

Stability Analysis for Memristive Recurrent Neural Network and Its Application to Associative Memory

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Abstract Memristor is a nonlinear resistor with variable resistance. This paper discusses dynamic properties of memristor and recurrent neural network (RNN) with memristors as connection weights. Firstly, it establishes that there exists a threshold voltage for memristor. Secondly, it presents a model for memristive recurrent neural network (MRNN) which has variable and bounded coefficients, and analyzes stability of memristive neural network by some maths tools. Thirdly, it gives a synthesis algorithm for associative memory based on memristive recurrent neural network. At last, three examples verify our results.

Key words Associative memory, memristor, memristive recurrent neural network (MRNN), stability

Citation Gang Bao, Yuanyuan Chen, Siyu Wen, Zhicen Lai. Stability analysis for memristive recurrent neural network and its application to associative memory. *Acta Automatica Sinica*, 2017, **43**(12): 2244–2252

DOI 10.16383/j.aas.2017.e170103

1 Introduction

Artificial neural networks are developed for solving some complex problems in control, optimal computation, pattern recognition, information processing, and associative memory [1]–[13]. American scientist Hopfield makes a great contribution for the development of neural network. That is the implementation of neural network by simple circuit devices, resistors, capacitors and amplifiers [14]. Hopfield neural network (HNN) can mimic the human's associative memory function and accomplish optimization. The key point is the weights of HNN which are implemented by resistors for simulating neuron synapse. While the bottleneck is that linear resistors cannot reflect variability of synapse for resistance of linear resistor being invariable.

Memristor [15], [16], the arising fourth circuit device, makes it better to simulate the variability of neuron synapse. Pershin and Ventra [17] gives their experimental research results that neurons with memristors as synapses can simulate the associative memory function of a dog. Hence, memristor is the advancing spot in the present physics research. Several models of memristor have been set up and its properties have been analyzed in [18]–[21]. Based on these analyses, memristor can be used to mimic synapse in neural computing architecture [22], construct memristor bridge synapse [23] and brain combined with the conventional complementary metal oxide semiconductor (CMOS) technology [24], set memristive neural network [25], [26] and implement memristor array for image processing [27] etc.

Some researchers derive mathematical model of memristive recurrent neural network (MRNN) by replacing resistors with memristors in Hopfield and cellular neural network circuit [28]–[30]. MRNN is modeled by state-dependent switched systems by simplifying the memris-

tance as two-valued device with different terminal voltage. With differential inclusion theory, Lyapunov-Krasovskii function and some other math tools, some sufficient conditions are derived for dynamics of MRNN, such as, convergence and attractivity [31]–[33], periodicity and dissipativity [34], dissipativity for stochastic and discrete case, global exponential almost periodicity, and complete stability [35], multi-stability [36], etc. Considering the trouble from the switching property of memristor, researchers derive some interesting results about exponential stabilization, reliable stabilization, and finite-time stabilization of MRNN by designing different state feedback controllers [37], [38] and sampled-data controller [39]. All of these results make a solid foundation for MRNN's application to associative memory.

Associative memory is a distinguished function of human brain which can be simulated by recurrent neural network (RNN). The design problem is that some given prototype patterns are to be stored by RNN, and then the stored patterns can be recalled by some prompt information. In the existing literatures [40]–[46], there are two design methods for associative memory. One is that prototype patterns are designed as multiple locally asymptotically stable equilibria and initial values are the recalling probes. Another is that a prototype pattern is designed as the unique globally asymptotically stable equilibrium point with one external input as the recalling probe. Different external inputs mean different equilibrium points, i.e., different prototype patterns.

To the best of our knowledge, the bottleneck of associative memory based on RNN is that capacity of RNN is limited and different storage task needs different RNN because resistance can not be changed. Furthermore, there are few works about associative memory based on MRNN. Hence, the contribution of this paper is obtaining a threshold voltage for memristor by simulation, presenting a novel type of MRNN with infinite number of sub neural networks, and design a program for associative memory based on MRNN. Compared with MRNN models in the existing literatures, the difference is that every coefficient of MRNN has infinite number of values, not two values. Furthermore, every coefficient can be changed by the external input. So the associative memory based on MRNN seems to solve the problem of storage capacity.

The rest of this paper is organized as the following sections. Memristor property analysis and some preliminaries

Manuscript received May 24, 2017; accepted October 12, 2017
 This work was supported by the National Natural Science Foundation of China (61125303), the Program for Science and Technology in Wuhan, China (2014010101010004), the Program for Changjiang Scholars and Innovative Research Team in University of China (IRT1245), China Three Gorges University Science Foundation (KJ2013B020), Hubei Key Laboratory of Cascaded Hydropower Stations Operation and Control Program (2013KJX12), and Hubei Science and Technology Support Program (2015BAA106)
 Recommended by Associate Editor Zhanshan Wang
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are stated in Section 2. Then, some sufficient conditions are given to ensure global stability and multi-stability of MRNN by some maths tools in Section 3, respectively. Next, design procedure for associative memory based on MRNN is given in Section 4. To elucidate our results, three simulation examples are presented in Section 5. At last, conclusion is drawn in Section 6.

2 Memristor Recurrent Neural Network Model

2.1 Memristor and Its Property

The definition of memristor [15] is a functional relation between charge q and magnetic flux φ , i.e., $g(\varphi, q) = 0$. Memristance of memristor is defined as the following formula with the assumption of linear dopant drift as follows

$$v(t) = \left(R_{\text{on}} \frac{w(t)}{D} + R_{\text{off}} \left(1 - \frac{w(t)}{D} \right) \right) i(t) \quad (1)$$

$$\frac{dw(t)}{dt} = \mu_V \frac{R_{\text{on}}}{D} i(t) \quad (2)$$

where $w(t)$, D , $i(t)$, $v(t)$, μ_V are the length of dopant region, the length of memristor, the current, voltage across the device and the average ion mobility, respectively.

The v - i simulation curve of memristor (1) with MATLAB is shown in Fig. 1.

