Robust H_∞ Consensus Control for High-order Discrete-time Multi-agent Systems With Parameter Uncertainties and External Disturbances

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Abstract The robust H_∞ consensus control problem of high-order discrete-time multi-agent systems with parameter uncertainties and external disturbances is studied, and a linear distributed consensus protocol is designed in this paper. Firstly, the robust H_∞ consensus control problem of high-order discrete-time multi-agent systems with parameter uncertainties and external disturbances is transformed to a robust H_∞ control problem of a set of independent uncertain systems. Secondly, a sufficient linear matrix inequality (LMI) condition is derived to insure that high-order discrete-time multi-agent systems with parameter uncertainties and external disturbances achieve robust consensus with a H_∞ performance level γ. Thirdly, convergence results are given as final consensus values of high-order discrete-time linear multi-agent systems with parameter uncertainties and without external disturbances. Finally, a contrast numerical experiment with and without parameter uncertainties is provided to demonstrate the correctness and effectiveness of the theoretical results.

Key words External disturbances, multi-agent systems (MASs), parameter uncertainties, robust H_∞ consensus control

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1 Introduction

In recent years, distributed coordination of multi-agent systems (MASs) has received great attention from many researchers due to its broad applications on MASs in many areas including formation control [1], [2], flocking [3], [4], distributed filtering [5], [6], synchronization of coupled chaotic oscillators [7]−[9]. Consensus is an essential problem of distributed coordination of MASs, which is to make each agent agree on some common values of interest through feedback of local information from neighboring agents.

The theoretical framework for posing and solving the consensus problem for MASs was first introduced in [10]−[12]. Their work mostly focused on the first-order and second-order consensus in MASs. Furthermore, the consensus problem of MASs has obtained a tremendous surge of interest and extensive development. These works can be generally divided into two categories depending on whether the agent models are continuous-time or discrete-time. The union of interaction topologies must contain a spanning tree if MASs are expected to achieve consensus asymptotically [13]. A framework of high-dimensional state space for the consensus problems of MASs was studied in [14], and then the consensus problems of high order or more general linear MASs models were discussed in [15]−[17]. The consensus problem of discrete-time MASs (D-MASs) based on general linear models was investigated in [18], [19]. The leader-following consensus problem of D-MASs based on general linear models was studied in [20]. The robust guaranteed cost consensus problem of general linear D-MASs models with parameter uncertainties and time-varying delays was investigated in [21].

With the development of the research, the H_∞ consensus control problem of MASs subject to external disturbances was considered in [22]−[24]. Robust H_∞ consensus control problems of first-order MASs with external disturbances and model uncertainties are discussed in [22]. The second-order robust H_∞ consensus control problem of MASs with measurement noises and asymmetric delays is studied in [23]. Distributed H_2 and H_∞ consensus control problems are investigated in [24] for MASs with linear dynamics subject to external disturbances. The robust H_∞ consensus control problem of high-order linear MASs with parameter uncertainties and external disturbances was studied in [25], which also considered the time-delay and switching topology simultaneously. Specifically, the aforementioned works were based on continuous-time models, while the study of discrete-time model cases is more widely applied in practice. In [26], H_∞ synchronization and state estimation problems were considered for an array of coupled discrete time-varying stochastic complex networks over a finite horizon. The robust H_∞ consensus control problem of high-order linear time-varying D-MASs with uncertainties/disturbances and missing measurements was investigated in [27]. The event-based H_∞ consensus control problem for high-order linear time-varying D-MASs over a finite horizon was studied in [28]. Nevertheless, although the robust H_∞ control consensus problem of high-order D-MASs with parameter uncertainties and external disturbances was addressed in [26]−[28], the final convergence value was not provided in these studies.

Motivated by the above, in this paper, the robust H_∞ control consensus problem of high-order D-MASs with parameter uncertainties and external disturbances is investigated by state space decomposition approach. We consider the leaderless consensus of the uncertain high-order D-MASs with fixed topologies. In this problem, if an appropriate consensus protocol is applied, the D-MASs should converge to a common value. Comparing with the existing works, the contribution of this paper is two-fold. On one hand, by state space decomposition approach, a sufficient linear matrix inequality (LMI) condition is given to guarantee that, high-order D-MASs subject to parameter uncertainties and external disturbances achieve robust consensus with a H_∞ performance index γ. On the other hand, with $\omega_x(k) \equiv 0$ or $\omega_x(k)$ interpreted as deterministic l_2 signal, final consensus values of high-order D-MASs with parameter uncertainties and external disturbances, which are first provided in this paper for the first time.

The rest of the paper is organized as follows. The problem formulation is presented in Section 2. In Section 3, the robust H_∞ consensus control problem of MASs (1) is transformed to a robust H_∞ control problem of a set of independent uncertain systems, and a sufficient LMI condition insuring the robust consensus, and a final consensus value of MASs (1) with protocol (4) are given. A numerical example is provided in Section 4 to verify the theoretical analysis. Some conclusions are finally drawn in Section 5

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which concludes the paper and proposes some possible future directions. The notions of graph theory and Kronecker product that will be used in this paper are summarized in Appendix A and Appendix B, respectively.

Notations: A matrix or a vector is said to be positive (respectively, non-negative) if all of its entries are positive (respectively, non-negative). A square matrix is called Schur stable if all of its eigenvalues lie in the open unit disk. Let diag$\{a_{11}, a_{22}, \ldots, a_{nn}\}$ be the diagonal matrix with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. The symbol $\otimes$ represents the Kronecker product. M^T denotes the transpose conjugate of matrix M. I is an appropriate dimensions identity matrix. $\mathbf{1}_N = [1, \ldots, 1]^T$ denotes an N-dimensional vector with all of its elements being 1.

