tau)] \tag{16}$$

where

$$\boldsymbol{v}_r(\tau) = col\{\boldsymbol{u}(\tau),\ \boldsymbol{w}(\tau)\} \tag{17}$$

with

$$\boldsymbol{u}(\tau) = \begin{cases} col\{\boldsymbol{u}_0(\tau), \boldsymbol{u}_1(\tau)\}, & 0 \leq \tau \leq N_d \\ col\{\boldsymbol{u}_0(\tau)\}, & \tau > N_d \end{cases} \tag{18}$$

$\check{\boldsymbol{v}}_r(\tau)$ is obtained from $\boldsymbol{v}_r(\tau)$ with $\boldsymbol{w}(\tau)$ and $\boldsymbol{u}_i(\tau)(i = 0, 1)$ replaced by $\check{\boldsymbol{w}}(\tau) = [0_m\ I_p]\check{\boldsymbol{v}}_0(\tau)$, $\check{\boldsymbol{u}}_0(\tau) = [I_m\ 0_p]\check{\boldsymbol{v}}_0(\tau)$, and $\check{\boldsymbol{u}}_1(\tau) = \check{\boldsymbol{v}}_1(\tau)$, respectively, while $\check{\boldsymbol{v}}_i(\tau)$ is given by

$$\check{\boldsymbol{v}}_0(\tau) = -[\mathcal{F}_0^\tau(0)]^{\mathrm{T}}\boldsymbol{x}(\tau) - \sum_{k=1}^{d} [\mathcal{F}_k^\tau(0)]^{\mathrm{T}} B_1 \check{\boldsymbol{u}}_1(k - 1 + \tau_d) \tag{19}$$

$$\check{\boldsymbol{v}}_1(\tau) = -[0_{m+p}\ I_m]\Big\{[\mathcal{F}_0^\tau(d)]^{\mathrm{T}}\boldsymbol{x}(\tau) + \sum_{k=1}^{d} [\mathcal{F}_k^\tau(d)]^{\mathrm{T}} B_1 \check{\boldsymbol{u}}_1(k - 1 + \tau_d)\Big\} \tag{20}$$

In (19) and (20),

$$\mathcal{F}_k^\tau(t) = \begin{cases} P_k^\tau[(\bar{\Phi}_{t+1,k}^\tau)^{\mathrm{T}}\Gamma_t(M_t^\tau)^{-1} - (\bar{\Phi}_{t,k}^\tau)^{\mathrm{T}} G^\tau(k) K_t^\tau], & 0 \leq t \leq k-1 \\ [I_n - P_k^\tau G^\tau(k)]\,\bar{\Phi}_{k,t}^\tau K_t^\tau, & k \leq t \leq N \end{cases} \tag{21}$$

with

$$\bar{\Phi}_j^\tau = \Phi^{\mathrm{T}} - K_j^\tau \Gamma_j^{\mathrm{T}},\ \bar{\Phi}_{j,j}^\tau = I,\ \bar{\Phi}_{j,t}^\tau = \bar{\Phi}_j^\tau \cdots \bar{\Phi}_{t-1}^\tau,\ t \geq j \tag{22}$$

$$G^\tau(k) = \sum_{j=1}^{k} (\bar{\Phi}_{j,k}^\tau)^{\mathrm{T}}\Gamma_{j-1}(M_{j-1}^\tau)^{-1}\Gamma_{j-1}^{\mathrm{T}}\bar{\Phi}_{j,k}^\tau,\ G^\tau(0) = 0 \tag{23}$$

and K_t^τ is defined as in (13). A careful observation will show us that $\bar{\Phi}_{j,t}^\tau$ is actually the state transition matrix corresponding to $\bar{\Phi}_j^\tau$.

Next, we will provide an important parameter $\tilde{M}_\tau$ in (16).

For $0 \leq \tau \leq N_d$,

$$\tilde{M}_\tau = \Theta_\tau^{\mathrm{T}} \bar{P}_{\tau+1}\Theta_\tau + \mathrm{diag}\{D_0^{\mathrm{T}} D_0, D_1^{\mathrm{T}} D_1; -\gamma^2 I_p\} \tag{24}$$

For $N_d < \tau \leq N$,

$$\tilde{M}_\tau = \Theta_\tau^{\mathrm{T}} \bar{P}_{\tau+1}\Theta_\tau + \mathrm{diag}\{D_0^{\mathrm{T}} D_0; -\gamma^2 I_p\} \tag{25}$$

In the above, $\Theta_\tau = \begin{cases} \mathrm{diag}\{B_0, B_1; G\}, & 0 \leq \tau \leq N_d \\ \mathrm{diag}\{B_0; G\}, & N_d < \tau \leq N \end{cases}$

$$\bar{P}_\tau = \begin{cases} \begin{bmatrix} \bar{P}_\tau(0,0) & \bar{P}_\tau^{\mathrm{T}}(1,0) \\ \bar{P}_\tau(1,0) & \bar{P}_\tau(1,1) \end{bmatrix}, & \tau \leq N_d \\ \bar{P}_\tau(0,0), & \tau > N_d \end{cases} \tag{26}$$

and $\bar{P}_\tau(i,j)(i \geq j)$ is given by

$$\begin{cases} \bar{P}_\tau(0,0) = P_0^\tau \\ \bar{P}_\tau(1,0) = P_d^\tau(\bar{\Phi}_{0,d}^\tau)^{\mathrm{T}} \\ \bar{P}_\tau(1,1) = P_d^\tau[I - G^\tau(d)P_d^\tau] \end{cases} \tag{27}$$

where $\bar{\Phi}_{0,d}^\tau$ and $G^\tau(d)$ are defined as in (22) and (23), respectively, and P_0^τ and P_d^τ can be obtained from (12).

Let

$$R_v = \begin{cases} I_s, & 0 \le \tau < d \\ \mathrm{diag}\{I_s,\ I_s\}, & \tau \ge d \end{cases} \tag{28}$$

$$H^\tau = \begin{cases} H_0, & 0 \le \tau < d \\ \mathrm{diag}\{H_0,\ H_1\}, & \tau \ge d \end{cases} \tag{29}$$

$$\bar{\boldsymbol{x}}(\tau) = \begin{cases} \boldsymbol{x}(\tau), & 0 \le \tau < d \\ col\{\boldsymbol{x}(\tau),\ \boldsymbol{x}(\tau_d)\}, & \tau \ge d \end{cases} \tag{30}$$

$$\boldsymbol{y}(\tau) = \begin{cases} \boldsymbol{y}_0(\tau), & 0 \le \tau < d \\ col\{\boldsymbol{y}_0(\tau),\ \boldsymbol{y}_1(\tau)\}, & \tau \ge d \end{cases} \tag{31}$$

Now, (16) allows us to write J_N^∞ as

$$\begin{aligned} J_N^\infty =\ & \boldsymbol{x}^{\mathrm{T}}(0)(\Pi_0^{-1} - \gamma^{-2}P_0)\boldsymbol{x}(0) - \\ & \gamma^{-2}\sum_{\tau=0}^{N} \begin{bmatrix} \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \end{bmatrix}^{\mathrm{T}} \tilde{M}_\tau \begin{bmatrix} \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \end{bmatrix} + \\ & \sum_{\tau=0}^{N} [\boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau)]^{\mathrm{T}} R_v^{-1} [\boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau)] \end{aligned} \tag{32}$$

