An Improved Control Algorithm of High-order Nonlinear Systems

DUAN Na[1] XIE Xue-Jun[1, 2] LIU Hai-Kuan[1]

Abstract This paper designs an improved output-feedback controller from the viewpoint of reducing the control effort at the premise of maintaining the desired control performance for a concrete example. The output-feedback controller guarantees the globally asymptotical stability of the closed-loop system by introducing a new rescaling transformation, adopting an effective reduced-order observer, and choosing ingenious Lyapunov function and appropriate design parameters. Simultaneously, from both the theoretical analysis and a concrete example, smaller critical values for gain parameter and rescaling transformation parameter are obtained to effectively reduce the control effort and the rate of change of controller than the design of the related papers.

Key words High-order nonlinear systems, control effort reduction, rescaling transformation, output-feedback control

Consider the following high-order nonlinear systems

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= \boldsymbol{F}_0(\boldsymbol{\zeta},\boldsymbol{\eta},v)\\ \dot{\eta}_i &= \eta_{i+1}^{p_i}+\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{f}_i(\boldsymbol{\zeta},\boldsymbol{\eta},v),\quad i=1,\cdots,n-1\\ \dot{\eta}_n &= v^{p_n}+\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{f}_n(\boldsymbol{\zeta},\boldsymbol{\eta},v)\\ y &= \eta_1\end{aligned}\tag{1}$$

where $v\in\mathbf{R}$, $y\in\mathbf{R}$, $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_n)\in\mathbf{R}^n$, $\boldsymbol{\zeta}\in\mathbf{R}^r$, and $\boldsymbol{\theta}$ are the control input, output, system state, unmodelled dynamics, and unknown parameter vector, respectively. $\boldsymbol{F}_0(\cdot)$ and $\boldsymbol{f}_i(\cdot)$ are nonlinear vector functions with the corresponding dimensions. p_i, $i=1,\cdots,n$, are positive odd numbers. When $p_i=1$, by using backstepping method and combining backstepping method with the other methods, such as small-gain theorem, the dynamic signal, and changing supply function, etc, the design of stabilizing controller has achieved remarkable development in recent years[1−3].

When $p_i\geq 1$, Lin first gave a new feedback design tool called adding a power integrator and studied systematically a series of control problems for system (1) in [4]. Compared with global stabilization by state-feedback, output-feedback stabilization is much more challenging. When $n=2$, $p_n=1$, and $\zeta=0$, for a class of system (1) whose Jacobian linearization is neither controllable nor observable, the output-feedback controller was studied in [5−6] by introducing a one-dimensional nonlinear observer. For the more general system (1) with $\zeta=f_i=0$, $i=1,\cdots,n$, $p_1=\cdots=p_{n-1}$, and $p_n=1$, Yang and Lin not only overcame the obstacle caused by unobservability of the Jacobian linearization of high-order system, but also provided an iterative design way to choose the gain parameter for general high-gain observer[7]. Furthermore, by introducing a rescaling transformation ingeniously, the designed gain parameter in [8] guaranteed the globally asymptotical stability of the closed-loop systems.

However, as one studies the above high-order nonlinear systems, the high order p_i, the high nonlinearities in themselves, the repeated use of some inequalities in the design of output-feedback controller, and the interaction between the observer gain parameters and the rescaling transformation parameters will unavoidably lead to a large control effort and a large rate of change of the controller. Naturally, one may ask the following interesting question:

How to reduce the control effort at the premise of maintaining the desired control performance, e.g., asymptotical stability?

To our knowledge, there are few results on the study for this question. In this paper, we will take a concrete system as an example to answer the question. In Section 1, the design of controller is given by exactly following the same method as that in [8]. In Section 2, by introducing a new rescaling transformation and choosing Lyapunov function ingeniously, an improved output-feedback controller is designed to guarantee the globally asymptotical stability of the closed-loop system. Furthermore, we give a comparison of two control schemes in terms of both the theoretical analysis and a concrete example. Comparing with the design in [8], by choosing the design parameter flexibly, we obtain the smaller critical values for gain parameter and rescaling transformation parameter, which provides more tradeoff (or degree of freedom) between the control effort and the control performance, and thus, reduces the control effort and the rate of change of controller more effectively.

Notations. The following notations will be used throughout the paper. $\mathbf{R}_+$ denotes the set of all nonnegative real numbers. For the variables $x_1,x_2,\cdots,x_n$, $\bar{\boldsymbol{x}}_i\triangleq(x_1,\cdots,x_i)$, $i=2,\cdots,n$. $\|\boldsymbol{x}\|$ denotes the Euclidean norm for vector $\boldsymbol{x}$. C^i denotes the set of all functions with continuous i-th partial derivatives.

1 A motivating example

Without loss of generality, we firstly give the design of output-feedback controller by following exactly the design procedure in [8] for the following interconnected systems:

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= \boldsymbol{F}_0(\boldsymbol{\zeta},\eta_1,v)\\ \dot{\eta}_1 &= \eta_2^p+f_1(\boldsymbol{\zeta},\eta_1)\\ \dot{\eta}_2 &= v+f_2(\boldsymbol{\zeta},\eta_1)\\ y &= \eta_1\end{aligned}\tag{2}$$

where $\boldsymbol{\zeta}\in\mathbf{R}^r$, $(\eta_1,\eta_2)^{\mathrm{T}}\in\mathbf{R}^2$, $v\in\mathbf{R}$, and $y\in\mathbf{R}$ are the system unmodeled dynamics, state, control input and output, respectively. $\boldsymbol{F}_0(\cdot):\mathbf{R}^r\times\mathbf{R}\times\mathbf{R}\rightarrow\mathbf{R}^r$ and $f_i:\mathbf{R}^r\times\mathbf{R}\rightarrow\mathbf{R}$ $(i=1,2)$ are C^0 functions, $p\geq 1$ is any odd number. We need the following assumptions in this paper.