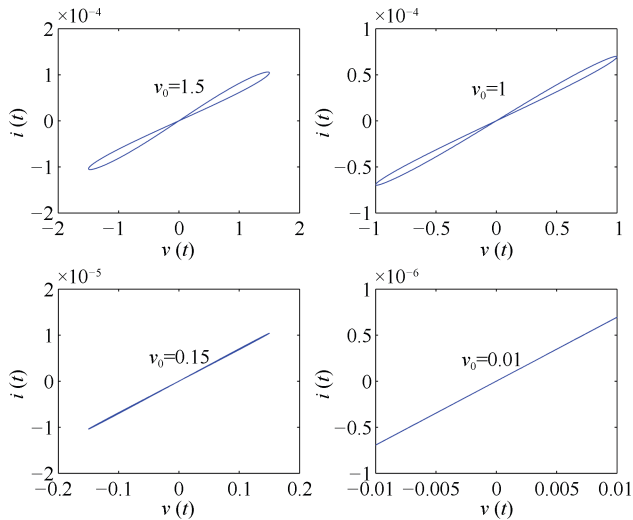


Fig. 1. The curve of $(v(t), i(t))$ under voltage sources with different amplitudes. The applied voltage source is $v(t) = v_0 \sin(\omega t)$, $v_0 = 1.5, 1, 0.15, 0.01$ V, $\omega = 2\pi$ rad/s and the other parameters are $s(t_0) = 0.1$, $t_0 = 0$ s, $R_{\text{on}} = 100 \Omega$, $r = 160$, $D = 10^{-6}$ cm, $\mu_V = 10^{-10}$ cm²/sV. From four subplots, there is a threshold voltage existing for one memristor.

From Fig. 1, it shows that a memristor will not change its resistance unless the terminal voltage exceeds a certain threshold value V_T as described in [20], [27]. It can be expressed by the following formula

$$R(w) = \begin{cases} R(w, u), & u > V_T \\ R_w, & u < V_T \end{cases} \quad (3)$$

where R_w is a constant between R_{on} and R_{off} ; $R(w, u)$ can be calculated by the following formula [10]

$$T_w = \frac{\Phi_D}{V_A R_{\text{off}}^2} [(R(w_0))^2 - (R(w))^2] \quad (4)$$

where $\Phi_D = (rD)^2/[2\mu_V(r-1)]$; V_A , T_w , and $R(w_0)$, $R(w)$ are voltage amplitude, time width, resistances of the device at the states w_0, w , respectively.

Remark 1: Simulation shows that there exists a threshold voltage for the memristor, i.e., memristance can be changed by terminal voltage with amplitude value being greater than threshold value. The result is consistent with the theoretical analysis in [20]. This property of memristor can reflect variability of neuron synapses. Furthermore, it makes memristor being suitable for constructing neural network with coefficients which can be changed according to our needs.

2.2 Model

In this section, we will firstly present the mathematical model for MRNN, and then give some concepts and lemmas in order to obtain our main results. MRNN is modelled by the following differential equation systems:

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -c_i x_i(t) + \sum_{j=1}^n a_{ij}(u_i) f(x_j(t)) \\ & + \sum_{j=1}^n b_{ij}(u_i) f(x_j(t - \tau(t))) + u_i \end{aligned} \quad (5)$$

where $i = 1, 2, \dots, n$, $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$ is the state vector; $A(u_i) = (a_{ij}(u_i))$, $B(u_i) = (b_{ij}(u_i))$ and $C = \text{diag}\{c_1, c_2, \dots, c_n\}$ are connection weight matrices; a_{ij}, b_{ij} are related to external inputs $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$; $c_i > 0, i = 1, 2, \dots, n, \forall t \geq t_0, \forall i, j \in \{1, 2, \dots, n\}$, $0 < \tau(t) \leq \tau$ is the time-varying delay; f is a bounded activation function satisfying the following condition

$$|f(r_1) - f(r_2)| \leq \mu |r_1 - r_2| \quad (6)$$

where $r_1, r_2, \mu \in \mathbb{R}$ and $\mu > 0$.

According to circuit theory and the property of memristor, it results that there exist some constants $\underline{a}_{ij}, \bar{a}_{ij}, \underline{b}_{ij}, \bar{b}_{ij}$ such that

$$\begin{cases} \underline{a}_{ij} \leq a_{ij}(u_i) \leq \bar{a}_{ij} \\ \underline{b}_{ij} \leq b_{ij}(u_i) \leq \bar{b}_{ij} \end{cases} \quad (7)$$

Remark 2: Compared with these models in [31]–[38], the difference of MRNN (5) is that coefficients $a_{ij}(u_i)$ and $b_{ij}(u_i)$, $i, j = 1, 2, \dots, n$ are continuous variable functions with respect to external inputs u_i . Memristor has multi resistances as demonstrated by real device experiments and circuit simulation in [47]. Hence, MRNN can be seen as a neural network with an infinite number of modes because $a_{ij}(u_i)$, $b_{ij}(u_i)$ belong to intervals $[\underline{a}_{ij}, \bar{a}_{ij}]$, and $[\underline{b}_{ij}, \bar{b}_{ij}]$, respectively. While the existing results are 2^{n^2+1} sub modes in [31], [32]. So MRNN (5) seems to model human neurons network more better.

2.3 Preliminaries

Let $u = (u_1, u_2, \dots, u_n)$ be the external input and denote $x(t; t_0, \phi, u)$ as the state of MRNN (5) with some u and initial value,

$$\phi(\vartheta) = (\phi_1(\vartheta), \phi_2(\vartheta), \dots, \phi_n(\vartheta))^T$$

where $\phi(\vartheta) \in C([t_0 - \tau, t_0], \mathcal{D})$, $\mathcal{D} \in \mathbb{R}^n$. Then, $x(t; t_0, \phi, u)$ is continuous and satisfies MRNN (5) and $x(s; t_0, \phi, u) = \phi(s)$, for $s \in [t_0 - \tau, t_0]$. For simplicity, let $x(t)$ be the state of MRNN (5).