In order to achieve an ideal representation, we make a fundamental transformation and then, get the matrix $\hat{M}_\tau$ as well as the expected quadratic form as follows:

$$\hat{M}_\tau = -\gamma^{-2} \begin{cases} \begin{bmatrix} & I_{2m} \\ I_p & \end{bmatrix}^{\mathrm{T}} \tilde{M}_\tau \begin{bmatrix} & I_{2m} \\ I_p & \end{bmatrix}, & 0 \le \tau \le N_d \\ \begin{bmatrix} & I_m \\ I_p & \end{bmatrix}^{\mathrm{T}} \tilde{M}_\tau \begin{bmatrix} & I_m \\ I_p & \end{bmatrix}, & \tau > N_d \end{cases} \tag{33}$$

$$\begin{aligned} J_N^\infty =\ & \boldsymbol{x}^{\mathrm{T}}(0)(\Pi_0^{-1} - \gamma^{-2}P_0)\boldsymbol{x}(0) + \\ & \sum_{\tau=0}^{N} \begin{bmatrix} \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \\ \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau) \end{bmatrix}^{\mathrm{T}} \times \\ & \begin{bmatrix} \hat{M}_\tau & \\ & R_v^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \\ \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau) \end{bmatrix} \end{aligned} \tag{34}$$

Remark 2. Equation (34) deserves such a troublesome transformation since it guarantees that we can find the causal controllers $\boldsymbol{u}_0(\tau)$ and $\boldsymbol{u}_1(\tau)$. Otherwise, only the casual $\boldsymbol{u}_0(\tau)$ can be obtained.

3 Main results

The main results will be stated in the following section.

Proposition 1. If $\hat{M}_\tau$ are invertible for any τ, then, it can be partitioned as

$$\hat{M}_\tau^{-1} = \begin{bmatrix} \Delta_\tau^{-1} & \bar{S}_\tau \\ \bar{S}_\tau^{\mathrm{T}} & (\bar{\Delta}_\tau)^{-1} \end{bmatrix} \tag{35}$$

where $\Delta_\tau^{-1}$is $p \times p$.

Remark 3. In order to guarantee that every element in performance functions as a lever, $\hat{M}_\tau$ tends to be invertible in practice. Therefore, the invertibility of $\hat{M}_\tau$ is reasonable.

Furthermore, we can use (35) to write (34) as

$$\begin{aligned} J_N^\infty =\ & \boldsymbol{x}^{\mathrm{T}}(0)(\Pi_0^{-1} - \gamma^{-2}P_0)\boldsymbol{x}(0) + \sum_{\tau=0}^{N} \begin{bmatrix} \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \\ \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau) \end{bmatrix}^{\mathrm{T}} \times \\ & \begin{bmatrix} Q_\tau^{\check{w}} & S_\tau \\ S_\tau^{\mathrm{T}} & Q_\tau^{v} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{w}(\tau) - \check{\boldsymbol{w}}(\tau) \\ \boldsymbol{u}(\tau) - \check{\boldsymbol{u}}(\tau) \\ \boldsymbol{y}(\tau) - H^\tau\bar{\boldsymbol{x}}(\tau) \end{bmatrix} \end{aligned} \tag{36}$$

where

$$Q_\tau^{\check{w}} = \Delta_\tau^{-1}, \quad 0 \le \tau \le N \tag{37}$$

$$S_\tau = \begin{cases} \begin{bmatrix} \bar{S}_\tau & 0 \end{bmatrix}, & 0 \le \tau < d \\ \begin{bmatrix} \bar{S}_\tau & 0 & 0 \end{bmatrix}, & \tau \ge d \end{cases} \tag{38}$$

$$Q_\tau^{v} = \begin{cases} \mathrm{diag}\{\bar{\Delta}_\tau^{-1},\ I_s\}, & 0 \le \tau < d \\ \mathrm{diag}\{\bar{\Delta}_\tau^{-1},\ I_s,\ I_s\}, & \tau \ge d \end{cases} \tag{39}$$

Note that $\boldsymbol{y}(\tau)$ is the observation of system (1) ~ (4) at time τ, which can be rewritten as

$$\boldsymbol{y}(\tau) = \begin{cases} H_0\boldsymbol{x}(\tau) + \boldsymbol{v}_0(\tau), & 0 \le \tau < d \\ \begin{bmatrix} H_0 & 0 \\ 0 & H_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\tau) \\ \boldsymbol{x}(\tau_d) \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_0(\tau) \\ \boldsymbol{v}_1(\tau) \end{bmatrix}, & \tau \ge d \end{cases} \tag{40}$$

3.1 Introduction of Krein space

For a technical reason, we associate the quadratic form (36) with Krein space state-space model as

$$\boldsymbol{X}(\tau+1) = (\Phi - GK_w^\tau)\boldsymbol{X}(\tau) + G\boldsymbol{W}(\tau) + \boldsymbol{C}(\tau) \tag{41}$$

$$\begin{bmatrix} \boldsymbol{U}(\tau) \\ \boldsymbol{Y}(\tau) \end{bmatrix} = \begin{bmatrix} \bar{K}^\tau \\ H^\tau \end{bmatrix} \bar{\boldsymbol{X}}(\tau) + \begin{bmatrix} \boldsymbol{s}_\tau(\boldsymbol{u}) \\ 0 \end{bmatrix} + \bar{\boldsymbol{V}}(\tau) \tag{42}$$

with $\boldsymbol{u} = \boldsymbol{u}(\cdot)$, where $\boldsymbol{u}(\cdot)$ is defined in (18), and

$$\begin{aligned} \boldsymbol{C}(\tau) =\ & B_0\boldsymbol{u}_0(\tau) + B_1\boldsymbol{u}_1(\tau_d) - \\ & G[0_m, I_p]\sum_{k=1}^{d} [\mathcal{F}_k^\tau(0)]^{\mathrm{T}} B_1\boldsymbol{u}_1(k-1+\tau_d) \end{aligned}$$

Note that $\boldsymbol{C}(\tau)$ only has access to the past inputs $\boldsymbol{u}_1(\tau-1), \cdots, \boldsymbol{u}_1(\tau-d)$, and the current input $\boldsymbol{u}_0(\tau)$.

The rest of the parameters in (41) ~ (42) are shown as

$$\begin{aligned} K_w^\tau &= -[0_m, I_p][\mathcal{F}_0^\tau(0)]^{\mathrm{T}} \\ \bar{K}^\tau &= \begin{cases} \begin{bmatrix} K_0^\tau & 0 \\ K_1^\tau & 0 \end{bmatrix}, & 0 \le \tau \le N_d \\ [K_0^\tau \ \ 0], & \tau > N_d \end{cases} \\ \boldsymbol{s}_\tau(\boldsymbol{u}) &= \begin{cases} col\{\boldsymbol{s}_{0,\tau}(\boldsymbol{u}),\ \boldsymbol{s}_{1,\tau}(\boldsymbol{u})\}, & 0 \le \tau \le N_d \\ col\{\boldsymbol{s}_{0,\tau}(\boldsymbol{u})\}, & \tau > N_d \end{cases} \\ \boldsymbol{U}(\tau) &= \begin{cases} col\{\boldsymbol{U}_0(\tau), \boldsymbol{U}_1(\tau)\}, & 0 \le \tau \le N_d \\ col\{\boldsymbol{U}_0(\tau)\}, & \tau > N_d \end{cases} \\ \bar{\boldsymbol{V}}(\tau) &= \begin{cases} col\{\boldsymbol{V}_u^0(\tau), \boldsymbol{V}_u^0(\tau), \boldsymbol{V}_0(\tau)\}, & 0 \le \tau \le d \\ col\{\boldsymbol{V}_u^0(\tau), \boldsymbol{V}_u^0(\tau), \boldsymbol{V}_0(\tau), \boldsymbol{V}_1(\tau)\}, & d \le \tau \le N_d \\ col\{\boldsymbol{V}_u^0(\tau), \boldsymbol{V}_0(\tau)\}, & \tau > N_d \end{cases} \end{aligned} \tag{43}$$