Assumption 1. There is a C^2 Lyapunov function $U_0(\boldsymbol{\zeta})$, which is positive definite and proper, such that for any $(\boldsymbol{\zeta},\eta_1,v)\in\mathbf{R}^r\times\mathbf{R}\times\mathbf{R}$,

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1. School of Electrical Engineering and Automation, Xuzhou Normal University, Xuzhou 221116, P. R. China 2. Institute of Automation, Qufu Normal University, Qufu 273165, P. R. China
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$$\frac{\partial U_0}{\partial \boldsymbol{\zeta}}\boldsymbol{F}_0(\boldsymbol{\zeta},\eta_1,v) \le -\|\boldsymbol{\zeta}\|^{p+1} + a_0|\eta_1|^{p+1}, \quad a_0 > 0$$

Assumption 2. There exists a real constant $c \ge 0$, such that the continuous functions $f_i(\boldsymbol{\zeta},\eta_1)$, $i=1,2$; $f_i(\boldsymbol{\zeta},\eta_1)$, $i=1,2$ satisfy

$$|f_i(\boldsymbol{\zeta},\eta_1)| \le c(\|\boldsymbol{\zeta}\|^p + |\eta_1|^p)$$

Example 1. To compare the following two control schemes, for (2), one chooses

$$\begin{aligned} F_0 &= -\zeta + f_0(\zeta,\eta_1,v), \quad f_0 = \frac{1}{3}\eta_1\cos(\zeta v), \\ f_1 &= f_2 = \frac{1}{100}(\zeta^3+\eta_1^3), \quad p=3, \quad r=1 \end{aligned} \tag{3}$$

Clearly,

$$|f_1| = |f_2| \le c(|\zeta|^3 + |\eta_1|^3), \quad c = 0.01 \tag{4}$$

satisfies Assumption 2. Choosing $U_0(\zeta) = \zeta^4$ and applying Lemma A1 in Appendix, it is easy to prove that $\dfrac{\partial U_0}{\partial \zeta}F_0 = 4\zeta^3\left(-\zeta + \dfrac{1}{3}\eta_1\cos(\zeta v)\right) \le -4\zeta^4 + 3\zeta^4 + 3^{-4}\eta_1^4 \le -\zeta^4 + a_0\eta_1^4$, thus, Assumption 1 is satisfied with $a_0 = 0.02$.

We first introduce the rescaling transformation (3.3) in [8]:

$$x_1 = \eta_1, \quad x_2 = \frac{\eta_2}{M^{1/3}}, \quad u = \frac{v}{M^{4/3}} \tag{5}$$

where $M \ge 1$ is design parameter. By (2), (3) and (5) are changed into

$$\begin{aligned} \dot{\zeta} &= -\zeta + g_0(\zeta,x_1,u) \\ \dot{x}_1 &= Mx_2^3 + g_1(\zeta,x_1) \\ \dot{x}_2 &= Mu + g_2(\zeta,x_1) \\ y &= x_1 \end{aligned} \tag{6}$$

where $g_0 = f_0, g_1 = f_1$, and $g_2 = \dfrac{f_2}{M^{1/3}}$. Choosing the first Lyapunov function $\hat{U}_0(\zeta) = kMU_0$, one has $\dot{\hat{U}}_0 \le M(-c_{00}\zeta^4 + c_{10}x_1^4)$, where $k \ge c_{00}, c_{10} \ge ka_0$.

The design of output-feedback controller is divided into two steps. In Step 1, one supposes that the unmeasured states are available for measurement, and a partial state-feedback controller is designed by combining a power integrator with backstepping technique. Then, by constructing a reduced-order observer with the gain parameter being determined later, using the certainty equivalence principle in [9], an output-feedback controller is designed.

Step 1. Setting $\xi_1 = x_1 - x_1^*, x_1^* = 0$, constructing Lyapunov function $U_1(\zeta,\xi_1) = \hat{U}_0 + \dfrac{1}{2}\xi_1^2$, choosing

$$x_2^* = -a_1\xi_1, \qquad a_1 = (c_{11} + c_{10} + \gamma_1 + c)^{\frac{1}{3}} \tag{7}$$

and using (4) ~ (6) and Lemma A1 in Appendix, one gets

$$\begin{aligned} \dot{U}_1 \le\ & M(-c_{00}\zeta^4 + c_{10}x_1^4 + \xi_1(x_2^3 - x_2^{*3}) + \xi_1 x_2^{*3}) + \\ & \xi_1 g_1 \le M(-c_{00}\zeta^4 + c_{10}\xi_1^4 + \xi_1(x_2^3 - x_2^{*3}) + \\ & \xi_1 x_2^{*3} + c\xi_1^4 + \varepsilon_1\zeta^4 + \gamma_1\xi_1^4) = \\ & M(-c_{01}\zeta^4 - c_{11}\xi_1^4 + \xi_1(x_2^3 - x_2^{*3})) \end{aligned} \tag{8}$$

where $\gamma_1 = 4^{-4}\cdot 3^3\varepsilon_1^{-3}c^4$, $\varepsilon_1 > 0$, $c_{01} = c_{00} - \varepsilon_1 > 0$, and $c_{11} > 0$ are design parameters.

Then, setting $\xi_2 = x_2 - x_2^* = x_2 + a_1\xi_1$, by (4) and (6), one obtains $\dot{\xi}_2 = M(u + a_1x_2^3) + (a_1g_1 + g_2)$, and $|a_1g_1 + g_2| \le (c + ca_1)(|\zeta|^3 + |\xi_1|^3)$. By choosing $U_2(\zeta,\xi_1,\xi_2) = U_1 + \frac{1}{2}\xi_2^2$, from (8), it follows

$$\begin{aligned} \dot{U}_2 \le\ & M(-c_{01}\zeta^4 - c_{11}\xi_1^4 + \xi_2(u - x_3^*) + \\ & \xi_2 x_3^* + \xi_1(x_2^3 - x_2^{*3}) + \\ & |\xi_2|(c + ca_1)(|\zeta|^3 + |\xi_1|^3) + a_1\xi_2x_2^3) \end{aligned} \tag{9}$$