Definition 1 [48]: The equilibrium point x^* of MRNN (5) is said to be locally exponentially stable in region \mathcal{D} , if there exist constants $\alpha > 0, \beta > 0$ such that $\forall t \geq t_0$

$$\|x(t; t_0, \phi, u) - x^*\| \leq \beta \|\phi - x^*\|_\infty \exp\{-\alpha(t - t_0)\}$$

where $x(t; t_0, \phi, u)$ is the solution of MRNN (5) with any external input u and initial condition $\phi(\vartheta) \in C([t_0 - \tau, t_0], \mathcal{D})$. \mathcal{D} is said to be a locally exponentially attractive set of the equilibrium point x^* . When $\mathcal{D} = \mathbb{R}^n$, x^* is said to be globally exponentially stable.

Lemma 1 [49]: Let \mathcal{D} be a bounded and closed set in \mathbb{R}^n , and H be a mapping on complete metric space $(\mathcal{D}, \|\cdot\|)$, where $\forall x, y \in \mathcal{D}$, $\|x - y\| = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$ is measurement in \mathcal{D} . If $H(\mathcal{D}) \subset \mathcal{D}$ and there exists a constant $\alpha < 1$ such that $\forall x, y \in \mathcal{D}$, $\|H(x) - H(y)\| \leq \alpha \|x - y\|$, then there exists a unique $x^* \in \mathcal{D}$ such that $H(x^*) = x^*$.

3 Stability Analysis for MRNN

Stability of MRNN is the foundation for its application to associative memory. So we discuss global stability and multi-stability of MRNN in the following subsections. Firstly, we analyze the differences between MRNN and traditional RNN. Traditional RNN is described by, for $i, j = 1, 2, \dots, n$,

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -c_i x_i(t) + \sum_{j=1}^n a_{ij} f(x_j(t)) \\ & + \sum_{j=1}^n b_{ij} f(x_j(t - \tau(t))) + u_i \end{aligned} \quad (8)$$

where c_i, a_{ij}, b_{ij}, u_i have the same means as those in (5).

Discussion: According to above analysis, coefficients $a_{ij}(u_i), b_{ij}(u_i)$ of MRNN (5) can take any values in $[\underline{a}_{ij}, \bar{a}_{ij}], [\underline{b}_{ij}, \bar{b}_{ij}]$. While the corresponding coefficients of RNN cannot be changed. So MRNN (5) is a family of neural networks with infinitely many modes or sub neural networks. Hence MRNN may have infinite number of globally or locally stable equilibrium points. Coefficients of the interval RNN [50] may be constants in different intervals because coefficients increments $\Delta a_{ij}, \Delta b_{ij}$ are caused by noises and implementation errors. This is different from MRNN but the systematic analysis method [50] can be as a reference for stability analysis of MRNN.

3.1 Global Stability Analysis

This subsection discusses global stability of MRNN (5). By using comparative principle and the existing stability criteria, it derives some sufficient conditions for global stability of (5). The following activation function will be

adopted in the rest of the paper

$$f(r) = \begin{cases} 4k - 3, & r \in [4k - 3, +\infty) \\ 2r - (4k - 3), & r \in [4k - 5, 4k - 3) \\ \dots, & \\ 2r - 5, & r \in [3, 5) \\ 1, & r \in [1, 3) \\ r, & r \in (-1, 1) \\ -1, & r \in (-3, -1] \\ 2r + 5, & r \in (-5, -3] \\ \dots, & \\ 2r + 4k - 3, & r \in (3 - 4k, 5 - 4k] \\ 3 - 4k, & r \in (-\infty, 3 - 4k]. \end{cases} \quad (9)$$

Obviously, $|f(r_1) - f(r_2)| \leq |r_1 - r_2|$, $r_1, r_2 \in \mathbb{R}$. In order to derive our result, the following lemma is needed.

Lemma 2: If the following three differential systems

$$\dot{y}(t) = g_1(y(t)) \quad (10)$$

$$\dot{y}(t) = g_2(y(t)) \quad (11)$$

$$\dot{y}(t) = g_3(y(t)) \quad (12)$$

have one common equilibrium point $y^* = 0$, $g_1(0) = g_2(0) = g_3(0) = 0$ and satisfying $g_1(y) \leq g_2(y) \leq g_3(y)$, then system (11) is globally exponentially stable if systems (10) and (12) are globally exponentially stable.

Proof: Take the same initial value $y(t_0) = 0$ for three systems, let y_1^*, y_2^* denote equilibrium points of (10) and (12), respectively. Then

$$\begin{aligned} g_1(y) & \leq g_2(y) \leq g_3(y) \\ \int_{t_0}^t g_1(y) & \leq \int_{t_0}^t g_2(y) \leq \int_{t_0}^t g_3(y) \\ y_1(t) & \leq y_2(t) \leq y_3(t). \end{aligned} \quad (13)$$

Hence, $|y_1(t)| \leq |y_2(t)| \leq |y_3(t)|$ or $|y_3(t)| \leq |y_2(t)| \leq |y_1(t)|$. And (10) and (11) are globally exponentially stable, there exist $\alpha_1, \alpha_3, \beta_1, \beta_3$, initial values ϕ_1, ϕ_3 satisfying

$$\begin{aligned} |y_1(t)| & \leq \|\phi_1\| \exp\{-\alpha_1(t - t_0)\} \\ |y_3(t)| & \leq \|\phi_3\| \exp\{-\alpha_3(t - t_0)\}. \end{aligned}$$

So there must exist α_2, β_2 and an initial value ϕ_2 , and the following inequality

$$|y_2(t)| \leq \|\phi_2\| \exp\{-\alpha_2(t - t_0)\}$$

is valid, i.e., (11) is globally exponentially stable. ■

Because the external inputs $u_i, i = 1, 2, \dots, n$ are just used to change the memristance, we assume that all of sub neural networks have the same external inputs $u_i, i = 1, 2, \dots, n$ in the following discussion.

Lemma 3 [48]: If for c_i, a_{ij} and $b_{ij}, \forall i, j \in \{1, 2, \dots, n\}$, $C - |A| - |B|$ is a nonsingular M matrix with $|A| = (|\mu_j a_{ij}|)_{n \times n}$ and $|B| = (|\omega_j b_{ij}|)_{n \times n}$, μ_j, ω_j are positive constants for $j = 1, 2, \dots, n$, then the corresponding equilibrium point of (14) is globally exponentially stable.