2 Problem Formulation

A high-order MAS can be described as a linear system, which has been presented in [15], and thus, consider a high-order identical D-MAS consisting of N agents indexed by $1, 2, \ldots, N$, distributed on an undirected communication graph G, in which the dynamics of agent i is described by a linear discrete-time system as follows

$$\begin{cases} \boldsymbol{x}_i(k+1) = (A + \Delta A)\boldsymbol{x}_i(k) \\ \qquad + (B + \Delta B)\boldsymbol{u}_i(k) + B_\omega \boldsymbol{\omega}_{i,x}(k) \\ \boldsymbol{z}_i(k) = C\boldsymbol{x}_i(k) \end{cases} \tag{1}$$

where $\boldsymbol{\omega}_{i,x}(k) \in \mathbb{R}^{m_2}$ is the external disturbance of agent i; $\boldsymbol{u}_i(k) \in \mathbb{R}^{m_1}$ and $\boldsymbol{x}_i(k) \in \mathbb{R}^d$ are the consensus protocols and states of agent i, respectively; $\boldsymbol{z}_{i,x}(k) \in \mathbb{R}^l$ is the controlled-output of agent i; $A \in {}^{d\times d}$, $B \in \mathbb{R}^{d\times m_1}$, $B_\omega \in \mathbb{R}^{d\times m_2}$ and $C \in \mathbb{R}^{l\times d}$ are known constant matrices, ΔA and ΔB are parameter uncertainties in the system matrix and input channel, which are assumed to be of the form

$$\begin{bmatrix} \Delta A & \Delta B \end{bmatrix} = DF\begin{bmatrix} E_1 & E_2 \end{bmatrix} \tag{2}$$

where D is a real constant matrix, and F is an unknown matrix function satisfying

$$F^T F \leq I. \tag{3}$$

The parameter uncertainties ΔA and ΔB are said to be admissible if both (2) and (3) hold. For the leaderless consensus problem of uncertain D-MASs (1), the following local consensus protocol is applied to each agent i

$$\boldsymbol{u}_i(k) = K\sum_{j\in N_i} a_{ij}(\boldsymbol{x}_j(k) - \boldsymbol{x}_i(k)) \tag{4}$$

where K is a constant gain matrix with appropriate dimensions, and a_{ij} being the graph edge weights. This protocol is distributed in nature as it only depends on the immediate neighbors N_i of agent (node) i. This is known as a local voting protocol because the control input of each agent depends on the difference between its state and all its neighbors. Then, the definition of consensus for high-order D-MASs (1) with consensus protocol (4) is given as follows.

Definition 1: For a given gain matrix K, system (1) is said to achieve consensus if for any given bounded initial condition, there exists a vector-valued $c(k)$ which is dependent on the initial condition such that $\lim_{k\to\infty}(x(k) - \mathbf{1}_N \otimes c(k)) = \mathbf{0}$, where $c(k)$ is called a final consensus value.

Let $\boldsymbol{x}(k) = [\boldsymbol{x}_1^T(k), \ldots, \boldsymbol{x}_N^T(k)]^T$, $\boldsymbol{\omega}_x(k) = [\boldsymbol{\omega}_{1,x}^T, \ldots, \boldsymbol{\omega}_{N,x}^T]^T$ and $\boldsymbol{z}(k) = [\boldsymbol{z}_1^T(k), \ldots, \boldsymbol{z}_N^T(k)]^T$, then the dynamics of high-order D-MASs (1) with the distributed consensus protocol (4) can be described by a closed-loop discrete-time networked dynamics as

$$\begin{cases} \boldsymbol{x}(k+1) = (I_N \otimes (A + \Delta A) \\ \qquad - L \otimes (B + \Delta B)K)\boldsymbol{x}(k) + (I_N \otimes B_\omega)\boldsymbol{\omega}_x(k) \\ \boldsymbol{z}(k) = (I_N \otimes C)\boldsymbol{x}(k). \end{cases} \tag{5}$$

The suboptimal robust H_∞ consensus control problem of system (5) is stated to find a distributed protocol (4) such that

1) with $\boldsymbol{\omega}_x(k) = 0$, the closed-loop system (5) is asymptotically stable for all admissible uncertain matrices F.

2) with $\boldsymbol{\omega}_x(k)$ interpreted as deterministic l_2 signal, the closed-loop transfer function from $\boldsymbol{\omega}_x(k)$ to $\boldsymbol{z}(k)$ of system (5), which is denoted by $T_{\omega z}$, satisfies $\|T_{\omega z}\|_\infty < \gamma$ for all admissible uncertain matrices F and a given allowable scalar $\gamma > 0$, where $\|T_{\omega z}\|_\infty$ is the H_∞ norm of $T_{\omega z}$, defined by $\|T_{\omega z}\|_\infty = \sup_{\omega_x \in \mathbb{R}^{Nd}} \bar{\sigma}(T_{\omega z}(j\omega))$.

In order to analyze the robust H_∞ consensus control problem of closed-loop D-MASs (6), we assume hereafter that the communication graph G is connected and give the following lemma about the graph theory.

Lemma 1 [29] : Let L be the Laplacian matrix of an undirected graph G , then zero is an eigenvalue of L. If, in addition, G is connected, the zero eigenvalue of L is simple, and all the other eigenvalues of L are positive and real.

Moreover, let λ_i $(i = 1, 2, \ldots, N)$ be eigenvalues of the Laplacian matrix $L \in \mathbb{R}^N$ for an undirected topology G, where $\lambda_1 = 0$ with the associated eigenvector $\bar{\boldsymbol{u}}_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. There exists an orthogonal matrix

$$U = \begin{bmatrix} \dfrac{1}{\sqrt{N}} & \dfrac{\mathbf{1}_{N-1}^T}{\sqrt{N}} \\ \dfrac{\mathbf{1}_{N-1}^T}{\sqrt{N}} & \bar{U} \end{bmatrix}$$

such that $U^T LU = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$.

Theorem 1: For a given $\gamma > 0$, system (5) is asymptotically stable and $\|T_{\omega z}\|_\infty < \gamma$, if and only if the following N systems are simultaneously asymptotically stable and the H_∞ norms of their transfer function matrices are all less than γ:

$$\begin{cases} \tilde{\boldsymbol{x}}_i(k+1) = (A + \Delta A - \lambda_i(B + \Delta B)K)\tilde{\boldsymbol{x}}_i(k) \\ \qquad + B_\omega \tilde{\boldsymbol{\omega}}_{i,x}(k) \\ \tilde{\boldsymbol{z}}_i(k) = C\tilde{\boldsymbol{x}}_i(k), \quad i = 1, 2, \ldots, N. \end{cases} \tag{6}$$

