# Asynchronous $H_\infty$ State Dependent Switching Control of Discrete-time Systems With Dwell Time

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**Abstract** For a class of switched linear systems, we propose a dwell time strategy depending on the state of systems. This switching strategy not only makes the asynchronous $H_\infty$ state-feedback switched systems stable but also shortens the active time. A new result on stability and $l_2$gain analysis for switched systems is given where the Lyapunov functions are allowed to be increasing during the running time of subsystems, at the same time, the Lyapunov functions do not contain the limit of $\mu$. By using the dwell time strategy depending on the state of systems, sufficient conditions for the desired $H_\infty$ controller of switched linear systems are derived. Then the result is expanded to nonlinear switched systems. A numerical example is provided to demonstrate the effectiveness of the proposed design approach.

**Key words** Asynchronous $H_\infty$ state-feedback switched systems, discrete-time systems, dwell time strategy

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## 1 Introduction

Switched systems, which are efficiently used to model many physical or man-made systems displaying features of switching, have been extensively studied over the past decades. Typically, switched systems consist of a finite number of subsystems (described by differential or difference equations) and an associated switching signal governing the switching among them. The switching signals may belong to a certain set and the sets may be various. This differentiates switched systems from general systems, since the solutions of the former are dependent on both system initial conditions and switching signals. Many physical processes exhibit switched and hybrid behavior [1]−[3], and switching frequently occurs in many engineering applications. Due to the theoretical development as well as practical applications, analysis and synthesis of switched system have recently gained considerable attention [4]−[10].

Recently, the $H_\infty$ control problem of switched systems has stirred renewed research interests [11]−[19]. The goal is to design a controller to stabilize a system while satisfying an $H_\infty$-norm bound constraint on disturbance attenuation. Sufficient conditions for designing a robust $H_\infty$ controller with time-varying norm-bounded uncertainty are studied by means of hybrid state feedback strategy in [12]. By average dwell time methods [13], investigated the $H_\infty$ controller of switched system with uncertain inputs.

While, considering the $H_\infty$ state feedback problem, a very common assumption is that the controller is switched synchronously with the switching of system modes, which is quite impractical. In engineering application, since it inevitably takes some time to identify the active subsystem and apply the matched controller, the real switching time of controllers may lag behind that of practical subsystems, that is to say, there exists asynchronous switching between the controllers and system mode. The necessities of considering asynchronous switching for efficient control design have been shown for mechanical or chemical systems in [20]. Recently, the asynchronous switching problem has been investigated, and some results are obtained in the studies of switched system [21]−[26]. The stabilization of asynchronous linear system has been included in [21]. Stability, $l_2$-gain and asynchronous control of discrete-time switched systems are considered in [23]. Then, the results are condensed to filter in [26], which discusses the stability and $l_2$-gain of switched linear systems.

The stability analysis of switched systems with dwell time has received a considerable attention in the last decade [27]−[30]. The method there do not guarantee any minimal dwell time. The practical case, however, is that some minimal time period between consecutive switching is required. Arbitrarily fast switching may cause large state transients at the switching points. A dwell time may be required for these transients to subside. This is one of the reasons why the area of switched systems with dwell time is becoming increasingly popular. At the same time, it is found that the most stabilizing switching law for many switched systems with unstable subsystems obeys some dwell time. Adding a dwell time constraint to a suboptimal switching law may thus achieve better results.

In this paper, the robust asynchronous $H_\infty$ state dependent switching of linear system with dwell time is considered. About switching signals, some works have been done in [27]−[30]. Combined stabilizing strategies are proposed in [27], and the result is improved in [27]. State-dependent switching laws have been first considered in [27]. The method there do not guarantee any dwell time. It has been shown in [27] that the most destabilizing switching law of a switching law for a switched system with stable subsystems applies a dwell time. Using the same analysis, it can be seen that the most stabilizing switching law for asynchronous switched systems obeys some dwell time. Adding a dwell time constant to switching law may achieve better results. Thus in our paper, minimal dwell time is introduced, which not only meets the need of minimal time between consecutive switching, but also compensates the possible increment introduced by the asynchronous phenomenon between the system modes and controllers. On the other hand, the dwell time strategies depending on state not only reduce the time of being active, but also make that the Lyapunov function does not have the limit of $\mu$. All of these motivated us to study the $H_\infty$ controller of asynchronous switched system, which is unstable within the unmatched interval of $(k_l k_{l+1}), \forall l \in \mathbb{N}$.