By (5) and (7), we have NN_1

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -c_i x_i(t) + \sum_{j=1}^n \bar{a}_{ij}(u_i) f(x_j(t)) \\ & + \sum_{j=1}^n \bar{b}_{ij}(u_i) f(x_j(t - \tau(t))) + u_i \end{aligned} \quad (14)$$

and NN_2

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -c_i x_i(t) + \sum_{j=1}^n \underline{a}_{ij}(u_i) f(x_j(t)) \\ & + \sum_{j=1}^n \underline{b}_{ij}(u_i) f(x_j(t - \tau(t))) + u_i \end{aligned} \quad (15)$$

for $i = 1, 2, \dots, n$.

Theorem 1: If coefficients of neural networks (14) and (15) satisfy that $C - |A| - |B|$ is a nonsingular M matrix with $|A| = (|a_{ij}|)_{n \times n}$ and $|B| = (|b_{ij}|)_{n \times n}$, then (5) is globally exponentially stable for $\forall a_{ij}(u_i) \in [\underline{a}_{ij}, \bar{a}_{ij}]$, $\forall b_{ij}(u_i) \in [\underline{b}_{ij}, \bar{b}_{ij}]$ and bounded external inputs u_i , $i, j = 1, 2, \dots, n$.

Proof: Because the activation $f(r)$ satisfy the Lipschitz condition and u_i , $i = 1, 2, \dots, n$ are bounded, there must exist one equilibrium for (5) at least by the Schauder fixed point theorem for $\forall a_{ij}(u_i) \in [\underline{a}_{ij}, \bar{a}_{ij}]$, $\forall b_{ij}(u_i) \in [\underline{b}_{ij}, \bar{b}_{ij}]$ and bounded external inputs u_i , $i, j = 1, 2, \dots, n$. Denote x^* , \underline{x}_i^* , \bar{x}_i^* equilibrium points of (5), (14), (15), respectively.

Let

$$\begin{aligned} z(t) = & (x_1(t) - x_1^*, x_2(t) - x_2^*, \dots, x_n(t) - x_n^*) \\ \bar{f}(z_i(t)) = & f(x_i(t) + x_i^*) - f(x_i^*). \end{aligned}$$

Hence,

$$\begin{aligned} \dot{z}_i(t) = & -c_i z_i(t) + \sum_{i=1}^n a_{ij}(u_i) \bar{f}(z_i(t)) \\ & + \sum_{i=1}^n a_{ij}(u_i) \bar{f}(z_i(t - \tau(t))). \end{aligned} \quad (16)$$

Let $V(t) = (V_1(t), V_2(t), \dots, V_n(t))$, $V_i(t) = |x_i(t)|$, then

$$\begin{aligned} D^+ V_i(t) \leq & -c_i V_i(t) + \sum_{i=1}^n |a_{ij}(u_i)| V_j(t) \\ & + \sum_{i=1}^n |b_{ij}(u_i)| V_j(t - \tau(t)). \end{aligned} \quad (17)$$

Let

$$\begin{aligned} \Psi(t) = & -c_i V_i(t) + \sum_{i=1}^n |a_{ij}(u_i)| V_j(t) \\ & + \sum_{i=1}^n |b_{ij}(u_i)| V_j(t - \tau(t)). \end{aligned} \quad (18)$$

Since

$$\underline{a}_{ij} \leq a_{ij}(u_i) \leq \bar{a}_{ij}, \quad \underline{b}_{ij} \leq b_{ij}(u_i) \leq \bar{b}_{ij}$$

then

$$|\underline{a}_{ij}| \leq |a_{ij}(u_i)| \leq |\bar{a}_{ij}|, \quad |\underline{b}_{ij}| \leq |b_{ij}(u_i)| \leq |\bar{b}_{ij}|$$

or

$$|\bar{a}_{ij}| \leq |a_{ij}(u_i)| \leq |\underline{a}_{ij}|, \quad |\bar{b}_{ij}| \leq |b_{ij}(u_i)| \leq |\underline{b}_{ij}|.$$

So

$$\begin{aligned} \Psi(t) \leq & -c_i V_i(t) + \sum_{i=1}^n |\bar{a}_{ij}| V_j(t) \\ & + \sum_{i=1}^n |\bar{b}_{ij}| V_j(t - \tau(t)) \end{aligned} \quad (19)$$

or

$$\begin{aligned} \Psi(t) \leq & -c_i V_i(t) + \sum_{i=1}^n |\underline{a}_{ij}| V_j(t) \\ & + \sum_{i=1}^n |\underline{b}_{ij}| V_j(t - \tau(t)). \end{aligned} \quad (20)$$

According to the condition of Theorem 1, (17), (19), (20) and Lemma 3, there must exist positive constants α and β satisfying $|V_i(t)| \leq \alpha \exp\{-\beta t\}$. Hence the conclusion of this theorem is valid. ■

Remark 3: When $\bar{a}_{ij} = a_{ij}$, $\bar{b}_{ij} = b_{ij}$, for $i, j = 1, 2, \dots, n$, the result in [48] can be obtained from Theorem 1. So we generalize the result of [47] to discuss global stability of MRNN (5) with infinite number of sub neural networks. Compared with the existing literatures, its main merit is that MRNN (5) has many globally exponentially stable equilibrium points. The systematic method in [43] can be used to derive sufficient conditions for global stability of (5) by virtue of many global stability criteria in the existing literatures.