By (7) and Lemmas A1 and A2 in Appendix, there exist positive real numbers $\varepsilon_{2i} (i = 1,2,3,4)$ satisfying

$$\begin{aligned} & \xi_1(x_2^3 - x_2^{*3}) = \xi_1\left((\xi_2 - a_1\xi_1)^3 - (-a_1\xi_1)^3\right) \le \\ & \qquad \xi_1\xi_2^3 + 3a_1^2\xi_1^3\xi_2 \le \varepsilon_{21}\xi_1^4 + \gamma_{21}\xi_2^4 \\ & (c + ca_1)|\zeta|^3|\xi_2| \le \varepsilon_{22}\zeta^4 + \gamma_{22}\xi_2^4 \\ & (c + ca_1)|\xi_1|^3|\xi_2| \le \varepsilon_{23}\xi_1^4 + \gamma_{23}\xi_2^4 \\ & a_1\xi_2x_2^3 \le a_1|\xi_2||\xi_2 - a_1\xi_1|^3 \le \\ & \qquad 4a_1|\xi_2|(|\xi_2|^3 + a_1^3|\xi_1|^3) \le \\ & \qquad \varepsilon_{24}\xi_1^4 + \gamma_{24}\xi_2^4 \end{aligned} \tag{10}$$

where

$$\begin{aligned} \gamma_{21} &= 3\cdot 4^{-\frac{4}{3}}\varepsilon_{210}^{-\frac{1}{3}} + \frac{3^7}{4^4}\varepsilon_{211}^{-3}a_1^8 \\ \gamma_{2i} &= \frac{3^3}{4^4}(c + ca_1)^4\varepsilon_{2i}^{-3}, \ i = 2,3 \\ \gamma_{24} &= 4a_1 + 27a_1^{16}\varepsilon_{24}^{-3} \end{aligned} \tag{11}$$

$\varepsilon_{21} = \varepsilon_{210} + \varepsilon_{211}$, ε_{210} and ε_{211} are positive numbers. Choosing the partial state-feedback controller

$$\begin{aligned} x_3^* &= -(a_2\xi_2)^3 = -(b_1x_1 + b_2x_2)^3, \ b_1 = a_1a_2 \\ b_2 &= a_2 = (c_{22} + \gamma_{21} + \gamma_{22} + \gamma_{23} + \gamma_{24})^{\frac{1}{3}} \end{aligned} \tag{12}$$

and substituting (10) into (9) leads to

$$\dot{U}_2 \le M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + \xi_2(u - x_3^*)) \tag{13}$$

where $c_{02} = c_{01} - \varepsilon_{22} > 0, c_{12} = c_{11} - \varepsilon_{21} - \varepsilon_{23} - \varepsilon_{24} > 0$, and $c_{22} > 0$ are design parameters.

Step 2. However, since η_2 and x_2 are unknown, a reduced-order observer is given below. Introducing an unmeasured variable $z_2 = x_2 - Lx_1$, where the gain constant $L \ge 1$ is to be determined later, by (6), one obtains

$$\dot{z}_2 = Mu - MLx_2^3 + g_2 - Lg_1 \tag{14}$$

from which, we construct the reduced-order observer

$$\dot{\hat{z}}_2 = Mu - ML\hat{x}_2^3 \tag{15}$$

where $\hat{x}_2$ is the estimate of x_2 with

$$\hat{x}_2 = \hat{z}_2 + Lx_1 \tag{16}$$

By (12), (16), and the certainty equivalence principle, we obtain the realizable output-feedback controller

$$u = -(b_1x_1 + b_2\hat{x}_2)^3 \tag{17}$$

Defining $e_2 = x_2 - \hat{x}_2 = z_2 - \hat{z}_2$, by (14) and (15), one gets

$$\dot{e}_2 = -ML(x_2^3 - \hat{x}_2^3) + g_2 - Lg_1 \tag{18}$$

According with (12), (17), $e_2 = x_2 - \hat{x}_2$, and Lemma A1 in Appendix, (13) is changed into

$$\begin{aligned}\dot{U}_2 \le\ & M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + \xi_2((b_1x_1 + b_2x_2)^3 - \\ & (b_1x_1 + b_2\hat{x}_2)^3) = \\ & M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + a_2^3\xi_2(\xi_2^3 - (\xi_2 - e_2)^3)) \le \\ & M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + 3a_2^3\xi_2^3 e_2 + a_2^3\xi_2 e_2^3) \le \\ & M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + (\varepsilon_{31} + \varepsilon_{32})\xi_2^4 + \\ & (\gamma_{31} + \gamma_{32})e_2^4) \end{aligned} \tag{19}$$

where

$$\gamma_{31} = \frac{3^7}{4^4}\varepsilon_{31}^{-3}a_2^{12}, \qquad \gamma_{32} = 3\cdot 4^{-\frac{4}{3}}\varepsilon_{32}^{-\frac{1}{3}}a_2^4 \tag{20}$$

and $\varepsilon_{3i}(i = 1, 2)$ are positive real numbers. Choosing $W(e_2) = \frac{1}{2}e_2^2$, by $e_2 = x_2 - \hat{x}_2$ and (18), one has

$$\begin{aligned}\dot{W} =\ & -MLe_2(x_2^3 - \hat{x}_2^3) + e_2(g_2 - Lg_1) = \\ & M\left(-Le_2((e_2 + \hat{x}_2)^3 - \hat{x}_2^3) + \frac{(e_2g_2 - Le_2g_1)}{M}\right)\end{aligned} \tag{21}$$

Using (4), (6), and Lemmas A1 and A3 in Appendix, there are positive real numbers $\varepsilon_{4i}(i = 1, 2, 3, 4)$, such that

$$\begin{aligned} & -Le_2((e_2 + \hat{x}_2)^3 - \hat{x}_2^3) = \\ & \qquad L(-((e_2 + \hat{x}_2) - \hat{x}_2)((e_2 + \hat{x}_2)^3 - \hat{x}_2^3)) \le -\frac{L}{4}e_2^4 \\ & \left|\frac{1}{M}Le_2g_1\right| \le \frac{1}{M}Lc|e_2|(|\zeta|^3 + |\xi_1|^3) \le \\ & \qquad \varepsilon_{41}\zeta^4 + \varepsilon_{42}\xi_1^4 + \frac{L^4}{M^4}(\gamma_{41} + \gamma_{42})e_2^4 \\ & \left|\frac{1}{M}e_2g_2\right| \le c|e_2|(|\zeta|^3 + |\xi_1|^3) \le \\ & \qquad \varepsilon_{43}\zeta^4 + \varepsilon_{44}\xi_1^4 + (\gamma_{43} + \gamma_{44})e_2^4 \end{aligned} \tag{22}$$