In these representations,

$$K_0^\tau = -[I_m, 0_p][\mathcal{F}_0^\tau(0)]^{\mathrm{T}} \tag{44}$$

$$K_1^\tau = -[0_{m+p}, I_m][\mathcal{F}_0^\tau(d)]^{\mathrm{T}} \tag{45}$$

$$\boldsymbol{U}_i(\tau) = K_i^\tau \boldsymbol{X}(\tau) + \boldsymbol{s}_{i,\tau}(\boldsymbol{u}) + \boldsymbol{V}_u^i,\ i = 0, 1 \tag{46}$$

$$\boldsymbol{s}_{0,\tau}(\boldsymbol{u}) = -[I_m, 0_p] \sum_{k=1}^{d} [\mathcal{F}_k^\tau(0)]^{\mathrm{T}} B_1 \boldsymbol{u}_1(k - 1 + \tau_d) \tag{47}$$

$$\boldsymbol{s}_{1,\tau}(\boldsymbol{u}) = -[0_{m+p}, I_m] \sum_{k=1}^{d} [\mathcal{F}_k^\tau(d)]^{\mathrm{T}} B_1 \boldsymbol{u}_1(k - 1 + \tau_d) \tag{48}$$

The initial state, $\boldsymbol{X}(0)$, and the driving and measurement disturbances, $\{\boldsymbol{W}(\tau)\}$ and $\{\bar{\boldsymbol{V}}(\tau)\}$, are Krein space variables impacting system (41) $\sim$ (42) and obey

$$\left\langle \begin{bmatrix} \boldsymbol{X}(0) \\ \boldsymbol{W}(\tau) \\ \bar{\boldsymbol{V}}(\tau) \end{bmatrix}, \begin{bmatrix} \boldsymbol{X}(0) \\ \boldsymbol{W}(r) \\ \bar{\boldsymbol{V}}(r) \end{bmatrix} \right\rangle = \begin{bmatrix} (\Pi_0 - \gamma^{-2} P_0)^{-1} & 0 \\ 0 & \begin{bmatrix} Q_\tau^{\tilde{w}} & S_\tau \\ S_\tau^{\mathrm{T}} & Q_\tau^v \end{bmatrix} \delta_{\tau r} \end{bmatrix} \tag{49}$$

Proposition 2. Assume that $\boldsymbol{\eta}_i = H\boldsymbol{\xi}_i + \boldsymbol{\nu}_i + \boldsymbol{\mu}$, where $\boldsymbol{\mu}, \boldsymbol{\nu}_i, \boldsymbol{\xi}_i$, and $\boldsymbol{\eta}_i$ are given constant vector, white noise with zero mean, state, and measurement, respectively. Then, its estimation and the corresponding innovation can be written as $\hat{\boldsymbol{\eta}}_i = H\hat{\boldsymbol{\xi}}_i + \boldsymbol{\mu}$ and $\boldsymbol{\omega}_i = H(\boldsymbol{\xi}_i - \hat{\boldsymbol{\xi}}_i) + \boldsymbol{\nu}_i$.

The proposition shows that innovation can not be impacted by the constant vector in the measurement.

Taking $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ in (36) as pseudo-measurements and recalling of Chapter 9 in [17] and Proposition 2, we can now establish the following results about J_∞^N.

Lemma 1. The minimum of J_∞^N is in the form of

$$J_\infty^N = \sum_{\tau=0}^{N} \begin{bmatrix} \boldsymbol{u}(\tau) - K^\tau \hat{\bar{\boldsymbol{x}}}(\tau \mid \tau - 1) - \boldsymbol{s}_\tau(\boldsymbol{u}) \\ \boldsymbol{y}(\tau) - H^\tau \hat{\bar{\boldsymbol{x}}}(\tau \mid \tau - 1) \end{bmatrix}^{\mathrm{T}} \times Q_{w_s}^{-1}(\tau) \begin{bmatrix} \boldsymbol{u}(\tau) - K^\tau \hat{\bar{\boldsymbol{x}}}(\tau \mid \tau - 1) - \boldsymbol{s}_\tau(\boldsymbol{u}) \\ \boldsymbol{y}(\tau) - H^\tau \hat{\bar{\boldsymbol{x}}}(\tau \mid \tau - 1) \end{bmatrix} \tag{50}$$

where

$$K^\tau = \begin{cases} \begin{bmatrix} K_0^\tau \\ K_1^\tau \end{bmatrix}, & 0 \le \tau \le N_d \\ [K_0^\tau], & \tau > N_d \end{cases} \tag{51}$$

$$\hat{\bar{\boldsymbol{x}}}(\tau \mid \tau - 1) = \begin{cases} col\{\hat{\boldsymbol{x}}(\tau|\tau - 1)\}, & 0 \le \tau < d \\ col\{\hat{\boldsymbol{x}}(\tau|\tau - 1),\ \hat{\boldsymbol{x}}(\tau_d|\tau - 1)\}, & \tau \ge d \end{cases} \tag{52}$$

In (50), the matrix $Q_{w_s}(\tau)$ is the covariance matrix of the innovations

$$\boldsymbol{w}_s(\tau) = \begin{cases} \mathcal{K}_\tau^{31} \left[\ \boldsymbol{X}(\tau) - \hat{\boldsymbol{X}}(\tau|\tau - 1)\ \right] + \bar{\boldsymbol{V}}(\tau), & 0 \le \tau < d \\ \mathcal{K}_\tau^{42} \begin{bmatrix} \boldsymbol{X}(\tau) - \hat{\boldsymbol{X}}(\tau|\tau - 1) \\ \boldsymbol{X}(\tau_d) - \hat{\boldsymbol{X}}(\tau_d|\tau - 1) \end{bmatrix} + \bar{\boldsymbol{V}}(\tau), & d \le \tau < N_d \\ \mathcal{H}_\tau^{32} \begin{bmatrix} \boldsymbol{X}(\tau) - \hat{\boldsymbol{X}}(\tau|\tau - 1) \\ \boldsymbol{X}(\tau_d) - \hat{\boldsymbol{X}}(\tau_d|\tau - 1) \end{bmatrix} + \bar{\boldsymbol{V}}(\tau), & N_d \le \tau \le N \end{cases} \tag{53}$$

where $\hat{\boldsymbol{X}}(\tau|\tau - 1)(\hat{\boldsymbol{X}}(\tau_d|\tau - 1))$ is the projection of $\boldsymbol{X}(\tau)(\boldsymbol{X}(\tau_d))$ onto the linear space $\mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}(i) \end{bmatrix}_{i=0}^{\tau-1} \right\}$.