Proof: Let λ_i $(i = 1, 2, \ldots, N)$ be eigenvalues of the Laplacian matrix $L \in \mathbb{R}^N$ for an undirected topology G, where $\lambda_1 = 0$ with the associated eigenvector $\bar{u}_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. There exists an orthogonal matrix

$$U = \begin{bmatrix} \dfrac{1}{\sqrt{N}} & \dfrac{\mathbf{1}_{N-1}^T}{\sqrt{N}} \\ \dfrac{\mathbf{1}_{N-1}^T}{\sqrt{N}} & \bar{U} \end{bmatrix}$$

such that $U^T LU = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\} = \Lambda$. Let

$$\begin{aligned} \tilde{\boldsymbol{x}}(k) &= \left(U^T \otimes I_d\right)\boldsymbol{x}(k) = \left[\tilde{\boldsymbol{x}}_c^T(k), \tilde{\boldsymbol{x}}_r^T(k)\right]^T \\ &= \left[\tilde{\boldsymbol{x}}_1^T(k), \tilde{\boldsymbol{x}}_2^T(k), \ldots, \tilde{\boldsymbol{x}}_N^T(k)\right]^T. \end{aligned} \tag{7}$$

Then, system (5) can be rewritten in terms of $\tilde{\boldsymbol{x}}(k)$ as

$$\begin{cases}\tilde{\boldsymbol{x}}(k+1)=(I_N\otimes(A+\Delta A)\\ \quad -\Lambda\otimes(B+\Delta B)K)\tilde{\boldsymbol{x}}(k)+(U^T\otimes B_\omega)\boldsymbol{\omega}_x(k)\\ \boldsymbol{z}(k)=(U\otimes C)\tilde{\boldsymbol{x}}(k).\end{cases}\tag{8}$$

Moreover, reformulate the disturbance variable $\boldsymbol{\omega}_x(k)$ and the performance variable $\boldsymbol{z}(k)$ via

$$\begin{aligned}\tilde{\boldsymbol{\omega}}_x(k)&=\left(U^T\otimes I_{m_2}\right)\boldsymbol{\omega}_x(k)\\ &=\left[\tilde{\omega}_{1,x}^T(k),\tilde{\omega}_{2,x}^T(k),\dots,\tilde{\omega}_{N,x}^T(k)\right]^T\end{aligned}\tag{9}$$

$$\tilde{\boldsymbol{z}}(k)=\left(U^T\otimes I_l\right)\boldsymbol{z}(k)=\left[\tilde{z}_1^T(k),\tilde{z}_2^T(k),\dots,\tilde{z}_N^T(k)\right]^T.\tag{10}$$

Subsequently, substituting (9) and (10) into (8) gives

$$\begin{cases}\tilde{\boldsymbol{x}}(k+1)=(I_N\otimes(A+\Delta A)\\ \quad -\Lambda\otimes(B+\Delta B)K)\tilde{x}(k)+(I_N\otimes B_{\boldsymbol{\omega}})\tilde{\boldsymbol{\omega}}_x(k)\\ \tilde{\boldsymbol{z}}(k)=(I_N\otimes C)\tilde{\boldsymbol{x}}(k).\end{cases}\tag{11}$$

Note that (11) is composed of N individual systems of (6). Denote by $\|T_{\tilde{\omega}\tilde{z}}\|_\infty$ and $\|T_{\tilde{\omega}_i\tilde{z}_i}\|_\infty$ the transfer function matrices of systems (11) and (5), respectively. Then, it follows from (5), (9), (10) and (11) that

$$\begin{aligned}T_{\tilde{\omega}\tilde{z}}&=\mathrm{diag}(T_{\tilde{\omega}_1\tilde{z}_1},T_{\tilde{\omega}_2\tilde{z}_2},\dots,T_{\tilde{\omega}_N\tilde{z}_N})\\ &=\left(U^T\otimes I_l\right)T_{\omega z}(U\otimes I_{m_2})\end{aligned}\tag{12}$$

which implies that

$$\|T_{\tilde{\omega}\tilde{z}}\|_\infty=\max_{i=2,3,\dots,N}\|T_{\tilde{\omega}_i\tilde{z}_i}\|_\infty=\|T_{\omega z}\|_\infty.\tag{13}$$

■

In addition, it is worth mentioning that,

$$\begin{aligned}\tilde{\boldsymbol{x}}(k)&=\left[\tilde{\boldsymbol{x}}_c^T(k),\tilde{\boldsymbol{x}}_r^T(k)\right]^T=\left[\tilde{\boldsymbol{x}}_c^T(k),\tilde{\boldsymbol{x}}_{r,2}^T(k),\dots,\tilde{\boldsymbol{x}}_{r,N}^T(k)\right]^T\\ \tilde{\boldsymbol{\omega}}_x(k)&=\left[\tilde{\boldsymbol{\omega}}_{c,x}^T(k),\tilde{\boldsymbol{\omega}}_{r,x}^T(k)\right]^T\\ &=\left[\tilde{\boldsymbol{\omega}}_{c,x}^T(k),\tilde{\boldsymbol{\omega}}_{r,2x}^T(k),\dots,\tilde{\boldsymbol{\omega}}_{r,Nx}^T(k)\right]^T\\ \tilde{\boldsymbol{z}}(k)&=\left[\tilde{\boldsymbol{z}}_c^T(k),\tilde{\boldsymbol{z}}_r^T(k)\right]^T=\left[\tilde{\boldsymbol{z}}_c^T(k),\tilde{\boldsymbol{z}}_{r,2}^T(k),\dots,\tilde{\boldsymbol{z}}_{r,N}^T(k)\right]^T.\end{aligned}$$

By Lemma 1, the discrete-time system (11) also can be rewritten as the following N subsystems

$$\begin{cases}\tilde{\boldsymbol{x}}_c(k+1)=(A+\Delta A)\tilde{\boldsymbol{x}}_c(k)+B_\omega\tilde{\boldsymbol{\omega}}_{c,x}(k)\\ \tilde{\boldsymbol{z}}_c(k)=C\tilde{\boldsymbol{x}}_c(k)\end{cases}\tag{14}$$

$$\begin{cases}\tilde{\boldsymbol{x}}_{r,i}(k+1)=(A+\Delta A-\lambda_i(B+\Delta B)K)\tilde{\boldsymbol{x}}_{r,i}(k)\\ \quad +B_\omega\tilde{\boldsymbol{\omega}}_{i,x}(k)\\ \tilde{\boldsymbol{z}}_{r,i}(k)=C\tilde{\boldsymbol{x}}_{r,i}(k),\quad i=2,3,\dots,N.\end{cases}\tag{15}$$

Obviously, if subsystems (15) are asymptotically stable, then D-MASs (5) reach consensus. Subsystem (14) determines the final consensus value of D-MASs (5), and the details of it will be discussed below.