Substitutes (22) into (21) leads to

$$\begin{aligned}\dot{W} \le\ & M\left((-Le_2((e_2+\hat{x}_2)^3 - \hat{x}_2^3)) + \left|\frac{1}{M}Le_2g_1\right| + \left|\frac{1}{M}e_2g_2\right|\right) \le \\ & M\Bigg((\varepsilon_{41} + \varepsilon_{43})\zeta^4 + (\varepsilon_{42} + \varepsilon_{44})\xi_1^4 - \\ & \left(\frac{L}{4} - \gamma_{43} - \gamma_{44} - \frac{L^4(\gamma_{41} + \gamma_{42})}{M^4}\right)e_2^4\Bigg)\end{aligned} \tag{23}$$

where $\gamma_{4i} = 4^{-4}\cdot 3^3\varepsilon_{4i}^{-3}c^4,\ i = 1, 2, 3, 4$ are positive real numbers. Lastly, considering

$$\begin{aligned}V(\zeta, \xi_1, \xi_2, e_2) =\ & U_2(\zeta, \xi_1, \xi_2) + W(e_2) = \\ & kMU_0 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}e_2^2\end{aligned} \tag{24}$$

and choosing the parameters

$$\begin{aligned} & M \ge M_1^* = \max\{1, L(\gamma_{41} + \gamma_{42})^{\frac{1}{4}}\} \\ & L \ge L_1^* = 4(\gamma_{31} + \gamma_{32} + \gamma_{43} + \gamma_{44} + 1 + c_3) \end{aligned} \tag{25}$$

$$\begin{aligned} & c_{00} > \varepsilon_1 + \varepsilon_{22} + \varepsilon_{41} + \varepsilon_{43}, \quad k \ge c_{00}, \quad c_{10} \ge ka_0 \\ & c_{11} > \varepsilon_{210} + \varepsilon_{211} + \varepsilon_{23} + \varepsilon_{24} + \varepsilon_{42} + \varepsilon_{44} \\ & c_{22} > \varepsilon_{31} + \varepsilon_{32}, \quad c_3 > 0 \end{aligned} \tag{26}$$

and using the definitions of c_{01} and c_{11} in (8), ε_{21} in (11), c_{02}, c_{12}, and c_{22} in (13), (19), (23), and (24), one gets

$$\begin{aligned}\dot{V} \le\ & M\Bigg(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + (\varepsilon_{41} + \varepsilon_{43})\zeta^4 + \\ & (\varepsilon_{42} + \varepsilon_{44})\xi_1^4 + (\varepsilon_{31} + \varepsilon_{32})\xi_2^4 - \\ & \left(\frac{L}{4} - \gamma_{31} - \gamma_{32} - \gamma_{43} - \gamma_{44} - L^4\frac{(\gamma_{41} + \gamma_{42})}{M^4}\right)e_2^4\Bigg) \le \\ & M(-c_0\zeta^4 - c_1\xi_1^4 - c_2\xi_2^4 - c_3e_2^4)\end{aligned} \tag{27}$$

and

$$\begin{aligned} c_0 &= c_{00} - \varepsilon_1 - \varepsilon_{22} - \varepsilon_{41} - \varepsilon_{43} > 0 \\ c_1 &= c_{11} - \varepsilon_{210} - \varepsilon_{211} - \varepsilon_{23} - \varepsilon_{24} - \varepsilon_{42} - \varepsilon_{44} > 0 \\ c_2 &= c_{22} - \varepsilon_{31} - \varepsilon_{32} > 0 \\ c_3 &> 0 \end{aligned} \tag{28}$$

Thus, the output-feedback controller consisting of (5), and (15) ∼ (17) guarantees the globally asymptotical stability of the closed-loop system (2), (3), (5), and (15) ∼ (17).

Remark 1. In this example, it is easy to obtain the following relationship

$$\begin{aligned} & a_1 \text{ in } (7) \xrightarrow{(11)} \gamma_{21}(a_1^8), \gamma_{22}(a_1^4), \gamma_{23}(a_1^4), \gamma_{24}(a_1^{16}) \xrightarrow{(12)} \\ & a_2, b_1, b_2 \xrightarrow{(20)} \gamma_{31}(a_2^{12}), \gamma_{32}(a_2^4) \xrightarrow{(25)} L_1^*, M_1^* \text{ and } L, M \\ & \xrightarrow{(15)} \hat{z}_2 \xrightarrow{(16)} \hat{x}_2 \xrightarrow{b_1, b_2, (5) \text{ and } (17)} v \end{aligned} \tag{29}$$

Generally speaking, because of the high-order $p = 3$ of (2), the nonlinearities in f_1 and f_2, and the repeated use of Lemmas A1∼A3 in Appendix in the design procedure of output-feedback controller, from (29) it is easy to find that when $a_1 > 1$ and $a_2 > 1$, $\gamma_{2i}(i = 1, 2, 3, 4), \gamma_{31}, \gamma_{32}$ will lead to the large critical values L_1^* and M_1^* of the gain parameter L_2 and rescaling transformation parameter M, which further lead to the large control effort and the rate of change of the controller. In the next section we will give an improved method to reduce L_1^* and M_1^*.

2 Design and analysis of an improved output-feedback controller[1]

Example 2. For the same system as in Example 1, one firstly introduces a new rescaling transformation

$$x_1 = \eta_1, \quad x_2 = \frac{\eta_2}{M^{1/3}}, \quad u = \frac{v}{k_0M^{4/3}} \tag{30}$$

where $M \ge 1$ is the design parameter. By (30), (2) and (3) are changed into

$$\begin{aligned} \dot{\zeta} &= -\zeta + g_0(\zeta, x_1, u) \\ \dot{x}_1 &= Mx_2^3 + g_1(\zeta, x_1) \\ \dot{x}_2 &= Mk_0u + g_2(\zeta, x_1) \\ y &= x_1 \end{aligned} \tag{31}$$

where $g_i(i = 0, 1, 2)$ are the same as those in Example 1. For $\hat{U}_0(\zeta) = kMU_0$, it is easy to get $\dot{\hat{U}}_0 \le M(-c_{00}\zeta^4 + c_{10}x_1^4)$, where $k \ge c_{00}, c_{10} \ge ka_0$. Similar to Example 1, we also have two steps to design the output-feedback controller.