The asynchronous $H_\infty$ control problem for a class of state dependent switching system with dwell time is investigated in this paper. Based on the dwell time approach depending on state, sufficient conditions are developed for

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the $H_\infty$ control system. It is noted that subsystems are allowed to be unstable within the unmatched interval of the interval $(k_l k_{l+1})$ , $\forall l \in \mathbb{N}$. Then, the corresponding solvability condition for the desired controller is established.

The remainder of this paper is organized as follows. The asynchronous $H_\infty$ control of switched systems is formulated in Section 2. Section 3 is devoted to derive the results on stability and $l_2$-gain analyses and formulate the problem of asynchronous $H_\infty$ control for discrete-time switched system. A numerical example is given in Section 4, and then, we make a conclusion about this article in Section 5.

*Notation:* The notations used throughout the paper are standard. $R^n$ denotes the $n$-dimensional Euclidean space; $N$represents the set of nonnegative integers; the notation $P > 0$ means that $P$ is real symmetric and positive definite; $l_2[0, \infty)$ is the space of square-integrable vector functions over $[0, \infty)$; $\|\cdot\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric matrices or long matrix expressions, we use a star $(*)$ to represent a term that is induced by symmetry.

## 2 Problem Description and Preliminaries

Consider a class of switched linear systems given by (1).

$$\begin{cases} x(k+1) = A_\sigma x(k) + B_\sigma u(k) + D_{1\sigma}\omega(k) \\ z(k) = C_\sigma x(k) + D_{2\sigma}\omega(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $u(k) \in \mathbb{R}^m$ is the control input; $\omega(k) \in \mathbb{R}^p$ is the disturbance input which belongs to $l_2[0, \infty)$ , $z(k) \in \mathbb{R}^q$ is the controlled output; $\sigma$ is a piece wise constant function of time $k$ called a switching signal, which takes its values in the finite set $\mathcal{I} = \{1, \dots, N\}$, and $N > 1$ is the number of subsystems.

In the paper, we design an $H_\infty$ state feedback controller with the following general structure.

$$u(k) = K_i x(k) \tag{2}$$

where $K_i \in \mathbb{R}^{m\times n}$ are matrices to be determined.

It is assumed that the subsystem is activated at the switching instant $k_l, \forall l \in \mathbb{N}$. Owing to the fact, real switching time of controllers exceeds or lags behind that of the practical subsystems, so the switching instant of the controller is $k_l + \Delta_l, \forall l \in \mathbb{N}$, where $\Delta_l > 0$ represents the unmatched time during which the switched system maybe unstable.

Therefore, by substituting $u(k)$ into system (1), we obtain the closed-loop system as

$$\begin{cases} x(k+1) = (A_i + B_i K_j)x(k) + D_{1i}\omega(k) \\ z(k) = C_i x(k) + D_{2i}\omega(k) \quad \forall k \in (k_l, k_l + \Delta_l) \\ x(k+1) = (A_i + B_i K_i)x(k) + D_{1i}\omega(k) \\ z(k) = C_i x(k) + D_{2i}\omega(k) \quad \forall k \in (k_l + \Delta_l, k_{l+1}). \end{cases} \tag{3}$$

Choose the Lyapunov functional candidate of the form

$$V_i(t) = x^T(k) P_i x(k). \tag{4}$$

The subsequent switching time/index sequences are defined as following Choose the Lyapunov functional candidate of the form

$$k_{l+1} = \begin{cases} k_l + \Delta_l + \Delta, & \text{if} \quad \mu_l < 1 \\ k_l + \Delta_l + \Delta + T_l, & \text{if} \quad \mu_l > 1 \end{cases} \tag{5}$$

where $\Delta = \dfrac{-\Delta_{\max}\ln(1+\beta)}{\ln(1-\alpha)}$

$$T_l = -\frac{\ln \mu_l}{\ln(1-\alpha)}$$

$$\Delta_{\max} \triangleq \max_{\forall l \in} \Delta_l$$

$$\mu_l = \frac{V_{\sigma(k_l^+)}(k_l)}{V_{\sigma(k_l^-)}(k_l)} \quad \forall l \in \mathbb{N}.$$

According to this strategy, when the $l$th subsystem is activated, it should be active for the minimal dwell time $\Delta$ to compensate the possible increment which is introduced by mismatching of controllers and subsystems. Besides, if $\mu_l > 1$ the subsystems will be active for another $T_l = \ln \mu_l / \ln \alpha$ to compensate the possible increment in switching instants, else the system will switch to the $j$th subsystem immediately.