Remark 1: The robust H_∞ leaderless consensus problem of uncertain D-MASs (1) is to design distributed consensus protocols $\boldsymbol{u}_i(k)$, $\forall i\in N_i$ such that the consensus is reached and $\|T_{\omega z}\|_\infty<\gamma$, simultaneously. Theorem 1 converts the robust H_∞ consensus control problem of D-MASs (5) into the robust H_∞ control problems of N subsystems (6), which is a set of independent systems having the same dimensions as a single agent in (1), thereby reducing the computational complexity significantly. The key tools leading to this result rely on the state space decomposition approach, as used in [15].

3 Main Results

Lemma 2: Given the pair $(K,\gamma>0)$, if the matrix inequality

$$\bar{A}_{\lambda_i}^TP\bar{A}_{\lambda_i}-P+PC^TCP+\gamma^{-2}B_\omega B_{\omega^T}<0\qquad i=1,2,\dots,N\tag{16}$$

admits a symmetric positive definite solution $P\in\mathbb{R}^{d\times d}$, where $\bar{A}_{\lambda_i}=A+\Delta A-\lambda_i(B+\Delta B)K=A_{\lambda_i}+DFE_{\lambda_i}$, $A_{\lambda_i}=A-\lambda_iBK$, $E_{\lambda_i}=E_1-\lambda_iE_2K$. Then, D-MASs (1) are said to achieve robust consensus with a H_∞ performance index γ.

Proof: Given $K,\gamma>0$, assume that $P=P^T>0$ satisfies the matrix inequality (16). In this case (dropping the quadratic semidefinite positive term in P) it follows

$$\bar{A}_{\lambda_i}^TP\bar{A}_{\lambda_i}-P\leq-\gamma^{-2}B_\omega B_{\boldsymbol{\omega}}^T<0,\quad i=1,2,\dots,N\tag{17}$$

(complying with the previous assumptions) that $\bar{A}_{\lambda_i}$ is asymptotically stable. To prove the H_∞-norm. Inequality, we proceed as follows. For each system (6), consider the closed-loop transfer function from $\tilde{\omega}_{i,x}(k)$ to $\tilde{\boldsymbol{z}}_i(k)$ given by

$$H_{\lambda_i}(s)=C(sI-\bar{A}_{\lambda_i})^{-1}(B+\Delta B),\quad i=1,2,\dots,N.\tag{18}$$

Defining $s=e^{j\omega}$, $\omega\in[-\pi,\pi]$ and the auxiliary transfer function $\bar{L}_{\lambda_i}(s)=sC(sI-\bar{A}_{\lambda_i})^{-1}PC$ after simple but tedious algebraic manipulations, inequality (17) can be factorized as

$$\begin{aligned}&CPC^T-\bar{L}_{\lambda_i}(s)-\bar{L}_{\lambda_i}(s^{-1})^T+\bar{L}_{\lambda_i}(s)\bar{L}_{\lambda_i}(s^{-1})^T\\ &\quad+\bar{A}_{\lambda_i}^TP\bar{A}_{\lambda_i}-P<-\gamma^{-2}B_\omega B_{\boldsymbol{\omega}}^T\leq0.\end{aligned}\tag{19}$$

which, after completing squares, becomes

$$\begin{aligned}&H_{\lambda_i}(s)H_{\lambda_i}(s^{-1})^T\leq\gamma^2I-\gamma^2CPC^T\\ &\quad-\gamma^2[I-\bar{L}_{\lambda_i}(s)][I-\bar{L}_{\lambda_i}(s^{-1})]^T<\gamma^2I\end{aligned}\tag{20}$$

meaning that $\|H_{\lambda_i}\|_\infty<\gamma$, which proves the lemma proposed. ■

Remark 2: In Lemma 2, a sufficient condition is given to guarantee D-MASs (5) achieving robust consensus with a H_∞ performance index γ. Nevertheless, it is not difficult to find that (16) is a nonlinear matrix inequality (NMI) and therein lies parameter uncertainties.

To cope with the uncertain matrices F and the nonlinear terms of (16), the following two lemmas are given.

Lemma 4 [30]: Given matrices Y, D and E of appropriate dimensions and with Y symmetric, then

$$Y+DFE+E^TF^TD^T<0\tag{21}$$

for all F satisfying $F^TF \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon DD^T + \varepsilon^{-1}E^TE < 0. \tag{22}$$

Lemma 5 (Schur complement) [31]: The linear matrix inequality

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$, depends affinely on x, is equivalent to one of the following conditions

1) $Q(x) > 0$, $R(x) - S(x)^TQ(x)^{-1}S(x) > 0$;

2) $R(x) > 0$, $Q(x) - S(x)R(x)^{-1}S(x)^T > 0$.

Theorem 2: Consider D-MASs (1) with a fixed, undirected and connected communication topology G. The distributed consensus protocol (4) globally asymptotically solves the robust consensus problem of D-MASs (1) with H_∞-norm consensus performance bound γ if there exist a scalar $\varepsilon > 0$, a matrix W with appropriate dimensions and a positive definite matrix X such that

$$\begin{bmatrix} -X + \varepsilon DD^T & AX - \lambda_i BW & 0 & 0 \\ (AX - \lambda_i BW)^T & -X + C^TC & (E_1X - \lambda_i E_2W)^T & XB_\omega \\ 0 & E_1X - \lambda_i E_2W & -\varepsilon I_d & 0 \\ 0 & B_\omega^T X & 0 & -\gamma^2 I_{m_2} \end{bmatrix} < 0 \tag{23}$$

where $i = 1, 2, \ldots, N$. Furthermore, if LMI (23) has a feasible solution ε, W, X, then the feedback gain matrix K of protocol (4) can be calculated by $K = WX^{-1}$.