3.2 Multi-stability of MRNN

Multi-stability of RNN means that RNN has coexisting multi attractors. Memory patterns can be stored by these attractors. Memory capacity of RNN is up to the number of attractors. Another factor affecting memory is the activation function $f(r)$. Zeng *et al.* has derived some sufficient conditions for multi-stability of n dimensional RNN with the activation function $f(r) = (|r + 1| - |r - 1|)/2$ which has 3^n equilibrium points and 2^n equilibrium points of them are locally exponentially stable. And then Zeng *et al.* [49] generalize their work to n dimensional RNN with the activation function (9). They derive that RNN with the activation function (9) has $(4k - 1)^n$ equilibrium points and $(2k)^n$ equilibrium points of them are locally exponentially stable in Ω_k , where

$$\begin{aligned} \Omega_k = & \left\{ \prod_{i=1}^n \ell^{(i)}, \ell^{(i)} = (-\infty, -(4k - 3)] \text{ or} \right. \\ & (- (4k - 3), -(4k - 5)] \text{ or } \dots \text{ or } (-3, -1] \text{ or} \\ & (-1, 1) \text{ or } [1, 3) \text{ or } \dots \text{ or} \\ & \left. [4k - 5, 4k - 3) \text{ or } [4k - 3, +\infty) \right\}. \end{aligned} \quad (21)$$

But there are limited number of equilibrium points and output patterns for RNN with these two kinds of activation functions. Hence, we discuss multi-stability of MRNN with the activation function (9).

Lemma 4 [49]: For the given integer $k \geq 1$, if $\forall i, j \in \{1, 2, \dots, n\}$, the following inequalities are valid for coefficients c_i , a_{ij} , b_{ij} and external inputs u_i of RNN with the activation function (9)

$$a_{ii} + b_{ii} - (4k - 3) \sum_{j=1, j \neq i}^n (|a_{ij} + b_{ij}|) - |u_i| > c_i \quad (22)$$

$$\begin{aligned} a_{ii} + b_{ii} + \sum_{j=1, j \neq i}^n (|a_{ij} + b_{ij}|) + \frac{|u_i|}{(4k - 3)} \\ < \left(1 + \frac{2}{(4k - 3)}\right) c_i \end{aligned} \quad (23)$$

then RNN with the activation function (9) has $(4k - 1)^n$ equilibrium points and $(2k)^n$ of them are locally exponentially stable.

Let $\check{a}_{ij} = \max\{|\underline{a}_{ij}|, \bar{a}_{ij}\}$, $\check{b}_{ij} = \max\{|\underline{b}_{ij}|, \bar{b}_{ij}\}$. Then, we have the following results.

Theorem 2: If the following inequalities are valid

$$\begin{aligned} \underline{a}_{ii} + \underline{b}_{ii} - (4k - 3) \sum_{j=1, j \neq i}^n \left(|\check{a}_{ij}| + |\check{b}_{ij}| \right) - |u_i| &> c_i \quad (24) \\ \bar{a}_{ii} + \bar{b}_{ii} + \sum_{j=1, j \neq i}^n \left(|\check{a}_{ij}| + |\check{b}_{ij}| \right) + \frac{|u_i|}{(4k - 3)} \\ &< \left(1 + \frac{2}{(4k - 3)} \right) c_i \quad (25) \end{aligned}$$

then for c_i , $a_{ij} \in [\underline{a}_{ij}, \bar{a}_{ij}]$ and $b_{ij} \in [\underline{b}_{ij}, \bar{b}_{ij}]$, $\forall i, j \in \{1, 2, \dots, n\}$, the corresponding MRNN (5) has $(4k - 1)^n$ equilibria located in Ω_k , $(2k)^n$ of them are locally exponentially stable.

Proof: In order to prove multi-stability of (5), it is sufficient to verify whether conditions (22) and (23) are valid or not. For $\underline{a}_{ii} \leq a_{ii}(u_i) \leq \bar{a}_{ii}$, we have

$$\begin{aligned} &a_{ii}(u_i) + b_{ii}(u_i) \\ &- (4k - 3) \sum_{j=1, j \neq i}^n \left(|a_{ij}(u_i) + b_{ij}(u_i)| \right) - |u_i| \\ &\geq \underline{a}_{ii} + \underline{b}_{ii} - (4k - 3) \sum_{j=1, j \neq i}^n \left(|a_{ij}(u_i) + b_{ij}(u_i)| \right) - |u_i| \\ &\geq \underline{a}_{ii} + \underline{b}_{ii} - (4k - 3) \sum_{j=1, j \neq i}^n \left(|\check{a}_{ij}| + |\check{b}_{ij}| \right) - |u_i| \\ &> c_i. \end{aligned}$$

And then

$$\begin{aligned} &\left(1 + \frac{2}{(4k - 3)} \right) c_i \\ &> \bar{a}_{ii} + \bar{b}_{ii} + \sum_{j=1, j \neq i}^n \left(|\check{a}_{ij}| + |\check{b}_{ij}| \right) + \frac{|u_i|}{(4k - 3)} \\ &\geq a_{ii}(u_i) + b_{ii}(u_i) + \sum_{j=1, j \neq i}^n \left(|\check{a}_{ij}| + |\check{b}_{ij}| \right) + \frac{|u_i|}{(4k - 3)} \\ &\geq a_{ii}(u_i) + b_{ii}(u_i) + \sum_{j=1, j \neq i}^n \left(|a_{ij}(u_i)| + |b_{ij}(u_i)| \right) \\ &\quad + \frac{|u_i|}{(4k - 3)}. \end{aligned}$$

Hence, (22) and (23) are valid for c_i , $a_{ij}(u_i), b_{ij}(u_i)$, $i, j = 1, 2, \dots, n$. By Lemma 4, the conclusion of Theorem 2 is valid. ■

Remark 4: In fact, (24) and (25) are minimum value and maximum value of (22) and (23), respectively. Hence, we generalize the systematic method [50], [51] to analyzing multi-stability of MRNN. Compared with results in [49], the conditions are more conservative. But MRNN has infinite number of sub neural networks, i.e, globally exponentially stable equilibrium points of MRNN (5) are infinite times $(2k)^n$. By virtue of the existing results for multi-stability of RNN, we can obtain many sufficient conditions for multi-stability of MRNN (5).

4 Associative Memory Synthesis

Based on the above analysis, we discuss associative memory design method based on MRNN (5). Memory patterns are described by bipolar value $\{-1, 1\}$. Associative memory is implemented by RNN circuit. So the activation function is taken as $f(r) = (|r + 1| - |r - 1|)/2$, $r \in \mathbb{R}$ and weight values are simulated by linear resistors. So associative memory can just remember bitmap, and storage capacity is limited. So our associative memory synthesis is based on MRNN with (9). It is able to memorize gray map and has infinite storage capacity. The key point of associative memory synthesis is the computation for weights value. So we firstly describe the synthesis problem, and then present our design method based on Zeng and Wang's work [43]. The activation function $F(r) = 0, r < 0, F(r) = f(r), r > 0$ where $f(r)$ is defined as (9). The purpose is to make the designed neural network be able to memorize gray map.