*Remark 1:* In this note, we propose a dwell time strategy depending on the state of subsystems. On one hand, when $\mu_l > 1$ the increment will be compensated by more specific decrement, in other words, the subsystem will be active for another $T_l$. When $\mu_l < 1$ the system will switch to another subsystem immediately. So we can get the conclusion that the active time of subsystem specified by dwell time strategy depending on the state of system is shorter than that of the system specified by normal dwell time strategy; on the other hand, it is easy to see that the Lyapunov function does not have the limit of $P_i < \mu P_j, \forall i, j \in \mathcal{I}$.

We give the following definition, which will play an important role in deriving our main results subsequently.

*Definition 1:* Given a constant $\gamma > 0$, the switched system (3) is said to be stabilized with $H_\infty$ disturbance attenuation $\gamma$ via switching if there exists a switching rule such that under this switching, it satisfies

1) System (3) with $\omega = 0$ is stable.

2) With zero-initial condition $x(0) = 0$, $\| z \|_2 < \gamma \| \omega \|_2$ holds for all nonzero $\omega \in l_2[\ 0 \quad \infty\ )$.

## 3 Main Results

### 3.1 Stability and Performance Analysis

In this section, sufficient conditions on stability with an $l_2$-gain are derived for system (3) via the dwell time strategies depending on the state of system.

*Theorem 1:* Given scalars $0 < \alpha < 1$, $\beta > 0$ and the switching time instants $k_0 < k_1 < \cdots < k_{l-1} < k_l$, during $[k_0, k_l]$, $l = 1, 2, \dots$, the closed-loop $H_\infty$ control system is stable under the switching signal as described in the formula (5), if $V_i(k)$ satisfies

$$\Delta V_i(k) \le \begin{cases} \beta V_i(k), & \forall t \in (k_l, k_l + \Delta_l) \\ -\alpha V_i(k), & \forall t \in (k_l + \Delta_l, k_{l+1}). \end{cases} \tag{6}$$

*Proof:* Denote $\Delta V_i(k) = V_i(k+1) - V_i(k)$. If $\mu_{l-1} > 1$, in other words the $l$th subsystem activates for $\Delta_l + \Delta + T_l$.

$$\begin{aligned} V_{\sigma(k_l)}(k_{l+1}) &\le (1-\alpha)^{\Delta + T_l} V_{\sigma(k_l)}(k_l + \Delta_l) \\ &\le (1+\beta)^{\Delta_l}(1-\alpha)^{\Delta+T_l} V_{\sigma(k_l)}(k_l) \\ &\le (1-\alpha)^{T_l}\mu_{l-1} V_{\sigma(k_{l-1})}(k_l) \\ &\le V_{\sigma(k_{l-1})}(k_l). \end{aligned} \tag{7}$$

If $\mu < 1$ , in other words, the $l$th subsystem activates for $\Delta_l + \Delta$.

$$\begin{aligned} V_{\sigma(k_l)}(k_{l+1}) &\le (1-\alpha)^{\Delta} V_{\sigma(k_l)}(k_l + \Delta_l) \\ &\le (1+\beta)^{\Delta_l}(1-\alpha)^{\Delta} V_{\sigma(k_l)}(k_l) \\ &\le V_{\sigma(k_l)}(k_l) \\ &\le V_{\sigma(k_{l-1})}(k_l). \end{aligned} \tag{8}$$

The switching signals stem from the requirement that the value of the Lyapunov function $V_{\sigma(k)}(k_{l+1})$ is less than $V_{\sigma(k_{l-1})}$. An alternative way to guarantee the decrease of $V$ is to require that the value of $V$, for another $T$ seconds after the switching, is less than the value it had just prior to the switching. The latter requirement is satisfied if the switching signals as described in (5) hold.

Therefore, we get the conclusion that the closed-loop system is stable.

*Remark 2:* Note that the switched systems are active in the intervals (consisting of matched and unmatched intervals) during which a subsystem may be unstable. In other words, the Lyapunov function gets increased in the unmatched intervals. However, the possible increment will be compensated by the more pronounced decrement (by limiting the lower bound of dwell time). Thus, we can get that the closed-loop system is stable.

Next, the $H_\infty$ performance of the system (3) is given in the following theorem.