Proof: By Lemma 5, matrix inequality (16) is equivalent to

$$\begin{bmatrix} -P^{-1} & \bar{A}_{\lambda_i} \\ \bar{A}_{\lambda_i}^T & -P + PC^TCP + \gamma^{-2}B_\omega B_\omega^T \end{bmatrix} < 0. \tag{24}$$

Moreover, the above inequality can be rewritten as

$$\begin{bmatrix} -P^{-1} & A_{\lambda_i} \\ A_{\lambda_i}^T & -P + PC^TCP + \gamma^{-2}B_\omega B_\omega^T \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & E_{\lambda_i} \end{bmatrix} + \begin{bmatrix} 0 & E_{\lambda_i} \end{bmatrix}^T F \begin{bmatrix} D \\ 0 \end{bmatrix}^T < 0. \tag{25}$$

It follows from the Lemma 4 that (25) can be expressed as

$$\begin{bmatrix} -P^{-1} + \varepsilon DD^T & A_{\lambda_i} \\ A_{\lambda_i}^T & -P + PC^TCP\gamma^{-2}B_\omega B_\omega^T + \varepsilon^{-1}E_{\lambda_i}^TE_{\lambda_i} \end{bmatrix} < 0. \tag{26}$$

Through Lemma 5 again, matrix inequality (26) is equivalent to

$$\begin{bmatrix} -P^{-1} + \varepsilon DD^T & A_{\lambda_i} & 0 \\ A_{\lambda_i}^T & -P + PC^TCP + \gamma^{-2}B_\omega B_\omega^T & E_{\lambda_i}^T \\ 0 & E_{\lambda_i} & -\varepsilon I \end{bmatrix} < 0. \tag{27}$$

Pre- and post-multiplying both sides of (27) by

$$\begin{bmatrix} I_d & 0 & 0 \\ 0 & P^{-1} & 0 \\ 0 & 0 & I_d \end{bmatrix}$$

letting $X = P^{-1}$, $W = KP^{-1}$, and applying Lemma 5 again yield LMI (23), where $i = 1, 2, \ldots, N$. ■

Remark 3: In Theorem 2, it can be noted that the NMI (16) is transformed to a LMI condition (23). Subsequently, high-order D-MASs (1) with the distributed consensus protocol (4) achieve robust consensus with a H_∞ performance index γ. Thereby the neighboring feedback matrix K also can be obtained. Then, the local consensus protocol (4) can be implemented by each agent in a fully distributed fashion requiring no global information of the communication topology.

Remark 4: From Theorem 2, we can also get that, the communication disturbances have effects on the performance of the control object, such as switching interaction topologies. In [32], [33], the time-varying formation tracking problems for second-order MASs with switching interaction topologies were studied. Switching topologies include two cases. One is that every interaction topology of MASs has a spanning tree; another is joint-contained spanning tree topologies. It should be mentioned that, this approach can be easily extended to the first case, and more details can be seen in our work [34]. There have been some difficulties to the joint-contained spanning tree case. We will consider it in the future.

Theorem 3: With $\boldsymbol{\omega}_x(k)$ interpreted as deterministic l_2 signal, when D-MASs (5) achieve robust consensus, the final consensus value $c(k)$ satisfies

$$\lim_{k\to\infty}\left(\boldsymbol{c}(k) - \mathbf{1}_N \otimes \left(\left(\frac{1}{N}(A+\Delta A)^k \sum_{i=1}^{N}\boldsymbol{x}_i(0)\right) + \sum_{i=0}^{k-1}\sum_{l=0}^{l=k-i-1}\sum_{j=1}^{j=N}\left(\frac{1}{N}(A+\Delta A)^i B_\omega \boldsymbol{\omega}_{x,j}(l)\right)\right)\right) = 0. \tag{28}$$

Proof: Let $\boldsymbol{x}_C(k) = (U \otimes I_d)[\tilde{\boldsymbol{x}}_c^T(k), 0]^T$ and $\boldsymbol{x}_{\bar{C}}(k) = (U \otimes I_d)[0, \tilde{\boldsymbol{x}}_r^T(k)]^T$, then by (7), $\boldsymbol{x}(k)$ can be uniquely decomposed as $\boldsymbol{x}(k) = \boldsymbol{x}_C(k) + \mathbf{x}_{\bar{C}}(k)$. As discussed above, we can know that if the system (5) achieves robust guaranteed cost consensus, the subsystem (15) should be Schur stable, which means that the response of system (15) due to $\boldsymbol{x}_{\bar{C}}(0)$ should satisfy $\lim_{k\to\infty}\boldsymbol{x}_{\bar{C}}(k) = 0$. Hence the final consensus value $c(k)$ is determined solely upon $\boldsymbol{x}_C(k)$. Since $[\tilde{\boldsymbol{x}}_c^T(k), 0]^T = \boldsymbol{e}_1 \otimes \tilde{\boldsymbol{x}}(k)$, we have $\boldsymbol{x}_C(0) = \bar{\boldsymbol{u}}_1 \otimes \tilde{\boldsymbol{x}}_c(0) = \bar{\boldsymbol{x}}_1 \otimes ((\boldsymbol{e}_1^T \otimes I_d)\tilde{\boldsymbol{x}}(0))$, and because $\tilde{\boldsymbol{x}}(0) = (U^T \otimes I_d)\boldsymbol{x}(0)$, then we can obtain $\boldsymbol{x}_C(0) = \bar{\boldsymbol{u}}_1 \otimes \tilde{\boldsymbol{x}}_c(0) = \bar{\boldsymbol{u}}_1 \otimes ((\boldsymbol{e}_1^T \otimes I_d)\tilde{\boldsymbol{x}}(0))$, that is to say

$$\begin{aligned} \boldsymbol{x}_C(0) &= \bar{\boldsymbol{u}}_1 \otimes \left((\boldsymbol{e}_1^T \otimes I_d) * (U^T \otimes I_d)\boldsymbol{x}(0)\right) \\ &= \bar{\boldsymbol{u}}_1 \otimes \left((\boldsymbol{e}_1^T U^T \otimes I_d)\boldsymbol{x}(0)\right) \\ &= \bar{\boldsymbol{u}}_1 \otimes \left((\frac{1}{\sqrt{N}}\mathbf{1}_N^T \otimes I_d)\boldsymbol{x}(0)\right) \\ &= \mathbf{1}_N^T \otimes \left(\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i(0)\right). \end{aligned}$$