Synthesis Problem: There are p memory patterns being denoted by vectors $\alpha^1, \alpha^2, \dots, \alpha^p$, $\alpha^i \in \{0, 1, 3, 5, \dots, 4k - 3\}^n$, $i = 1, 2, \dots, p$. Compute coefficients c_i , a_{ij}, b_{ij} and u_i in order that $\alpha^1, \alpha^2, \dots, \alpha^p$, α^i are stable memory vectors of MRNN (9).

Design procedure:

Step 1: Use vectors $\alpha^1, \alpha^2, \dots, \alpha^p$, $\alpha^i \in \{0, 1, 3, 5, \dots, 4k - 3\}^n$ (n , the dimension of MRNN) presenting the desired memory pattern. If $p \leq (2k)^n$, then go to Step 2 computing coefficients c_i , a_{ij}, b_{ij} and u_i . If $p = q(2k)^n + \gamma$, then divide $\alpha^1, \alpha^2, \dots, \alpha^p$ into $q + 1$ groups. Go to Step 2 and compute coefficients for each group.

Step 2: For the desired memory vectors, do the following:

- 1) $-\tilde{U}_k S(l)^T (S(l) S(l)^T)^{-1} = (t_{ij})_{n \times n}$ ($l \leq n$);

- 2) Take $\sigma_i > 1$, $i = 1, 2, \dots, n$ and choose a_{ij}, b_{ij} satisfying $a_{ii} + b_{ii} - \sigma_i = t_{ii}$ and $a_{ij} + b_{ij} = t_{ij}$; where $\tilde{U}_l, S(l)$ are defined in [41].

Step 3: If $p = (2k)^n$, compute memristance M_{ij} according to a_{ij}, b_{ij} ; if $p = q(2k)^n + \gamma$, compute memristance M_{ij} according to $|a_{ij}|_{\max}, |b_{ij}|_{\max}$ where

$$|a_{ij}|_{\max} = \max_{1 \leq \delta \leq q+1} |a_{ij}^\delta|$$

$$|b_{ij}|_{\max} = \max_{1 \leq \delta \leq q+1} |b_{ij}^\delta|.$$

Remark 5: Compared with the work in [41], we do not require that p , the number of desired memory vectors, is less than or equal to $(2k)^n$. Hence, we generalize Zeng and Wang's work [43]. And we choose the activation function (9) in order to make the designed associative memory MRNN be able to memorize gray map not bitmap. This is one difference from the existing work. Another merit is that the designed MRNN has infinite number of equilibrium points, i.e., MRNN can be used to implement large storage capacity associative memory. For example, RNN with $f(r) = (|r + 1| - |r - 1|)$ only has 2^n memory patterns in $\{-1, 1\}^n$ and RNN with (9) only has $(4)^n$ memory patterns in $\{-5, -1, 1, 5\}^n$ when $k = 2$. MRNN breaks this bottleneck for it has variable coefficients and infinite memory patterns.

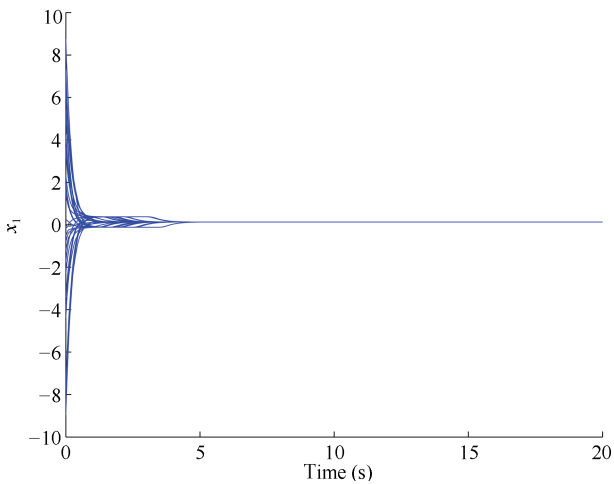
5 Illustrative Examples

Example 1: Consider the following MRNN with activation function $f(r), r \in \mathbb{R}$ (9) with $k = 1, n = 2$.

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_{11}f(x_1(t)) + a_{11}f(x_2(t)) \\ \quad + b_{11}f(x_1(t - 0.1)) \\ \quad + b_{12}f(x_2(t - 0.1)) + 0.8 \\ \dot{x}_2(t) = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) \\ \quad + b_{21}f(x_1(t - 0.2)) \\ \quad + b_{22}f(x_2(t - 0.2)) + 0.4 \end{cases} \quad (26)$$

where

$$\begin{aligned} -3 \leq a_{11} \leq -2, \quad \frac{1}{3} \leq a_{12} \leq \frac{1}{2} \\ -\frac{1}{3} \leq a_{21} \leq \frac{1}{2}, \quad -3 \leq a_{22} \leq -2 \\ 0.9 \leq b_{11} \leq 1, \quad -\frac{1}{4} \leq b_{12} \leq \frac{1}{4} \\ \frac{1}{4} \leq b_{21} \leq \frac{1}{2}, \quad 0.6 \leq b_{22} \leq 1 \\ c_1 = c_2 = 4.8. \end{aligned}$$



According to Theorem 1, every sub neural network of MRNN (26) is globally exponentially stable. Let $a_{11} = a_{22} = -3, a_{12} = a_{21} = 1/2, b_{11} = b_{22} = 1, b_{12} = 1/4, b_{21} = 1/2$, and simulate with 50 initial values. The dynamic characteristics are shown in Figs. 2–4.