[1] For the sake of simplicity and the consistency of comparison, in this section, we use the same signals as those in Example 1, although their concrete expressions may be different.

Step 1. First, suppose that η_2 and x_2 are available to measurement. Setting $\xi_1 = x_1 - x_1^*, x_1^* = 0$, choosing $U_1(\zeta, \xi_1, k_1) = \hat{U}_0 + \frac{1}{2}k_1\xi_1^2$ and

$$x_2^* = -a_1(k_1)\xi_1, a_1(k_1) = \left(\frac{(c_{11} + c_{10} + \gamma_1)}{k_1 + c}\right)^{\frac{1}{3}} \quad (32)$$

by (4), (30), (31), and Lemma A1 in Appendix, one arrives at

$$\begin{aligned}\dot{U}_1 &\le M\left(-c_{00}\zeta^4 + c_{10}\xi_1^4 + k_1\xi_1(x_2^3 - x_2^{*3}) + k_1\xi_1 x_2^{*3}\right) + \\ &k_1\xi_1 g_1 \le M\left(-c_{01}\zeta^4 - c_{11}\xi_1^4 + k_1\xi_1(x_2^3 - x_2^{*3})\right) \quad (33)\end{aligned}$$

where parameters $c_{01} = c_{00} - \varepsilon_1 > 0$, $\gamma_1 = 4^{-4} \cdot 3^3\varepsilon_1^{-3}c^4k_1^4$, $\varepsilon_1 > 0$, and $c_{11} > 0$ will be determined later.

Then, setting $\xi_2 = x_2 - x_2^* = x_2 + a_1\xi_1$, by (4) and (31), one gets $\dot{\xi}_2 = M(k_0u + a_1x_2^3) + (a_1g_1 + g_2)$, and $|a_1g_1 + g_2| \le (c + ca_1)(|\zeta|^3 + |\xi_1|^3)$. Choosing $U_2(\zeta, \bar{\boldsymbol{\xi}}_2, \bar{\boldsymbol{k}}_2) = U_1 + \frac{1}{2}k_2\xi_2^2$, by (33), one obtains

$$\begin{aligned}\dot{U}_2 &\le M(-c_{01}\zeta^4 - c_{11}\xi_1^4 + k_2k_0\xi_2(u - x_3^*) + \\ &k_2k_0\xi_2x_3^* + k_1\xi_1(x_2^3 - x_2^{*3}) + \\ &k_2|\xi_2|(c + ca_1)(|\zeta|^3 + |\xi_1|^3) + a_1k_2\xi_2x_2^3) \quad (34)\end{aligned}$$

Similar to (10), there exist positive real numbers $\varepsilon_{2i} (i = 1, 2, 3, 4)$ satisfying

$$\begin{aligned}k_1\xi_1(x_2^3 - x_2^{*3}) &\le \varepsilon_{21}\xi_1^4 + \gamma_{21}(k_1)\xi_2^4 \\ (c + ca_1)k_2|\zeta|^3|\xi_2| &\le \varepsilon_{22}\zeta^4 + \gamma_{22}(k_1, k_2)\xi_2^4 \\ (c + ca_1)k_2|\xi_1|^3|\xi_2| &\le \varepsilon_{23}\xi_1^4 + \gamma_{23}(k_1, k_2)\xi_2^4 \\ a_1k_2\xi_2x_2^3 &\le \varepsilon_{24}\xi_1^4 + \gamma_{24}(k_1, k_2)\xi_2^4 \quad (35)\end{aligned}$$

where

$$\begin{aligned}\gamma_{21}(k_1) &= 3 \cdot 4^{-\frac{4}{3}}\varepsilon_{210}^{-\frac{1}{3}}k_1^{\frac{4}{3}} + \frac{3^7}{4^4}\varepsilon_{211}^{-3}a_1^8k_1^4 \\ \gamma_{2i}(\bar{\boldsymbol{k}}_2) &= \frac{3^3}{4^4}(c + ca_1)^4\varepsilon_{2i}^{-3}k_2^4, \quad i = 2, 3 \\ \gamma_{24}(\bar{\boldsymbol{k}}_2) &= 4a_1k_2 + 27a_1^{16}\varepsilon_{24}^{-3}k_2^4 \quad (36)\end{aligned}$$

$\varepsilon_{21} = \varepsilon_{210} + \varepsilon_{211}$, ε_{210} and ε_{211} are positive real numbers. Choose

$$\begin{aligned}&x_3^* = -(a_2(k_0, \bar{\boldsymbol{k}}_2)\xi_2)^3 = -(b_1(k_0, \bar{\boldsymbol{k}}_2)x_1 + b_2(k_0, \bar{\boldsymbol{k}}_2)x_2)^3 \\ &b_1 = a_1a_2, \quad b_2 = a_2 \\ &a_2(k_0, \bar{\boldsymbol{k}}_2) = (c_{22} + \gamma_{21} + \gamma_{22} + \gamma_{23} + \gamma_{24})^{1/3} \quad (37)\end{aligned}$$

Substituting (35) and (37) into (34) yields

$$\dot{U}_2 \le M\left(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + \xi_2(u - x_3^*)\right) \quad (38)$$

where $c_{02} = c_{01} - \varepsilon_{22} > 0, c_{12} = c_{11} - \varepsilon_{21} - \varepsilon_{23} - \varepsilon_{24} > 0$, $k_2k_0 = 1$, and $c_{22} > 0$ are some parameters to be determined.

Remark 2. There are two points to be emphasized.

1) There do exist such constants k_0, k_1, k_2 and a_1, a_2 such that (38) holds. For example, that $k_0 = k_1 = k_2 = 1$ is exactly the design procedure of Example 1.