The estimators $\hat{\boldsymbol{x}}(\tau|\tau - 1)$ and $\hat{\boldsymbol{x}}(\tau_d|\tau - 1)$ in (52) are obtained from the projections of $\boldsymbol{x}(\tau)$ and $\boldsymbol{x}(\tau_d)$ onto the linear space $\mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{u}(i) \\ \boldsymbol{y}(i) \end{bmatrix}_{i=0}^{\tau-1} \right\}$, respectively.

3.2 Reorganizing innovations

Observing (50) of the minimal value, we realize that the estimator $\hat{\bar{\boldsymbol{x}}}(\tau|\tau - 1)$ and the innovation covariance matrix $Q_{w_s}(\tau)$ are the key to design of the controller. At this stage, we face with the problem that the standard Kalman filter formulation is not applicable to computation of $\hat{\bar{\boldsymbol{x}}}(\tau|\tau - 1)$ and $Q_{w_s}(\tau)$. To conquer it, we shall reorganize the delay-measurements and define reorganized innovation, which, in fact, is the core of the innovation analysis method.

In view of (41) $\sim$ (43), it is not hard to find that

$$\mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}(i) \end{bmatrix}_{i=0}^{\tau} \right\} = \begin{cases} \mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}_{i=0}^{\tau} \right\}, & 0 \le \tau < d \\ \mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_f(i) \end{bmatrix}_{i=0}^{\tau_d}, \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}_{i=\tau_d+1}^{\tau} \right\}, & d \le \tau \le N_d \\ \mathcal{L}\left\{ \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_f(i) \end{bmatrix}_{i=0}^{\tau_d}, \begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}_{i=\tau_d+1}^{N_d}, \begin{bmatrix} \boldsymbol{U}_0(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}_{i=N_d+1}^{\tau} \right\}, & N_d < \tau \le N \end{cases} \tag{54}$$

where

$$\boldsymbol{Y}_f(i) = \begin{bmatrix} \boldsymbol{Y}_0(i) \\ \boldsymbol{Y}_1(i + d) \end{bmatrix} + \begin{bmatrix} H_0 \\ H_1 \end{bmatrix} \boldsymbol{X}(i) + \boldsymbol{V}_f(i) \tag{55}$$

$$\boldsymbol{V}_f(i) = col\{\boldsymbol{V}_0(i), \boldsymbol{V}_1(i + d)\},\ \langle \boldsymbol{V}_f(i), \boldsymbol{V}_f(i) \rangle = I_{2s} \tag{56}$$

$$\begin{cases} \left\langle \begin{bmatrix} \boldsymbol{V}_u^0(i) \\ \boldsymbol{V}_u^1(i) \end{bmatrix}, \begin{bmatrix} \boldsymbol{V}_u^0(i) \\ \boldsymbol{V}_u^1(i) \end{bmatrix} \right\rangle = \bar{\Delta}_i^{-1}, & 0 \le i \le N_d \\ \langle \boldsymbol{V}_u(i), \boldsymbol{V}_u(i) \rangle = \bar{\Delta}_i^{-1}, & i > N_d \end{cases} \tag{57}$$

To avoid confusion, we have to emphasize that just like $\boldsymbol{U}(i)$, the orders of $\bar{\Delta}_i^{-1}$ in (57) vary with index i.

Once the delay information is reorganized, it becomes the delay-free measurement so that we can apply it to design the controller directly. $\begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_f(i) \end{bmatrix}$, $\begin{bmatrix} \boldsymbol{U}(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}$, and $\begin{bmatrix} \boldsymbol{U}_0(i) \\ \boldsymbol{Y}_0(i) \end{bmatrix}$ are the so-called reorganized measurements. Equations (41), (55), and (43) form a state-space model without delay. Now, let us define the innovation sequence associated with the reorganized measurements.

$$\boldsymbol{w}^2(i) = \begin{bmatrix} \boldsymbol{U}_0(\overline{N}_i) \\ \boldsymbol{Y}_0(\overline{N}_i) \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{U}}_0(\overline{N}_i|\overline{N}_i - 1, N_d, N_{2d} + i) \\ \hat{\boldsymbol{Y}}_0(\overline{N}_i|\overline{N}_i - 1, N_d, N_{2d} + i) \end{bmatrix} = \mathcal{K}_{\overline{N}_i}^{21} \boldsymbol{E}^2(i) + \begin{bmatrix} \boldsymbol{V}_u^0(\overline{N}_i) \\ \boldsymbol{V}_0(\overline{N}_i) \end{bmatrix}, \quad i = 1, \cdots, d \tag{58}$$

$$\boldsymbol{w}^1(i) = \begin{bmatrix} \boldsymbol{U}(\overline{\tau}_i) \\ \boldsymbol{Y}_0(\overline{\tau}_i) \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{U}}(\overline{\tau}_i|\overline{\tau}_i - 1, \overline{\tau}_i - 1, \tau_{2d} + i) \\ \hat{\boldsymbol{Y}}_0(\overline{\tau}_i|\overline{\tau}_i - 1, \overline{\tau}_i - 1, \tau_{2d} + i) \end{bmatrix} = \mathcal{K}_{\overline{\tau}_i}^{31} \boldsymbol{E}^1(i) + \begin{bmatrix} \boldsymbol{V}_u^0(\overline{\tau}_i) \\ \boldsymbol{V}_u^1(\overline{\tau}_i) \\ \boldsymbol{V}_0(\overline{\tau}_i) \end{bmatrix}, \quad i = 1, \cdots, N_\tau \tag{59}$$

$$\boldsymbol{w}^0(i) = \left[\begin{array}{c}\boldsymbol{U}(i)\\ \boldsymbol{Y}_f(i)\end{array}\right] - \left[\begin{array}{c}\hat{\boldsymbol{U}}(i|i-1,i-1,i-1)\\ \hat{\boldsymbol{Y}}_f(i|i-1,i-1,i-1)\end{array}\right] = \mathcal{K}_i^{41}\boldsymbol{E}^0(i) + \left[\begin{array}{c}\boldsymbol{V}_u^0(i)\\ \boldsymbol{V}_u^1(i)\\ \boldsymbol{V}_f(i)\end{array}\right], \quad i=0,\cdots,\tau_d \tag{60}$$

with initial estimation value, $\left[\begin{array}{c}\hat{\boldsymbol{U}}(0|-1,-1,-1)\\ \hat{\boldsymbol{Y}}_f(0|-1,-1,-1)\end{array}\right] = 0$, and one-step estimation error,

$$\begin{aligned}
\boldsymbol{E}^2(i) &= \boldsymbol{X}(N_d+i) - \hat{\boldsymbol{X}}(N_d+i|N_d+i-1,N_d,N_{2d}+i)\\
\boldsymbol{E}^1(i) &= \boldsymbol{X}(\tau_d+i) - \hat{\boldsymbol{X}}(\tau_d+i|\tau_d+i-1,\tau_d+i-1,\tau_{2d}+i)\\
\boldsymbol{E}^0(i) &= \boldsymbol{X}(i) - \hat{\boldsymbol{X}}(i|i-1,i-1,i-1)
\end{aligned}$$

In the above equations, $\hat{\boldsymbol{X}}(j|t,s,r)$, $t\geq s\geq r$, is the estimation of $\boldsymbol{X}(j)$ utilizing the measurement sequence

$$\left\{\left[\begin{array}{c}\boldsymbol{U}(i)\\ \boldsymbol{Y}_f(i)\end{array}\right]_{i=0}^{r}, \left[\begin{array}{c}\boldsymbol{U}(i)\\ \boldsymbol{Y}_0(i)\end{array}\right]_{i=r+1}^{s}, \left[\begin{array}{c}\boldsymbol{U}_0(i)\\ \boldsymbol{Y}_0(i)\end{array}\right]_{i=s+1}^{t}\right\}$$

It is easy to verify that elements in the reorganized innovation $\{\boldsymbol{w}^i(\cdot)\}(i=0,1,2)$ are mutually uncorrelated.