Likewise, let $\boldsymbol{\omega}_{C,x}(k) = (U \otimes I_d)[\tilde{\boldsymbol{\omega}}_{c,\boldsymbol{x}}^T(k), 0]^T$ and $\tilde{\boldsymbol{\omega}}_{\bar{C},x}(k) = (U \otimes I_d)[0, \tilde{\boldsymbol{\omega}}_{r,x}^T(k)]^T$, then by (9), $\boldsymbol{x}(k)$ can be uniquely decomposed as $\boldsymbol{\omega}_x(k) = \boldsymbol{\omega}_{C,x}(k) + \boldsymbol{\omega}_{\bar{C},x}(k)$. If the system (5) achieves robust guaranteed cost consensus, the response of system (15) due to $\boldsymbol{\omega}_{\bar{C},x}(0)$ also should satisfy $\lim_{k\to\infty}\boldsymbol{\omega}_{\bar{C},x}(k) = 0$. Since $[\tilde{\boldsymbol{\omega}}^T_{c,x}(k), 0]^T = e_1 \otimes \tilde{\boldsymbol{\omega}}_{\boldsymbol{x}}(k)$, we have $\tilde{\boldsymbol{\omega}}_{C,\boldsymbol{x}}(k) = \bar{u}_1 \otimes \tilde{\boldsymbol{\omega}}_{c,x}(k) = \bar{u}_1 \otimes ((e_1^T \otimes I_d)\tilde{\boldsymbol{\omega}}_x(k))$, and because $\tilde{\boldsymbol{\omega}}_x(k) = (U^T \otimes I_d)\boldsymbol{\omega}_x(k)$, then we can obtain $\boldsymbol{\omega}_{C,x}(k) = \bar{u}_1 \otimes \tilde{\boldsymbol{\omega}}_{c,x}(k) = \bar{u}_1 \otimes ((e_1^T \otimes I_d)\tilde{\boldsymbol{\omega}}_x(k))$, i.e.,

$$\begin{aligned}\boldsymbol{\omega}_{C,\boldsymbol{x}}(k) &= \bar{\boldsymbol{u}}_1 \otimes \Big((\boldsymbol{e}_1^T \otimes I_d) * (U^T \otimes I_d)\boldsymbol{\omega}_x(k)\Big)\\ &= \bar{\boldsymbol{u}}_1 \otimes \Big((\boldsymbol{e}_1^T U^T \otimes I_d)\boldsymbol{\omega}_x(k)\Big)\\ &= \bar{\boldsymbol{u}}_1 \otimes \Big((\frac{1}{\sqrt{N}}\boldsymbol{1}_N^T \otimes I_d)\boldsymbol{\omega}_x(k)\Big)\\ &= \boldsymbol{1}_N^T \otimes \left(\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\omega}_{x,i}(k)\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}\boldsymbol{x}_C(k) &= (A+\Delta A)^k \boldsymbol{x}_C(0)\\ &\quad + \boldsymbol{1}_N \otimes \sum_{i=0}^{k-1}\Big((A+\Delta A)^i B_\omega \tilde{\boldsymbol{\omega}}_{c,x}(k-i-1)\Big)\\ &= \boldsymbol{1}_N \otimes \Bigg(\Bigg(\frac{1}{N}(A+\Delta A)^k \sum_{i=1}^{N}\boldsymbol{x}_i(0)\Bigg)\\ &\quad + \sum_{i=0}^{k-1}\sum_{l=0}^{l=k-i-1}\Big((A+\Delta A)^i B_\omega \boldsymbol{\omega}_{C,x}(l)\Big)\Bigg)\\ &= \boldsymbol{1}_N \otimes \Bigg(\Bigg(\frac{1}{N}(A+\Delta A)^k \sum_{i=1}^{N}\boldsymbol{x}_i(0)\Bigg)\\ &\quad + \sum_{i=0}^{k-1}\sum_{l=0}^{l=k-i-1}\sum_{j=1}^{j=N}\left(\frac{1}{N}(A+\Delta A)^i B_\omega \boldsymbol{\omega}_{x,j}(l)\right)\Bigg)\end{aligned}$$

then the final consensus value $\boldsymbol{c}(k)$ satisfies $\lim_{k\to\infty}(\boldsymbol{c}(k) - \boldsymbol{x}_C(k)) = 0$, $k = 0, 1, 2, \ldots$. ■

Corollary 1: With $\boldsymbol{\omega}_x(k) \equiv 0$, when multi-agent system (5) achieves robust consensus, the final consensus value $\boldsymbol{c}(k)$ satisfies

$$\lim_{k\to\infty}\left(\boldsymbol{c}(k) - \boldsymbol{1}_N \otimes \left(\frac{1}{N}(A+\Delta A)^k \sum_{i=1}^{N}\boldsymbol{x}_i(0)\right)\right) = 0. \tag{29}$$

Proof: This proof can be easily obtained from the proof of Theorem 3. ■

Remark 5: With $\boldsymbol{\omega}_x(k)$ interpreted as deterministic l_2 signal, the final consensus value $\boldsymbol{c}(k)$ of system (5) is given by Theorem 3. The final consensus value $\boldsymbol{c}(k)$ can be divided into two parts, one is $((A+\Delta A)^k/N)\sum_{i=1}^{N}\boldsymbol{x}_i(0)$, which is related to the system matrix $A+\Delta A$ and initial state $x(0)$, the other is $\sum_{i=0}^{k-1}\sum_{j=1}^{j=N}\sum_{l=0}^{l=k-i-1}(1/N(A + \Delta A)^i B_\omega \boldsymbol{\omega}_{x,j}(l))$, which is related to the external disturbance $\boldsymbol{\omega}_x(k)$. This implies that the external disturbance $\boldsymbol{\omega}_x(k)$ has an effect on the final consensus value, and which is also related to the system matrix $A+\Delta A$, and initial state $\boldsymbol{x}(0)$. This condition is different from that of high-order D-MASs without parameter uncertainties and external disturbances, which is discussed in [34]. With $\boldsymbol{\omega}_x(k) \equiv 0$, the final consensus value $\boldsymbol{c}(k)$ of system (5) is given by Corollary 1. That is, in this case, the final consensus value is only related to the system matrix $A+\Delta A$, and initial state $\boldsymbol{x}(0)$.