2) The purpose of introducing k_0, k_1, and k_2 is to control the values of γ_1 and $\gamma_{2i} (i = 1, 2, 3, 4)$ effectively, such that a_2, b_1, and b_2 in (37) are as small as possible, and thus, the critical values of gain parameters and rescaling transformation parameters can be reduced effectively.

Step 2. Since η_2 and x_2 are unmeasurable, one has to design a reduced-order observer. Introducing the unmeasurable variables $z_2 = x_2 - Lx_1$, one has

$$\dot{z}_2 = Mk_0u - MLx_2^3 + g_2 - Lg_1 \quad (39)$$

where the gain $L \ge 1$ is the parameter to be determined. Choose the following observer

$$\dot{\hat{z}}_2 = Mk_0u - ML\hat{x}_2^3 \quad (40)$$

where

$$\hat{x}_2 = \hat{z}_2 + Lx_1 \quad (41)$$

Defining $e_2 = x_2 - \hat{x}_2 = z_2 - \hat{z}_2$, by (39) and (40), one arrives at

$$\dot{e}_2 = -ML(x_2^3 - \hat{x}_2^3) + g_2 - Lg_1 \quad (42)$$

Choosing $W(e_2, k_3) = \frac{1}{2}k_3e_2^2$, one obtains

$$\dot{W} = Mk_3\left(-Le_2\left((e_2 + \hat{x}_2)^3 - \hat{x}_2^3\right) + \frac{(e_2g_2 - Le_2g_1)}{M}\right) \quad (43)$$

Similar to (22), by Lemmas A1 and A3 in Appendix, there exist positive real numbers $\varepsilon_{4i} (i = 1, 2, 3, 4)$ satisfying

$$\begin{aligned}&-k_3Le_2\left((e_2 + \hat{x}_2)^3 - \hat{x}_2^3\right) \le -\frac{k_3L}{4}e_2^4 \\ &\left|\frac{1}{M}k_3Le_2g_1\right| \le \frac{k_3}{M}Lc|e_2|(|\zeta|^3 + |\xi_1|^3) \le \\ &\quad \varepsilon_{41}\zeta^4 + \varepsilon_{42}\xi_1^4 + \frac{L^4}{M^4}(\gamma_{41}(k_3) + \gamma_{42}(k_3))e_2^4 \\ &\left|\frac{1}{M}k_3e_2g_2\right| \le ck_3|e_2|(|\zeta|^3 + |\xi_1|^3) \le \\ &\quad \varepsilon_{43}\zeta^4 + \varepsilon_{44}\xi_1^4 + (\gamma_{43}(k_3) + \gamma_{44}(k_3))e_2^4 \quad (44)\end{aligned}$$

where $\gamma_{4i}(k_3) = 4^{-4} \cdot 3^3\varepsilon_{4i}^{-3}c^4k_3^4 (i = 1, 2, 3, 4)$ are positive real numbers. Substituting (44) into (43) results in

$$\begin{aligned}\dot{W} &\le M\Bigg((\varepsilon_{41} + \varepsilon_{43})\zeta^4 + (\varepsilon_{42} + \varepsilon_{44})\xi_1^4 - \\ &\left(\frac{k_3L}{4} - \gamma_{43} - \gamma_{44} - \frac{L^4(\gamma_{41} + \gamma_{42})}{M^4}\right)e_2^4\Bigg) \quad (45)\end{aligned}$$

Similar to (17), the output-feedback controller is constructed as

$$u = -(b_1(k_0, \bar{\boldsymbol{k}}_2)x_1 + b_2(k_0, \bar{\boldsymbol{k}}_2)\hat{x}_2)^3 \quad (46)$$

Similar to (19), (38) is changed into

$$\begin{aligned}\dot{U}_2 &\le M(-c_{02}\zeta^4 - c_{12}\xi_1^4 - c_{22}\xi_2^4 + (\varepsilon_{31} + \varepsilon_{32})\xi_2^4 + \\ &(\gamma_{31} + \gamma_{32})e_2^4) \quad (47)\end{aligned}$$

where

$$\gamma_{31}(k_0, \bar{\boldsymbol{k}}_2) = \frac{3^7}{4^4}\varepsilon_{31}^{-3}a_2^{12}, \gamma_{32}(k_0, \bar{\boldsymbol{k}}_2) = 3 \cdot 4^{-\frac{4}{3}}\varepsilon_{32}^{-\frac{1}{3}}a_2^4 \quad (48)$$

and $\varepsilon_{3i} (i = 1, 2)$ are positive real numbers. Considering

$$\begin{aligned}V(\zeta, \bar{\boldsymbol{\xi}}_2, e_2, k_0, \bar{\boldsymbol{k}}_3) &= U_2(\zeta, \bar{\boldsymbol{\xi}}_2, k_0, \bar{\boldsymbol{k}}_2) + W(e_2, k_3) = \\ &kMU_0 + \frac{1}{2}k_1\xi_1^2 + \frac{1}{2}k_2\xi_2^2 + \frac{1}{2}k_3e_2^2 \quad (49)\end{aligned}$$

and choosing the parameters

$$M \geq M_2^* = \max\left\{1, L(\gamma_{41}+\gamma_{42})^{\frac{1}{4}}\right\}$$
$$L \geq L_2^* = \frac{4(\gamma_{31}+\gamma_{32}+\gamma_{43}+\gamma_{44}+1+c_3)}{k_3} \quad (50)$$

and $c_{ii}(i=0,1,2)$, c_{10} and c_3 in (26), by (45), (47), and (49), one has

$$\dot{V} \leq M(-c_0\zeta^4 - c_1\xi_1^4 - c_2\xi_2^4 - c_3e_2^4) \quad (51)$$

and c_0, c_1 and c_2 are the same as those in (28), $c_3 > 0$ is parameter to be determined. From (49) and (51), we immediately obtain the main result in this paper.

Theorem 1. Consider system (2) and (3). By appropriately choosing the design parameters $c_{i,i}(i=0,1,2)$, c_3, $k_j(j=0,1,2,3)$, M_2^* and L_2^*, when $M \geq M_2^*$, $L \geq L_2^*$, the output-feedback controller (30), (40), (41), and (46) guarantees that the closed-loop system consisting of (2), (3), (30), (40), (41), and (46) is globally asymptotically stable.