3.3 Innovation covariance matrix and estimator

In this subsection, we shall provide the general form of the optimal estimator $\hat{\boldsymbol{x}}(\tau|t,s,r)(t\geq s\geq r)$ and the innovation covariance matrix $Q_{w_s}(\tau)$.

If we denote the covariance matrices of the estimation error $\boldsymbol{E}^i(j)$ as P_j^i, then, the covariance matrices $Q_{w^i}(j)$ of the innovation $\boldsymbol{w}^i(j)$ in (58) ∼ (60) can be given by

$$Q_{w^2}(j) = \mathcal{K}_{j+N_d}^{21}P_j^2(\mathcal{K}_{j+N_d}^{21})^{\rm T} + {\rm diag}\{\bar{\Delta}_{j+N_d}^{-1}, I_s\} \tag{61}$$

$$Q_{w^1}(j) = \mathcal{K}_{j+\tau_d}^{31}P_j^1(\mathcal{K}_{j+\tau_d}^{31})^{\rm T} + {\rm diag}\{\bar{\Delta}_{j+\tau_d}^{-1}, I_s\} \tag{62}$$

$$Q_{w^0}(j) = \mathcal{K}_j^{41}P_j^0(\mathcal{K}_j^{41})^{\rm T} + {\rm diag}\{\bar{\Delta}_j^{-1}, I_{2s}\} \tag{63}$$

In the following, we will see that although $Q_{w^i}(j)$ depend on P_j^i, they, in turn, help to update P_j^i.

Lemma 2. Let $\Psi_\tau = \Phi + GK_w^\tau$. The error covariance matrices $P_\tau^i, i=0,1,2$, can be calculated, respectively, as

$$\begin{aligned}
P_{\tau+1}^0 = &\ \Psi_\tau P_\tau^0\Psi_\tau^{\rm T} + GQ_\tau^{\bar{w}}G^{\rm T} - \Psi_\tau P_\tau^0(\mathcal{K}_\tau^{41})^{\rm T}\times\\
&\ Q_{w^0}^{-1}(\tau)\mathcal{K}_\tau^{41}P_\tau^0\Psi_\tau^{\rm T}, P_0^0 = (\Pi_0^{-1}-\gamma^{-2}P_0)^{-1}
\end{aligned} \tag{64}$$

$$\begin{aligned}
P_{i+1}^1 = &\ \Psi_{\overline{\tau}_i}P_i^1\Psi_{\overline{\tau}_i}^{\rm T} + GQ_{\overline{\tau}_i}^{\bar{w}}G^{\rm T} - \Psi_{\overline{\tau}_i}P_i^1(\mathcal{H}_{\overline{\tau}_i}^{31})^{\rm T}\times\\
&\ Q_{w^1}^{-1}(i)\mathcal{H}_{\overline{\tau}_i}^{31}P_i^1\Psi_{\overline{\tau}_i}^{\rm T}, P_0^1 = P_{\tau_d+1}^0, i>0
\end{aligned} \tag{65}$$

$$\begin{aligned}
P_{i+1}^2 = &\ \Psi_{\overline{N}_i}P_i^2\Psi_{\overline{N}_i}^{\rm T} + GQ_{\overline{N}_i}^{\bar{w}}G^{\rm T} - \Psi_{\overline{N}_i}P_i^2(\mathcal{K}_{\overline{N}_i}^{21})^{\rm T}\times\\
&\ Q_{w^2}^{-1}(i)\mathcal{K}_{\overline{N}_i}^{21}P_i^2\Psi_{\overline{N}_i}^{\rm T}, P_0^2 = P_{N_d+1}^1, i>0
\end{aligned} \tag{66}$$

The proof of Lemma 2 is straightforward and similar to that of [16], thus, it is omitted.

For $i,j\geq 0$, let

$$\begin{aligned}
R_{\underline{\tau}_j,\overline{\tau}_i}^{00} &= \langle\boldsymbol{X}(\underline{\tau}_j),\boldsymbol{E}^0(\underline{\tau}_i)\rangle, \quad R_{\underline{\tau}_j,i}^{01} = \langle\boldsymbol{X}(\underline{\tau}_j),\boldsymbol{E}^1(i)\rangle\\
R_{\underline{\tau}_j,i}^{02} &= \langle\boldsymbol{X}(\underline{\tau}_j),\boldsymbol{E}^2(i)\rangle, \quad R_{j,\underline{\tau}_i}^{10} = \langle\boldsymbol{X}(\overline{\tau}_j),\boldsymbol{E}^0(\underline{\tau}_i)\rangle\\
R_{j,i}^{11} &= \langle\boldsymbol{X}(\overline{\tau}_j),\boldsymbol{E}^1(i)\rangle, \quad R_{j,i}^{12} = \langle\boldsymbol{X}(\overline{\tau}_j),\boldsymbol{E}^2(i)\rangle\\
R_{j,i}^{20} &= \langle\boldsymbol{X}(\overline{N}_j),\boldsymbol{E}^0(\underline{\tau}_i)\rangle, \quad R_{j,i}^{21} = \langle\boldsymbol{X}(\overline{N}_j),\boldsymbol{E}^1(i)\rangle\\
R_{j,i}^{22} &= \langle\boldsymbol{X}(\overline{N}_j),\boldsymbol{E}^2(i)\rangle
\end{aligned}$$

be the cross-covariance matrices of the state $\boldsymbol{X}(\cdot)$ and the state estimation error $\boldsymbol{E}^i(\cdot)(i=0,1,2)$. Clearly, these cross-covariance matrices can be calculated directly now.

After having made the above preparation, we give the explicit expressions for the innovation covariance matrix $Q_{w_s}(\tau)$ and the optimal estimator $\hat{\boldsymbol{x}}(\tau_l|\tau,\tau_d)$ via a theorem.