Remark 6: It should be pointed out that, in [26]−[28], by recursive linear matrix inequalities (RLMIs) techniques, the robust H_∞ consensus control problem of high-order D-MASs (1) with uncertainties/disturbances was investigated over a finite horizon. They were concerned about the boundedness of the consensus error but did not actually guarantee its convergence. Different from [26]−[28], we consider the infinite time horizon case, which took care of the consensusability of D-MASs rather than consensus errors. In Theorems 1 and 2, a sufficient LMI condition is given to guarantee that high-order D-MASs (1) with parameter uncertainties and external disturbances achieve robust consensus with a performance level γ. Comparing to related works [25]−[28], this approach has a favorable decoupling feature. Specifically, note that the H_∞ performance level $\gamma_{\min}$ of network (6), consisting of N agents in D-MASs (1) under consensus protocols (4), is actually equal to the minimal H_∞ norm of a single agent (1) by means of a state feedback controller of the form $\boldsymbol{u}_i = K\boldsymbol{x}_i$, independent of the communication topology G as long as it is connected. In addition, final consensus values of high-order D-MASs (1) with parameter uncertainties and external disturbances are first given in this paper. In addition, practical consensus problems for general high-order linear time-invariant swarm systems with interaction uncertainties and time-varying external disturbances on directed graphs were investigated in [35]. The authors paid attention to the output consensus of continuous-time high-order linear time-invariant swarm systems. However, the state consensus problem of discrete-time multi-agent systems is addressed in this paper. Moreover, the external disturbance was solved by the Lyapunov-Krasovskii functional approach and the linear matrix inequality technique in the literature, but we use the H_∞ control method to deal with it.

4 Simulations

In this section, a numerical example is given to illustrate the effectiveness of the proposed theoretical results. We apply the above proposed consensus protocol (4) to achieve state alignment among 8 agents. The dynamics of them are described by (1), where

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{bmatrix},\quad B = \begin{bmatrix} 0.2\\ -0.4\\ 1\end{bmatrix}$$

$$D = \begin{bmatrix} 0.1 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.3\end{bmatrix},\quad E_1 = \begin{bmatrix} 0.1 & 0.3 & 0\\ 0.2 & 0.4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0.2\\ 0.1\\ 0.3\end{bmatrix},\quad F = \begin{bmatrix} r_1 & 0 & 0\\ 0 & r_2 & 0\\ 0 & 0 & r_3\end{bmatrix}$$

and r_1, r_2 and r_3 are uncertain parameters which satisfy $-1 \le r_1 \le 1$, $-1 \le r_2 \le 1$ and $-1 \le r_3 \le 1$. Then, D-MASs (1) can be rewritten as

$$\boldsymbol{x}_i(k+1) = (A + DFE_1)\boldsymbol{x}_i(k) + (B + DFE_2)\boldsymbol{u}_i(k). \tag{30}$$

We apply the consensus protocol (4) to achieve consensus among the above those 8 agents under a fixed topology G, which is shown in Fig. 1.

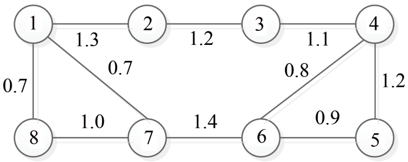


Fig. 1. The interaction topology G of 8 agents.

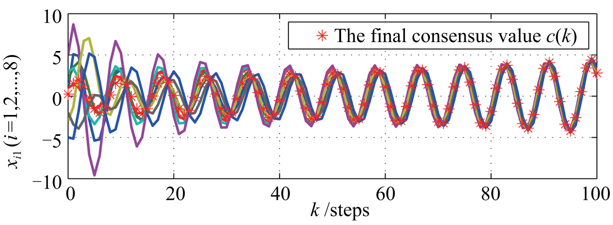
Assume that the initial state values of the all agents $1,\dots,8$ are randomly produced with $x_1(0) = [1, 5, -2]^T$, $x_2(0) = [2, 4, 3]^T$, $x_3(0) = [1, 1, 2]^T$, $x_4(0) = [3, 2, 1]^T$, $x_5(0) = [5, 6, -2]^T$, $x_6(0) = [-3, 3, 4]^T$, $x_7(0) = [-2, -4, -3]^T$, $x_8(0) = [-5, -2, -1]^T$, and let $r_1 = 0.15$, $r_2 = 0.25$, $r_3 = 0.15$ and $\tau_{\max} = 3$. Each agent uses protocol (4). Let $\gamma = 1$ and suppose that the exogenous disturbance inputs are selected as $\omega_{i,x}(k) = 0.1ie^{-0.5k}\sin(k)$. By Theorem 2, we can get that

$$K = \begin{bmatrix} 0.0983 & -0.0884 & 0.2803 \end{bmatrix}$$

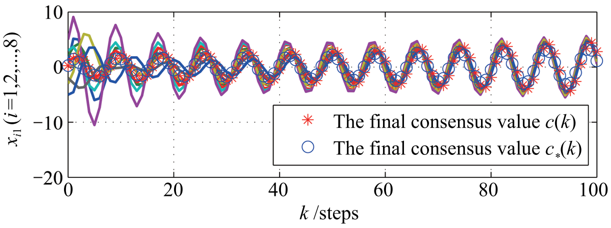
$$X = \begin{bmatrix} 0.8086 & 0.0443 & 0.2130 \\ 0.0443 & 0.6593 & -0.0279 \\ 0.2130 & -0.0279 & 1.1160 \end{bmatrix}, \quad \varepsilon = 4.2304$$

$$W = \begin{bmatrix} 0.1353 & -0.0617 & 0.3362 \end{bmatrix}$$

In Figs. 2−4, the simulation results are given. The state trajectories of uncertain D-MASs (1) with and without external disturbances are shown in Figs. 2−4 (b) and (a), respectively. Final consensus values $c(k)$ and $c_*(k)$, which are produced by Corollary 1 and Theorem 3, are marked by the red asterisk and blue circle, respectively.



(a) State trajectories of D-MAS (5) without external disturbances



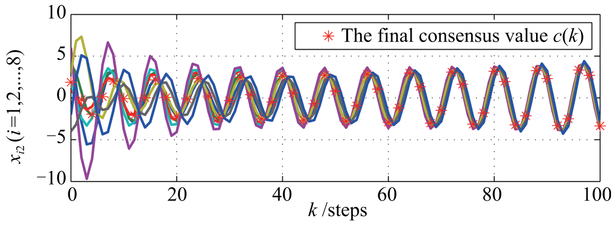
(b) State trajectories of D-MAS (5) with external disturbances

Fig. 2. The state 1 trajectories of D-MASs (1).