Remark 3. Similar to (29) in Example 1, one can obtain the following relationship

$$\begin{aligned}&a_1(k_1) \text{ in } (32) \xrightarrow{(36)} \gamma_{21}(a_1^8,k_1), \gamma_{22}(a_1^4,\bar{\boldsymbol{k}}_2), \gamma_{23}(a_1^4,\bar{\boldsymbol{k}}_2),\\ &\gamma_{24}(a_1^{16},\bar{\boldsymbol{k}}_2) \xrightarrow{(37)} a_2(k_0,\bar{\boldsymbol{k}}_2), b_1(k_0,\bar{\boldsymbol{k}}_2), b_2(k_0,\bar{\boldsymbol{k}}_2)\\ &\xrightarrow{(48)} \gamma_{31}(a_2^{12}), \gamma_{32}(a_2^4) \xrightarrow{(50)} L_2^*, M_2^* \text{ and } L, M \xrightarrow{(40)}\\ &\hat{z}_2 \xrightarrow{(41)} \hat{x}_2 \xrightarrow{b_1,b_2,(30) \text{ and } (46)} v\end{aligned} \quad (52)$$

From the above relationship, it is easy to see that the critical values L_2^* and M_2^* will be reduced recursively by choosing $k_i(i=0,1,2,3)$ flexibly, $\gamma_{2i}(i=1,2,3,4)$, $a_2(k_0,\bar{\boldsymbol{k}}_2)$, $b_1(k_0,\bar{\boldsymbol{k}}_2), b_2(k_0,\bar{\boldsymbol{k}}_2)$; $\gamma_{31}(a_2^{12})$ and $\gamma_{32}(a_2^4)$; This implies that the larger ranges of L and M (see (50)) can be provided, thus, more tradeoff (or degree of freedom) between control effort and control performance can be achieved.

3 Simulation and comparison

We compare the control effort of the above two control schemes.

In Example 1, choosing the design parameters $c_{10}=0.1$, $c_{00}=5$, $c_{11}=15$, $c_{22}=5.2$, $c_3=1$, $k=5$, $\varepsilon_1=0.1$, $\varepsilon_{210}=3$, $\varepsilon_{211}=8.98$, $\varepsilon_{22}=2.6$, $\varepsilon_{23}=0.01$, $\varepsilon_{24}=2.8$, $\varepsilon_{31}=5$, $\varepsilon_{32}=0.1$, $\varepsilon_{41}=\varepsilon_{43}=1$, and $\varepsilon_{42}=\varepsilon_{44}=0.1$, (26) holds. By (28), one gets $c_0=0.3$, $c_1=0.01$, $c_2=0.1$, and $c_3=1$. By (7), (11), (12), and (20), one has $a_1=2.472\,2$, $a_2=133.793\,6$, $b_1=330.768\,0$, $b_2=a_2=133.793\,6$, $\gamma_{21}=16.789\,7$, $\gamma_{22}=8.722\,4\times10^{-9}$, $\gamma_{23}=0.153\,3$, $\gamma_{24}=2.395\,0\times10^6$, $\gamma_{31}=2.248\,7\times10^{24}$, and $\gamma_{32}=3.261\,7\times10^8$. Using (25), one obtains $L_1^*=8.994\,6\times10^{24}$ and $M_1^*=2.883\,2\times10^{23}$.

In Example 2, choosing the same parameters as those in Example 1, and $k_0=1\,000, k_1=0.1, k_2=0.001$, and $k_3=100$, by (28), one gets $c_0=0.3, c_1=0.01, c_2=0.1$, and $c_3=1$. Applying (32), (36), (37) and (48), one has $a_1=5.325\,2$, $a_2=1.867\,6$, $b_1=9.945\,2$, $b_2=a_2=1.867\,6$, $\gamma_{21}=0.778\,1$, $\gamma_{22}=9.605\,0\times10^{-20}$, $\gamma_{23}=1.688\,2\times10^{-12}$, $\gamma_{24}=0.535\,6$, $\gamma_{31}=123.032\,6$, and $\gamma_{32}=12.382\,7$. Furthermore, by (50), it is easy to obtain $L_2^*=9.759\,6 \ll 8.994\,6\times10^{24}=L_1^*$, and $M_2^*=31.283\,9 \ll 2.883\,2\times10^{23}=M_1^*$.

In Example 2, for the initial values $\zeta(0)=5$, $\eta_1(0)=-0.1$, $\eta_2(0)=1.5$, $\hat{z}_2(0)=1.53$, and $L=9.76$, and $M=31.29$, Fig. 1 gives the responses of the closed-loop system (2), (3), (30), (40), (41) and (46). In Example 1, one chooses the same initial value as those in Example 2, and $L=8.994\,6\times10^{24}, M=2.883\,2\times10^{23}$. Fig. 2 gives the comparison of the control efforts between Example 1 and Example 2.