Theorem 1. Considering the state-space model (41) ∼ (42) in Krein space, the innovation covariance matrix Q_{w_s} is given as

1) For $0\leq\tau<d$,

$$Q_{w_s}(\tau) = \mathcal{K}_\tau^{31}Q_e^0(\mathcal{K}_\tau^{31})^{\rm T} + {\rm diag}\{\bar{\Delta}_\tau^{-1}, I_s\} \tag{67}$$

with $Q_e^0 = P_\tau^0$;

2) For $d\leq\tau\leq N_d$,

$$Q_{w_s}(\tau) = \mathcal{K}_\tau^{42}Q_e^1(\mathcal{K}_\tau^{42})^{\rm T} + {\rm diag}\{\bar{\Delta}_\tau^{-1}, I_{2s}\} \tag{68}$$

with

$$Q_e^1 = \left[\begin{array}{cc} P_d^1 & R_{d,\tau_d}^{10}\\ (R_{d,\tau_d}^{10})^{\rm T} & \bar{P}_{\tau_d}^{01}\end{array}\right]$$

$$\begin{aligned}
\bar{P}_{\tau_d}^{01} = &\ P_{\tau_d}^0 + R_{\tau_d,\tau_d}^{00}(\mathcal{K}_{\tau_d}^{31})^{\rm T}Q_{w^0}(\tau_d)\mathcal{K}_{\tau_d}^{31}(R_{\tau_d,\tau_d}^{00})^{\rm T} +\\
&\ \sum_{i=1}^{d-1}R_{\tau_d,i}^{01}(\mathcal{K}_{\overline{\tau}_i}^{41})^{\rm T}Q_{w^1}(i)\mathcal{K}_{\overline{\tau}_i}^{41}(R_{\tau_d,i}^{01})^{\rm T}
\end{aligned}$$

3) For $\tau>N_d$,

$$Q_{w_s}(\tau) = \mathcal{H}_\tau^{32}Q_e^2(\mathcal{H}_\tau^{32})^{\rm T} + {\rm diag}\{\bar{\Delta}_\tau^{-1}, I_{2s}\} \tag{69}$$

with

$$Q_e^2 = \left[\begin{array}{cc} P_{\tau-N_d}^2 & R_{\tau-N_d,\tau_d}^{20}\\ (R_{\tau-N_d,\tau_d}^{20})^{\rm T} & \bar{P}_{\tau_d}^{02}\end{array}\right]$$

$$\begin{aligned}
\bar{P}_{\tau_d}^{02} = &\ P_{\tau_d}^0 + R_{\tau_d,\tau_d}^{00}(\mathcal{K}_{\tau_d}^{31})^{\rm T}Q_{w^0}(\tau_d)\mathcal{K}_{\tau_d}^{31}(R_{\tau_d,\tau_d}^{00})^{\rm T} +\\
&\ \sum_{i=1}^{N_\tau}R_{\tau_d,i}^{01}(\mathcal{K}_{\overline{\tau}_i}^{41})^{\rm T}Q_{w^1}(i)\mathcal{K}_{\overline{\tau}_i}^{41}(R_{\tau_d,i}^{02})^{\rm T} +\\
&\ \sum_{i=1}^{\tau-N_d}R_{\tau_d,i}^{02}(\mathcal{H}_{\overline{N}_i}^{31})^{\rm T}Q_{w^2}(i)\mathcal{H}_{\overline{N}_i}^{31}(R_{\tau_d,i}^{02})^{\rm T}
\end{aligned}$$

Meanwhile, the optimal estimator $\hat{\boldsymbol{x}}(\tau_l|\tau) = \hat{\boldsymbol{x}}(\tau_l|\tau,\tau_d)$ (l is an integer) is updated via the following recursions

1) For $k>0$ and $N\geq N_{2d}+k+1\geq 0$,

$$\begin{aligned}
&\hat{\boldsymbol{x}}(N_d+k+1|N_d+k,N_d,N_{2d}+k+1) =\\
&\quad\Psi_{N_d+k}\hat{\boldsymbol{x}}(N_d+k|N_d+k-1,N_d,N_{2d}+k)+\\
&\quad\boldsymbol{C}(N_d+k) + \Psi_{N_d+k}R_{k,k}^{22}(\mathcal{K}_{N_{2d}+k}^{21})^{\rm T}Q_{w^2}^{-1}\boldsymbol{w}^2(k)
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\hat{\boldsymbol{x}}(N_d+1|N_d,N_d,N_{2d}+1) =\\
&\quad\Psi_{N_d}\hat{\boldsymbol{x}}(N_d|N_d-1,N_d-1,N_{2d}) + \boldsymbol{C}(N_d)+\\
&\quad\Psi_{N_d}R_{d-1,d-1}^{11}(\mathcal{K}_{N_{2d}}^{31})^{\rm T}Q_{w^1}^{-1}(d_1)\boldsymbol{w}^1(d_1)
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\hat{\boldsymbol{x}}(N_{2d}+k+1|N_d+k,N_d,N_{2d}+k+1) =\\
&\quad\hat{\boldsymbol{x}}(N_{2d}+k+1|N_{2d}+k,N_{2d}+k,N_{2d}+k)+\\
&\quad R_{N_{2d}+k+1,i}^{00}(\mathcal{K}_{N_{2d}+k+1}^{41})^{\rm T}Q_{w^0}^{-1}\boldsymbol{w}^0(N_{2d}+k+1)+\\
&\quad\sum_{i=1}^{d-k-1}R_{N_{2d}+k+1,i}^{01}(\mathcal{K}_{N_{2d}+k+1+i}^{31})^{\rm T}Q_{w^1}^{-1}\boldsymbol{w}^1(i)+\\
&\quad\sum_{i=1}^{k}R_{N_{2d}+k-1,i}^{02}(\mathcal{K}_{N_d+i}^{21})^{\rm T}Q_{w^2}^{-1}\boldsymbol{w}^2(i)
\end{aligned} \tag{72}$$

2) For $k > 0$ and $N_d \geq \tau_{2d} + k + 1 \geq 0$,

$$\begin{aligned}&\hat{\boldsymbol{x}}(\tau_d + k + 1|\tau_d + k, \tau_d + k, \tau_{2d} + k + 1) = \\ &\quad \Psi_{\tau_d+k}\hat{\boldsymbol{x}}(\tau_d + k|\tau_d + k - 1, \tau_d + k - 1, \tau_{2d} + k) + \\ &\quad \boldsymbol{C}(\tau_d + k) + R^{11}_{k,k}(\mathcal{K}^{41}_{\tau_d+k})^{\rm T} Q^{-1}_{w^1}\boldsymbol{w}^1(k) \end{aligned} \tag{73}$$

$$\begin{aligned}&\hat{\boldsymbol{x}}(\tau_d + 1|\tau_d, \tau_d, \tau_d) = \\ &\quad \Psi_{\tau_d}\hat{\boldsymbol{x}}(\tau_d|\tau_d - 1, \tau_d - 1, \tau_d - 1) + \boldsymbol{C}(\tau_d) + \\ &\quad R^{00}_{\tau_d,\tau_d}(\mathcal{K}^{41}_{\tau_d})^{\rm T} Q^{-1}_{w^0}\boldsymbol{w}^0(\tau_d) \end{aligned} \tag{74}$$

$$\begin{aligned}&\hat{\boldsymbol{x}}(\tau_{2d} + k + 1|\tau_d + k, \tau_d + k, \tau_{2d} + k + 1) = \\ &\quad \hat{\boldsymbol{x}}(\tau_{2d} + k + 1|\tau_{2d} + k, \tau_{2d} + k, \tau_{2d} + k) + \\ &\quad R^{00}_{\tau_{2d}+k+1,k+1}(\mathcal{K}^{41}_{\tau_{2d}+k+1})^{\rm T} Q^{-1}_{w^0}\boldsymbol{w}^0(\tau_{2d} + k + 1) + \\ &\quad \sum_{i=1}^{d-1} R^{01}_{\tau_{2d}+k+1,i}(\mathcal{K}^{41}_{\tau_{2d}+k+1+i})^{\rm T} Q^{-1}_{w^1}\boldsymbol{w}^1(i) \end{aligned} \tag{75}$$

3) one-step predict formula is shown as

$$\begin{aligned}\hat{\boldsymbol{x}}(\tau + 1|\tau, \tau, \tau) = {} & \Psi_\tau\hat{\boldsymbol{x}}(\tau|\tau - 1, \tau - 1, \tau - 1) + \\ & \boldsymbol{C}(\tau) + \Psi_\tau R^{00}_{\tau,\tau} Q^{-1}_{w^0}\boldsymbol{w}^0(\tau) \end{aligned} \tag{76}$$

with initial value $\hat{\boldsymbol{x}}(0| - 1, -1, -1) = 0$.