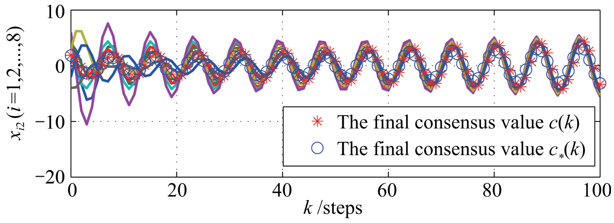
From Figs. 2−4 (a), it can be seen that the state trajectories of D-MASs (1) with $\boldsymbol{\omega}_x(k) \equiv 0$ asymptotically converge to the common value $\boldsymbol{c}(k)$, which is related to r_j $(j = 1, 2, 3)$. The final consensus value of D-MASs (1) with parameter uncertainties is $\mathbf{1}_N \otimes ((A + DFE_1)^k (1/N \times \sum_{i=1}^{N} \boldsymbol{x}_i(0)))$. This is in accord with Corollary 1. Nevertheless, in Figs. 2−4 (b), we can know that the common value of D-MASs (1) is related to $\omega_x(k)$, and if $\omega_x(k) \neq 0$, $c(k)$ is altered and asymptotically converges to

$$\mathbf{1}_N \otimes \left((A + DFE_1)^k \left(\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}_i(0) \right) + \sum_{i=0}^{k-1} \sum_{j=1}^{j=N} \sum_{l=0}^{l=k-i-1} \left(1 \; N(A + \Delta A)^i B_\omega \boldsymbol{\omega}_{\boldsymbol{x},j}(l) \right) \right)$$

which is in accordance with Theorem 3. By Definition 1, it is clear that D-MASs (1) achieves robust consensus with protocol (4). Therefore, the correctness and validity of proposed protocols and theorems are demonstrated.

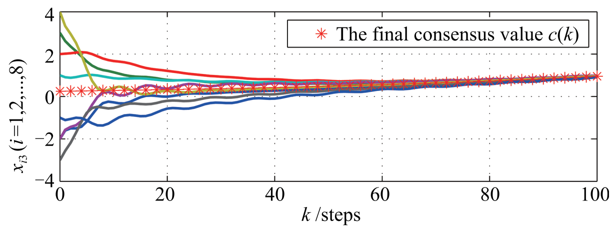


(a) State trajectories of D-MAS (5) without external disturbances

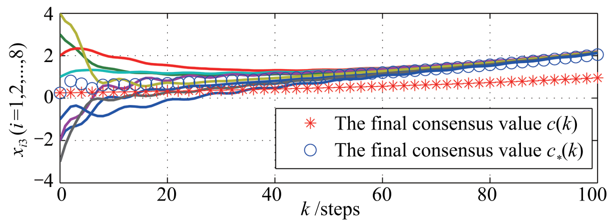


(b) State trajectories of D-MAS (5) with external disturbances

Fig. 3. The state 2 trajectories of D-MASs (1).



(a) State trajectories of D-MAS (5) without external disturbances



(b) State trajectories of D-MAS (5) with external disturbances

Fig. 4. The state 3 trajectories of D-MASs (1).

5 Conclusions

The robust H_∞ consensus control problem of high-order D-MASs with parameter uncertainties and external disturbances is investigated in this paper. A sufficient LMI condition is obtained to guarantee that D-MASs (1) achieve robust consensus with protocol (4). Meanwhile, the convergence result is given as a final consensus value. Finally, an illustrative example is given to demonstrate the correctness and effectiveness of the theoretical results. Further research will be conducted on the consensus problem of D-MASs with switching topologies and time-delays.

Appendix A Graph

Let a weighted digraph (or directed graph) $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order N represents an interaction topology of a network of agents, with the set of nodes $\mathcal{V} = \{v_1, \dots, v_N\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with nonnegative adjacency elements a_{ij}.

The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, \dots, N\}$. An edge of G is denoted by $e_{ij} = (v_i, v_j)$, where v_i and v_j are called the initial and terminal nodes. It implies that node v_j can receive information from node v_i, but not necessarily vice versa. The adjacency elements associated with the edges of the graph are positive if $e_{ij} \in \mathcal{E}$ while $a_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$. Furthermore, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. A cluster is any subset $J \subseteq \mathcal{V}$ of the nodes of the graph. The set of neighbors of a cluster N_J is defined by $N_J = \cup_{v_i \in J} N_i = \{v_j \in \mathcal{V} : v_i \in J, (v_i, v_j) \in \mathcal{E}\}$. The in-degree and out-degree of node v_i are defined as $\deg_{\rm in}(v_i) = \sum_{j=1}^n a_{ji}$ and $\deg_{\rm out}(v_i) = \sum_{j=1}^n a_{ij}$, respectively, The degree matrix of the digraph G is a diagonal matrix $\Delta = [\Delta_{ij}]$, where

$$\Delta_{ij} = \begin{cases} 0, & i \neq j \\ \deg_{\rm out}(v_i), & i = j. \end{cases}$$

The graph Laplacian matrix associated with the digraph G is defined as $\mathcal{L}(G) = L = \Delta - \mathcal{A}$.

Appendix B Kronecker Product

Given matrices $P = (p_{ij})_{n\times n} \in \mathbb{R}^{m\times n}$ and $Q = (q_{ij})_{n\times n} \in \mathbb{R}^{p\times q}$, their Kronecker product is defined as

$$P \otimes Q = [p_{ij}Q] \in \mathbb{R}^{mp\times nq}$$

in [36]. For matrices A, B, C and D, with appropriate dimensions, we have the following conditions.

1) $(\gamma A) \otimes B = A \otimes (\gamma B)$, where γ is a constant;

2) $(A + B) \otimes C = A \otimes C + B \otimes C$;

3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

4) $(A \otimes B)^T = A^T \otimes B^T$;

5) Suppose that A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

6) If A and B are symmetric, so is $(A \otimes B)$;

7) If A and B are symmetric positive definite (respectively, positive semidefinite), so is $(A \otimes B)$;

8) Suppose that A has the eigenvalues β_i with associated eigenvectors $f_i \in \mathbb{R}^p$, $i = 1, \dots, p$, and B has the eigenvalues ρ_i with associated eigenvectors $g_j \in \mathbb{R}^p$, $j = 1, \dots, q$. Then the pq eigenvalues of $(A \otimes B)$ are $\beta_i \rho_j$ with associated eigenvectors $f_i \otimes g_j$, $i = 1, \dots, p$, $j = 1, \dots, q$.

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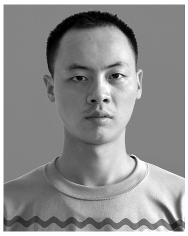

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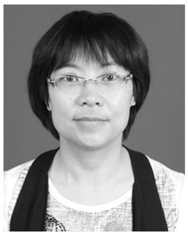

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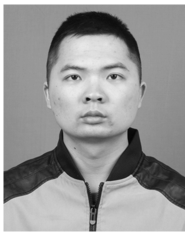

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