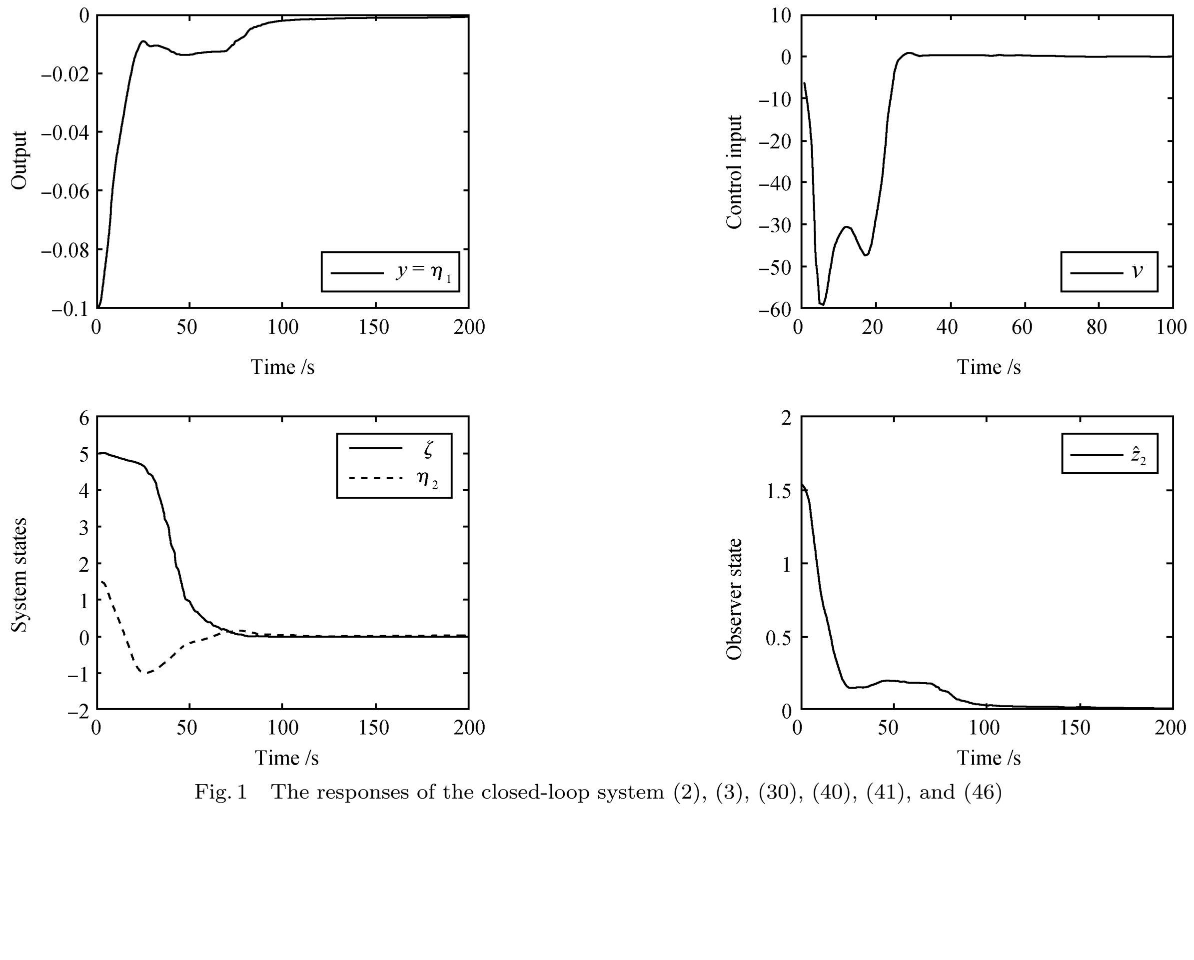

Fig. 1 The responses of the closed-loop system (2), (3), (30), (40), (41), and (46)

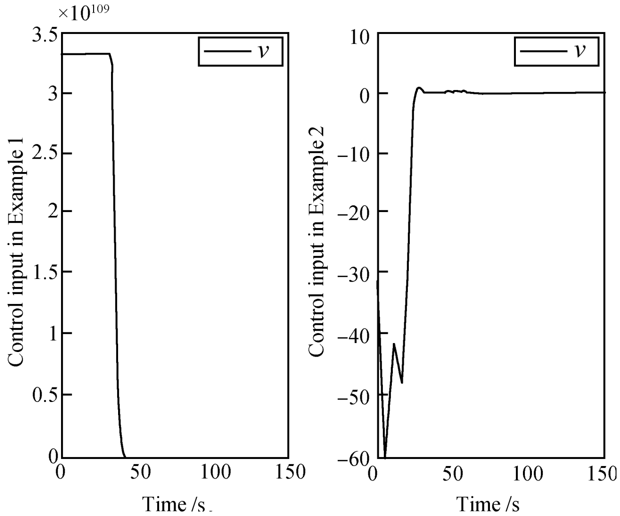

Fig. 2 The comparison of the control efforts between Examples 1 and 2

Fig. 1 verifies the effectiveness of the improved control scheme. Fig. 2 demonstrates that the control effort and the rate of change of the controller in Example 1 are too large!

4 Conclusion

Comparing with the design in [8], the improved controller provides more tradeoff between the control effort and the control performance, such as Example 2, $L_2 \geq L_2^* = 9.759\,6$, $L_2^* \ll L_1^*$, $M \geq M_2^* = 31.283\,9$, and $M_2^* \ll M_1^*$, and thus, reduces the control effort and the rate of the change of the controller more effectively.

An important problem under investigation is that, on the premise of $|v| \leq a$, for some special nonlinear functions $f_i(\boldsymbol{\zeta}, \boldsymbol{\eta}, v)$, $i = 1, \cdots, n$, how to find the maximum variation neighborhood on the design parameters in a unified way for system (1), where $a > 0$ is a constant.

Appendix

Lemma A1. Let x and y be real variables. Then for any positive integers m, n and positive real number a, there is a positive real number d, such that

$$ax^m y^n \leq d|x|^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}} a^{\frac{(m+n)}{n}} d^{-\frac{m}{n}} |y|^{m+n}$$

Lemma A2. For any positive real numbers $x_1, \cdots, x_n$ and p, one has

$$(x_1 + \cdots + x_n)^p \leq \max\{n^{p-1}, 1\}(x_1^p + \cdots + x_n^p)$$

Lemma A3. For all x and $y \in \mathbf{R}$ and any odd positive integer p, the following inequality holds:

$$-(x-y)(x^p - y^p) \leq -\frac{1}{2^{p-1}}(x-y)^{p+1}$$

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DUAN Na Ph. D. candidate at Qufu Normal University and an assistant at Xuzhou Normal University. Her research interest covers nonlinear adaptive control and stochastic nonlinear control.
E-mail: duanna08@163.com

XIE Xue-Jun Received his Ph. D. degree from the Institute of Systems Science, Chinese Academy of Sciences in 1999. From March 2001 to March 2003, he did postdoctoral research at Northeastern University. His main research interest is adaptive control for complex systems. Corresponding author of this paper.
E-mail: xxj@mail.qfnu.edu.cn

LIU Hai-Kuan Professor at Xuzhou Normal university. His research interest covers robust control, power source technology, and computer integrated system with application.