For a simple expression of the optimal estimator, we omit the time index of innovation covariance matrix, which is identical with that of the adjacent innovation when it does not cause confusion.

3.4 Solutions for the H_∞ measurement control problem

By the virtue of the discussions in the previous sections, we shall formulate the H_∞ measurement-feedback control-law and the sufficient and necessary condition thereof.

Theorem 2. Consider the state-space model (1) ~ (4). For a given $\gamma > 0$, a measurement-feedback H_∞ controller $\boldsymbol{u}_i(t) = \boldsymbol{F}_i(\boldsymbol{y}(j)|_{0\leq j\leq t})$ $(i = 0, 1)$ that makes (9) hold exists if and only if

1) $\Pi_0^{-1} - \gamma^{-2}P_0 > 0$;

2) $\Delta_\tau > 0$ for all $\tau = 0, 1, \cdots, N$;

3) The matrix $Q^v_\tau - S_\tau(Q^{\tilde{w}}_\tau)^{-1}S^{\rm T}_\tau$ has the same inertia as $Q_{\boldsymbol{w}_s}$ for all $\tau = 0, 1, \cdots, N$.

Moreover, for $\tau \leq N_d$, the central controller is given by

$$\begin{aligned}\boldsymbol{u}_0(\tau) = {} & K^\tau_0\hat{\boldsymbol{x}}(\tau|\tau - 1) + \\ & [I_m\ 0_m]\, Q_{uy}Q^{-1}_y \left(\boldsymbol{y}(\tau) - H^\tau\hat{\hat{\boldsymbol{x}}}(\tau \mid \tau - 1)\right) + \boldsymbol{s}_{0,\tau}(\boldsymbol{u}) \\ \boldsymbol{u}_1(\tau) = {} & K^\tau_1\hat{\boldsymbol{x}}(\tau|\tau - 1) + \\ & [0_m\ I_m]\, Q_{uy}Q^{-1}_y \left(\boldsymbol{y}(\tau) - H^\tau\hat{\hat{\boldsymbol{x}}}(\tau \mid \tau - 1)\right) + \boldsymbol{s}_{1,\tau}(\boldsymbol{u})\end{aligned}$$

For $\tau > N_d$, the central controller is given by

$$\begin{aligned}\boldsymbol{u}_0(\tau) = {} & K^\tau_0\hat{\boldsymbol{x}}(\tau|\tau - 1) + \\ & Q_{uy}Q^{-1}_y \left(\boldsymbol{y}(\tau) - H^\tau\hat{\hat{\boldsymbol{x}}}(\tau \mid \tau - 1)\right) + \boldsymbol{s}_{0,\tau}(\boldsymbol{u})\end{aligned}$$

where

$$Q_{uy} = \begin{cases} \left[\begin{array}{cc} (K^\tau_0)^{\rm T} & (K^\tau_1)^{\rm T} \end{array}\right]^{\rm T} P^0_\tau H^{\rm T}_0, & 0 \leq \tau < d \\ \left[\begin{array}{cc} K^\tau_0 P^1_d H^{\rm T}_0 & K^\tau_0 R^{10}_{d,\tau_d} H^{\rm T}_1 \\ K^\tau_1 P^1_d H^{\rm T}_0 & K^\tau_1 R^{10}_{\tau,\tau_d-1} H^{\rm T}_1 \end{array}\right], & d \leq \tau \leq N_d \\ [K^\tau_0 P^1_d H^{\rm T}_0 \quad K^\tau_0 R^{10}_{\tau,\tau_d-1} H^{\rm T}_1], & \tau > N_d \end{cases}$$

$$Q_y = \begin{cases} H_0 P^0_\tau H^{\rm T}_0 + I_r, & 0 \leq \tau < d \\ \left[\begin{array}{cc} H_0 P^1_d H^{\rm T}_0 & H_0 R^{10}_{d,\tau_d} H^{\rm T}_1 \\ H_1 (R^{10}_{d,\tau_d})^{\rm T} H^{\rm T}_0 & H_1 \bar{P}^{01}_{\tau_d} H^{\rm T}_1 \end{array}\right] + I_{2r}, & d \leq \tau \leq N_d \\ \left[\begin{array}{cc} H_0 P^2_{\tau-N_d} H^{\rm T}_0 & H_0 R^{20}_{\tau-N_d,\tau_d} H^{\rm T}_1 \\ H_1 (R^{20}_{\tau-N_d,\tau_d})^{\rm T} H^{\rm T}_0 & H_1 \bar{P}^{02}_{\tau_d} H^{\rm T}_1 \end{array}\right] + I_{2r}, & \tau > N_d \end{cases}$$

Proof. The sufficient and necessary condition of the existence of the H_∞ measurement-feedback controller $\boldsymbol{u}_i(t) = \boldsymbol{F}_i(\boldsymbol{y}(j)|_{0\leq j\leq t})(i = 0, 1)$ can be referred to [17] directly. As for the central controller, it will be clear after we make an LDU decomposition for Q_{w_s} in (50). □

Different from the delay-free case, the present controller involves something else besides the state-estimation, which originates from the delay-input joining the original state equation. Actually, the H_∞ state-feedback controller[16] has already included the additional term besides the state.

Towards the end, let us analyze the relationship between [4, 16] and the paper. In fact, three of them share similar ideas to some extent.

Remark 4. Reference [4] is clearly a special case of our paper, so is its result. In addition, when the perfect and uncontaminated states can be observed directly, namely, the estimator of states is accurate, the result of [16] is also a special case of this paper.

This paper achieves the causal and central H_∞ controller of the time-invariant system with single I/O delay. Especially, the idea can be extended to the time-varying or multi-delay systems.

Remark 5. The present approach can be extended to deal with the multi-delay and vary-time case trivially.

4 Numerical example

In order to display that the present controller is effective, we still take the model in [18] with respect to the network congestion control and follow almost all of Zhang's[16] parameters.

Generally speaking, at time instant t, the high priority sources ξ_t, ξ_{t-1}, ξ_{t-h_i}, and ξ_{t-h_i-1}, and the queue lengths q_t and q_{t-h_i} can be obtained readily. Therefore, we can introduce measurement equations, such as

$$y_0(t) = \boldsymbol{H}_0\boldsymbol{x}(t) + v_0(t) \tag{77}$$

$$y_1(t) = \boldsymbol{H}_1\boldsymbol{x}(t - h_1) + v_1(t) \tag{78}$$

with the single delay $h_1 = d = 5$, $\boldsymbol{H}_0 = [0\ 1\ 0]$, and $\boldsymbol{H}_1 = [1\ 0\ 0]$.