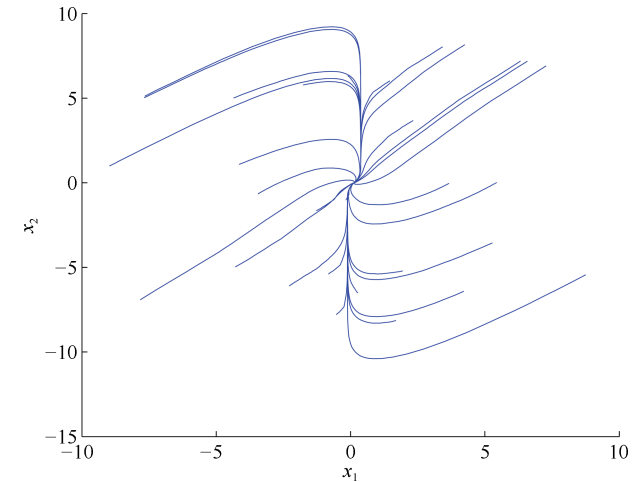


Fig. 4. Phase plot of $x_1(t)$ and $x_2(t)$ of MRNN (26).

Example 2: Consider a MRNN with activation function $f(r), r \in \mathbb{R}$ (9) with $k = 2, n = 2$.

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_{11}f(x_1(t)) + a_{11}f(x_2(t)) \\ \quad + b_{11}f(x_1(t - 0.1)) \\ \quad + b_{12}f(x_2(t - 0.1)) + 0.05 \\ \dot{x}_2(t) = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) \\ \quad + b_{21}f(x_1(t - 0.2)) \\ \quad + b_{22}f(x_2(t - 0.2)) - 0.04 \end{cases} \quad (27)$$

where

$$\begin{aligned} 0.5 \leq a_{11} \leq 0.7, \quad 0.01 \leq a_{12} \leq 0.02 \\ -0.02 \leq a_{21} \leq -0.01, \quad 0.4 \leq a_{22} \leq 0.6 \\ 0.4 \leq b_{11} \leq 0.5, \quad -0.02 \leq b_{12} \leq -0.01 \\ 0.005 \leq b_{21} \leq 0.01, \quad 0.6 \leq b_{22} \leq 0.7. \end{aligned}$$

Fig. 2. Transient behaviors of $x_1(t)$ of MRNN (26).

According to Theorem 2, every sub neural network of MRNN (26) has 7^2 isolated equilibrium points and 4^2 of them are locally exponentially stable. Take maximum values for $a_{ij}, b_{ij}, i, j = 1, 2$ and simulate with 50 initial values. The dynamics characteristics are shown in Fig. 5.

Example 3: The same example has been introduced by Lu and Liu [52], Zeng and Wang [43] for associative memory synthesis. The desired memory patterns are three letters “I, L, U” and number “7” as plotted by gray Fig. 6.

These four desired patterns can be denoted by memory vectors

$$\begin{aligned} \alpha^1 &= (1, 1, 1, 1, 5, 5, 5, 5, 1, 1, 1, 1) \\ \alpha^2 &= (5, 5, 5, 5, 1, 1, 1, 1, 5, 1, 1, 1, 5) \\ \alpha^3 &= (5, 5, 5, 5, 1, 1, 1, 1, 5, 5, 5, 5, 5) \\ \alpha^4 &= (5, 1, 1, 1, 5, 1, 1, 5, 5, 5, 5, 5, 5). \end{aligned}$$

The objective is to design one 12 dimension MRNN with $\alpha^1, \alpha^2, \alpha^3, \alpha^4$ being stable memory vectors. Obviously, the number of stable memory vectors is less than

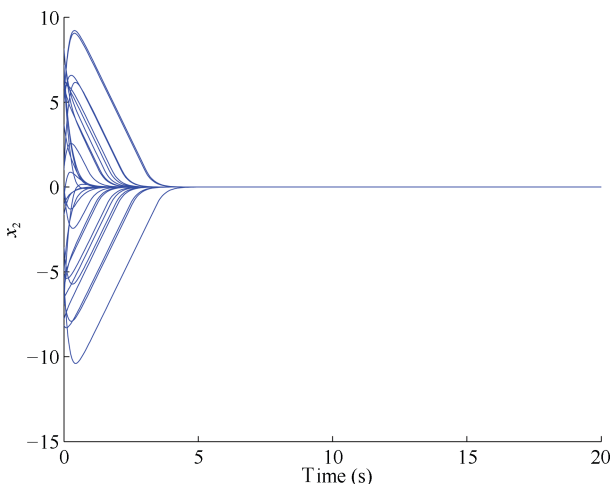


Fig. 3. Transient behaviors of $x_2(t)$ of MRNN (26).

$(2k)^n$ ($k = 2, n = 12$). For $l = 12$, then we add eight vectors $\alpha^5, \dots, \alpha^8$ such that

stable region Ω_k . According to associative memory synthesis program, we can obtain

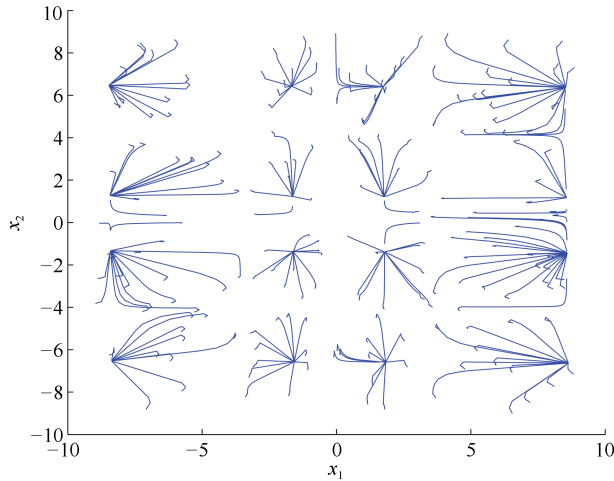


Fig. 5. Transient behaviors of $x_1(t)$ and $x_2(t)$ of MRNN (27).