*Theorem 2:* Given scalars $0 < \alpha < 1$, $\beta > 0$ and the switching time instants $k_0 < k_1 < \cdots < k_{l-1} < k_l$, during $[k_0, k_l]$, $l = 1, 2, \ldots$, the closed-loop $H_\infty$ control system is stable and has an $l_2$-gain no greater than $\gamma_s$, if $V_i(k)$ satisfies

$$\Delta V_i(k) \le \begin{cases} \beta V_i(k) - \Gamma(k) & \forall t \in (k_l, k_l + \Delta_l) \\ -\alpha V_i(k) - \Gamma(k) & \forall t \in (k_l + \Delta_l, k_{l+1}) \end{cases} \tag{9}$$

where

$$\Gamma(k) = z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k)$$
$$\gamma_s = \tilde{\alpha}^{-(\Delta + T_{\max}/2)}\gamma$$
$$\mu_{\max} = \max_{i,j\in\mathcal{I}, i\neq j} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}.$$

In order to study the asynchronous $H_\infty$ controller for system (1), for conciseness, the time $t_1, t_2, \ldots, t_k, \ldots$ is set to switching instant. Without loss of generality, we assume that $\mu_{l-2} > 1, \mu_{l-1} < 1, \ldots$ (for other situations we can get the same result through the same proof process), in other words $k_l = k_{l-1} + \Delta_{l-1} + T_{l-1}, k_{l+1} = k_l + \Delta_l, \ldots$.

$$\begin{aligned}
&V_{\sigma(k_l)}(k_{l+1}) \\
&\le (1-\alpha)^{\Delta} V_{\sigma(k_l)}(k_l + \Delta_l) \\
&\quad - \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s) \\
&\le (1-\alpha)^{\Delta}(1+\beta)^{\Delta_l} V_{\sigma(k_l)}(k_l) \\
&\quad - \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s) \\
&\le V_{\sigma(k_{l-1})}(k_l) \\
&\quad - \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s) \\
&\le V_{\sigma(k_{l-2})}(k_{l-1}) \\
&\quad - \sum_{s=k_{l-1}}^{k_{l-1}+\Delta_{l-1}-1} (1-\alpha)^{\Delta+T_{l-1}}(1+\beta)^{k_{l-1}+\Delta_{l-1}-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_{l-1}+\Delta_{l-1}}^{k_l-1} (1-\alpha)^{k_l-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s) \\
&\le V_{\sigma(k_0)}(k_0) \\
&\quad - \sum_{s=k_0}^{k_0+\Delta_0-1} (1-\alpha)^{\Delta}(1+\beta)^{k_0+\Delta_0-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_0+\Delta_0}^{k_1-1} (1-\alpha)^{k_1-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_{l-1}}^{k_{l-1}+\Delta_{l-1}-1} (1-\alpha)^{\Delta+T_{l-1}}(1+\beta)^{k_{l-1}+\Delta_{l-1}-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_{l-1}+\Delta_{l-1}}^{k_l-1} (1-\alpha)^{k_l-s-1}\Gamma(s) - \cdots \\
&\quad - \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\Gamma(s) \\
&\quad - \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s).
\end{aligned}$$

Under zero initial condition, we know that

$$\begin{aligned}
&\sum_{s=k_0}^{k_0+\Delta_0-1} (1-\alpha)^{\Delta}(1+\beta)^{k_0+\Delta_0-s-1}\Gamma(s) \\
&+ \sum_{s=k_0+\Delta_0}^{k_1-1} (1-\alpha)^{k_1-s-1}\Gamma(s) \\
&+ \sum_{s=k_{l-1}}^{k_{l-1}+\Delta_{l-1}-1} (1-\alpha)^{\Delta+T_{l-1}}(1+\beta)^{k_{l-1}+\Delta_{l-1}-s-1}\Gamma(s) \\
&+ \sum_{s=k_{l-1}+\Delta_{l-1}}^{k_l-1} (1-\alpha)^{k_l-s-1}\Gamma(s) + \cdots \\
&+ \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\Gamma(s) \\
&+ \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1}\Gamma(s) \le 0.
\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}
&\sum_{s=k_0}^{k_0+\Delta_0-1} (1-\alpha)^{\Delta}(1+\beta)^{k_0+\Delta_0-s-1} z(s)^T z(s) \\
&+ \sum_{s=k_0+\Delta_0}^{k_1-1} (1-\alpha)^{k_1-s-1} z(s)^T z(s) \\
&+ \sum_{s=k_{l-1}}^{k_{l-1}+\Delta_{l-1}-1} (1-\alpha)^{\Delta+T_{l-1}}(1+\beta)^{k_{l-1}+\Delta_{l-1}-s-1} \\
&z(s)^T z(s) \\
&+ \sum_{s=k_{l-1}+\Delta_{l-1}}^{k_l-1} (1-\alpha)^{k_l-s-1} z(s)^T z(s) \\
&+ \sum_{s=k_l}^{k_l+\Delta_l-1} (1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1} z(s)^T z(s) \\
&+ \sum_{s=k_l+\Delta_l}^{k_{l+1}-1} (1-\alpha)^{k_{l+1}-s-1} z(s)^T z(s) \\
&\le \sum_{s=k_0}^{k_0+\Delta_0-1} (1-\alpha)^{\Delta}(1+\beta)^{k_0+\Delta_0-s-1}\gamma^2\omega^T(s)\omega(s) \\
&+ \sum_{s=k_0+\Delta_0}^{k_1-1} (1-\alpha)^{k_1-s-1}\gamma^2\omega^T(s)\omega(s)
\end{aligned}$$