For a prescribed $\gamma > 0$, the H_∞ congestion control with measurement-feedback desires to find source rate $\bar{v}_{1,t}$ so that

$$\sup_{\{q_0,\bar{v}_{1,\cdot},w(\cdot),v_0(\cdot),v_1(\cdot)\}} J(q_0, \bar{v}_{1,\cdot-d}, w(\cdot), v_0(\cdot), v_1(\cdot)) < \gamma^2 \tag{79}$$

where

$$\begin{aligned}&J(q_0, \bar{v}_{1,\cdot-d}, w(\cdot), v_0(\cdot), v_1(\cdot)) = \\ &\quad \frac{\sum_{t=0}^N \left[(q_t - \bar{q})^2 + (\bar{v}_{1,t-d} - \mu)^2\right]}{\sum_{t=0}^N \left[w(t)^2 + v_0(t)^2 + v_1(t)^2\right]}\end{aligned} \tag{80}$$

In practice, the performance (79) aims to keep the queue in the buffer close to the target length $\bar{q}$ and the source rate close to the nominal service rate μ, i.e., control input u_1 in the neighborhood of 0.

The simulation result with respect to Theorem 2 can be seen in Figs. 1 and 2. It is shown that the controller is effective. Moreover, the desired performance is achieved.

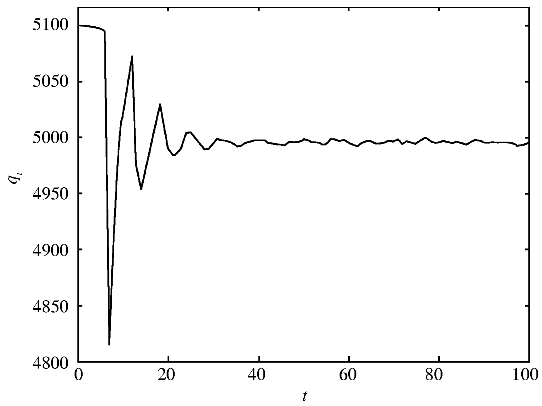

Fig. 1 Queue length response under the H_∞ controller

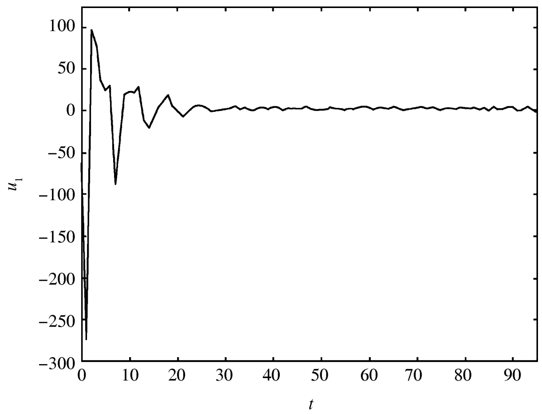

Fig. 2 H_∞ measurement-feedback controller

5 Conclusion

The paper provides a solution to the H_∞ measurement-feedback control problem for systems with single input-delay and measurement-delay via introducing pseudo-measurements as well as Krein space. It is testified that the reorganizing technique is very effective to conquer the difficulty aroused by delays. The results also show us that the solution has almost the same structure as the full-information control-law, where the states are replaced by their optimal estimations now. Furthermore, we see that the present separation principle is slightly different from the one first presented in [19], and the former is more natural in the underlying setting. More fortunately, the idea can be readily extended to the time-varying or multiple delays cases.

References

1 Zames G. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms and approximate inverse. *IEEE Transactions on Automatic Control*, 1981, **26**(2): 301−320

2 Doyle J C. *Advances in Multivariables Control*. Minneapolis: ONR/Honeywell Workshop, 1984

3 Kimura H. *Chain-scattering Approach to H_∞ Control*. Boston, Massachusetts: Birkhauser, 1996

4 Liu M, Zhang H S, Duan G R. H_∞ measurement feedback control for time delay system via Krein space. In: Proceedings of American Control Conference. Oregon, Portland: IEEE, 2005. 4010−4015

5 Tadmor G, Mirkin L. H_∞ control and estimation with preview — part I: matrix ARE solutions in continous time. *IEEE Transactions on Automatic Control*, 2005, **50**(1): 19−28

6 van Keulen B. *H_∞ Control for Distributed Parameter Systems: A State Space Approach*. Boston, Massachusetts: Birkhauser, 1993

7 Meisma G, Mirkin L, Zhong Q C. Control of systems with I/O delay via reduction to a one-block problem. *IEEE Transactions on Automatic Control*, 2002, **47**(11): 1890−1895

8 Zhou K M, Doyle J C. *Essentials of Robust Control*. New Jersey: Prentice-Hall, 1999

9 Meinsma G, Zwart H. On H_∞ control for dead-time systems. *IEEE Transactions on Automatic Control*, 2000, **45**(2): 272−285

10 Mirkin L. On the extraction of dead-time controllers and estimators from delay-free parametrizations. *IEEE Transactions on Automatic Control*, 2003, **48**(4): 543−553

11 Meinsma G, Mirkin L. H_∞ control of systems with multiple I/O delays via decomposition to adobe problem. *IEEE Transactions on Automatic Control*, 2005, **50**(2): 199−211

12 Zhang W H, Chen B S. State feedback H_∞ control for a class of nonlinear stochastic systems. *SIAM Journal of Control and Optimization*, 2006, **44**(6): 1973−1991

13 Mirkin L, Tadmor G. H_∞ control of system with I/O delay: a review of some problem-oriented methods. *IMA Journal of Mathematical Control and Information*, 2002, **19**(1-2): 185−199

14 Kojima A, Ishijima S. Formulars on preview and delayed H_∞ control. *IEEE Transactions on Automatic Control*, 2006, **51**(12): 1920−1937

15 Tadmor G, Mirkin L. H_∞ control and estimation with preview — part II: fixed-size ARE solutions in discrete time. *IEEE Transactions on Automatic Control*, 2005, **50**(1): 29−39

16 Zhang H S, Xie L H, Duan G R. H_∞ control of discrete-time systems with multiple input delays. *IEEE Transactions on Automatic Control*, 2007, **52**(2): 271−283

17 Hassibi B, Sayed A H, Kailath T. *Indefinite Quadratic Estimation and Control: a Unified Approach to H_2 and H_∞ Theories*. New York: Society for Industrial Mathematics, 1998

18 Altman E, Basar T, Srikant R. Congestion control as a stochastic control problem with action delays. *Automatica*, 1999, **35**(12): 1937−1950

19 Doyle J C, Glover K, Khargonekar P P, Francis B A. State-space solutions to standard H_2 and H_∞ control problem. *IEEE Transactions on Automatic Control*, 1989, **34**(8): 831−847

WANG Hong-Xia Ph. D. candidate in control science at Harbin Institute of Technology. She received her B. Sc. degree in applied mathematics in 2003, and M. Sc. degree in computational mathematics in 2005, both from Harbin Institute of Technology. Her research interest covers H_∞ preview control, H_∞ control, and time delay systems.
E-mail: whx1123@126.com

ZHANG Huan-Shui Received his Ph. D. degree from Northeastern University in 1997. From 1998 to 2001, he worked as a postdoctoral fellow at Nanyang Technological University and from 2001 to 2003 a research fellow at Hong Kong Polytechnic University. Since 1999, he has been a professor at Shandong University. His research interest covers optimal estimation, robust filtering and control, time delay systems, singular systems, wireless communication, and signal processing. Corresponding author of this paper.
E-mail: hszhang@sdu.edu.cn