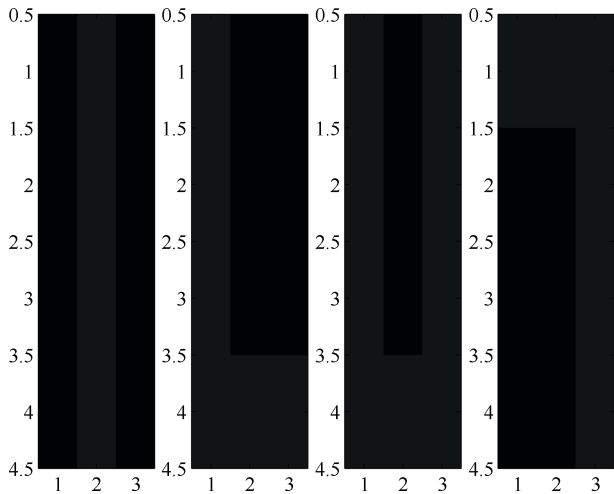


Fig. 6. Three letters "I, L, U" and number "7" being presented by gray map.

$$S(12) = \begin{bmatrix} 1 & 5 & 5 & 5 & 1 & 5 & 5 & 5 & 1 & 5 & 5 & 5 \\ 1 & 5 & 5 & 1 & 1 & 1 & 5 & 1 & 5 & 5 & 5 & 5 \\ 1 & 5 & 5 & 1 & 1 & 5 & 1 & 5 & 5 & 1 & 5 & 5 \\ 1 & 5 & 5 & 1 & 1 & 5 & 5 & 1 & 5 & 5 & 1 & 5 \\ 5 & 1 & 1 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 5 & 5 & 5 & 1 & 5 & 5 & 1 & 5 & 1 \\ 5 & 1 & 5 & 1 & 5 & 1 & 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 5 & 1 & 5 & 1 & 1 & 1 & 5 & 1 & 1 \\ 5 & 1 & 5 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 5 & 1 & 5 & 1 & 5 & 1 & 1 & 1 & 1 & 5 & 1 \end{bmatrix}$$

is an invertible matrix. Choose $u_i = 1.825$ (external inputs), $i = 1, 2, \dots, 12$, $\lambda_i^l = 1.5$ ($i = 1, 2, \dots, 12$; $l = 1, 2, 3, 4$), $\lambda_i^l = 0.1$ ($i = 1, 2, \dots, 12$; $l = 5, 6, \dots, 12$). The function of λ_i^l is to make these memory vectors be in the

$$W = \begin{bmatrix} 0.5020 & 0.2342 & 1.1002 & 0.0348 \\ -0.6667 & 0.6667 & -0.8889 & 0.4444 \\ 0.8354 & -0.0992 & 1.9891 & -0.4097 \\ -0.3333 & 0.3333 & -0.6667 & 0.6667 \\ 5.0000 & -5.0000 & 10.0000 & 0.0000 \\ 5.0000 & -5.0000 & 10.0000 & 0.0000 \\ -12.5307 & 1.4877 & -6.5031 & -0.5215 \\ -7.5307 & -3.5123 & -16.5031 & -0.5215 \\ 7.5307 & 3.5123 & 13.1697 & -2.8119 \\ -10.0000 & 10.0000 & -13.3333 & 6.6667 \\ 0.0000 & -0.0000 & -6.6667 & 3.3333 \\ 7.5307 & 3.5123 & 19.8364 & -6.1452 \\ \\ 0.1227 & -0.3640 & 0.9202 & 0.5215 \\ -0.2222 & 0.4444 & -0.2222 & 0 \\ 0.3449 & -0.1418 & 1.1425 & 0.5215 \\ -0.0000 & 0.0000 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 10.0000 & 0 \\ 0.0000 & -10.0000 & 10.0000 & 0 \\ -1.8405 & 5.4601 & -3.8037 & 12.1779 \\ 8.1595 & 5.4601 & -13.8037 & 2.1779 \\ -1.4928 & -8.7935 & 10.4703 & -2.1779 \\ -3.3333 & 6.6667 & -3.3333 & 0.0000 \\ 3.3333 & 3.3333 & -6.6667 & -0.0000 \\ -4.8262 & -12.1268 & 17.1370 & -2.1779 \\ \\ 0.1227 & 0.9202 & 0.2342 & 0.7209 \\ 0.0000 & -0.4444 & 0.4444 & -0.0000 \\ 0.7894 & 1.3647 & 0.1230 & 1.0542 \\ -0.6667 & -0.0000 & 0.3333 & -0.3333 \\ 0.0000 & 10.0000 & -5.0000 & 5.0000 \\ 0.0000 & 10.0000 & 5.0000 & 5.0000 \\ -1.8405 & -3.8037 & 1.4877 & -5.8129 \\ 8.1595 & -13.8037 & -3.5123 & -10.8129 \\ 1.8405 & 7.1370 & 0.1789 & 10.8129 \\ 0.0000 & 3.3333 & -3.3333 & -0.0000 \\ 0.0000 & -3.3333 & 3.3333 & 0.0000 \\ -8.1595 & 10.4703 & 6.8456 & 10.8129 \end{bmatrix}$$

where $W = (t_{ij})$, $t_{ij} = a_{ij} + b_{ij}$. It is easy to verify that $\alpha^1, \alpha^2, \alpha^3, \alpha^4$ are stable memory vectors according to Theorem 2. Take $c_i = 1, i = 1, 2, \dots, n$, $a_{ij} = b_{ij}$, $u_i = 0.425$, then we have the MRNN with these desired patterns as stable memory vectors.

6 Concluding Remarks

In this paper, we have introduced MRNN which is a family of recurrent neural networks. Some sufficient conditions are derived to assure its mono-stability and multi-stability. In the existing literature on neural network, the largest number of equilibrium points is $(4k-1)^n$ and $(2k)^n$ of them are locally exponentially stable. In fact, associative memory output patterns are up to the activation function. This point affects the storage capacity of associative memory. Our MRNN with coefficients in intervals cannot be limited by output value of the activation. Hence MRNN can increase the storage capacity of associative memory. This is the main merit which is different from traditional artificial

neural network. So self-adaptive and self-organization recurrent neural network can be realized with memristor [26] in the future.

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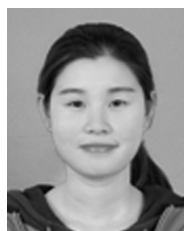
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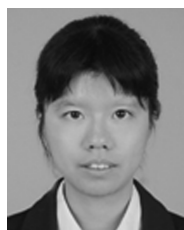
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