$$+\sum_{s=k_{l-1}}^{k_{l-1}+\Delta_{l-1}-1}(1-\alpha)^{\Delta+T_{l-1}}(1+\beta)^{k_{l-1}+\Delta_{l-1}-s-1}$$
$$\gamma^2\omega^T(s)\omega(s)$$
$$+\sum_{s=k_{l-1}+\Delta_{l-1}}^{k_l-1}(1-\alpha)^{k_l-s-1}\gamma^2\omega^T(s)\omega(s)$$
$$+\sum_{s=k_l}^{k_l+\Delta_l-1}(1-\alpha)^{\Delta}(1+\beta)^{k_l+\Delta_l-s-1}\gamma^2\omega^T(s)\omega(s)$$
$$+\sum_{s=k_l+\Delta_l}^{k_{l+1}-1}(1-\alpha)^{k_{l+1}-s-1}\gamma^2\omega^T(s)\omega(s).$$

So we can get

$$\begin{aligned}&(1-\alpha)^{(\Delta+T_{\max})}\sum_{s=k_0}^{k_{l+1}}z^T(s)z(s)\\ \leq\ &(1+\beta)^{\Delta_{\max}}\sum_{s=k_0}^{k_{l+1}}\gamma^2\omega^T(s)\omega(s)\\ \leq\ &(1-\alpha)^{-\Delta}\sum_{s=k_0}^{k_{l+1}}\gamma^2\omega^T(s)\omega(s).\end{aligned}\tag{10}$$

When $l\to\infty$ , we can get

$$\sum_{s=k_0}^{\infty}z^T(s)z(s)\leq\sum_{s=k_0}^{\infty}(1-\alpha)^{-(2\Delta+T_{\max})}\gamma^2\omega^T(s)\omega(s).\tag{11}$$

■

*Remark 3:* The proof of disturbance attenuation level is different from [24], in which the result is got under zero initial condition $V_i(k_l)=0,\forall l\in\mathcal{I}$. In this paper, we provided a better result about weighted $l_2$-gain under zero initial condition $V_i(k_0)=0$; besides the result is suitable for any positive number $\Delta_{\max}$, which does not contain limit of $\mathcal{T}_{\max}>1$ in [24].

### 3.2 $H_\infty$ Controller Design

Now, based on the conditions on stability with a $l_2$-gain in Theorem 1 and Theorem 2, sufficient conditions for the existence of controller (2) are presented in the following theorem. Then, the admissible $H_\infty$ controller parameters can be given.

*Theorem 3:* Consider switched linear system (3) and let $0<\alpha<1,\beta>0$ be given constants. If there exist matrices $S_i$, $V_i$ and $U_i,\forall i\in\mathcal{I}$, such that for $\forall i,j\in\mathcal{I},i\neq j$, the following inequalities hold

$$\Phi_i=\begin{bmatrix}-I & 0 & C_iV_i & D_{2i}\\ * & -S_i & A_iV_i+B_iU_i & D_{1i}\\ * & * & (1-\alpha)(S_i-V_i-V_i^T) & 0\\ * & * & * & -\gamma^2I\end{bmatrix}<0\tag{12}$$

$$\Phi_{ij}=\begin{bmatrix}-I & 0 & C_iV_j & D_{2i}\\ * & -S_i & A_iV_j+B_iU_j & D_{1i}\\ * & * & (1+\beta)(S_i-V_j-V_j^T) & 0\\ * & * & * & -\gamma^2I\end{bmatrix}<0\tag{13}$$

then there exists a set of mode-dependent stabilizing controllers with asynchronous delay such that system (3) is stable with an $H_\infty$ performance index $\gamma_s$ for the switching signal (5). Moreover, if (13) and (14) have a feasible solution, the admissible controller can be given by

$$K_i=U_iV_i^{-1}.\tag{14}$$

*Proof:* When $t\in(t_k+\Delta_k,t_{k+1})$ , we can get

$$\begin{aligned}\Delta V_i(k)\ &+\alpha V_i(k)\\ &=V_i(k+1)-(1-\alpha)V_i(k)\\ &=x^T(k)\begin{bmatrix}-P_i^{-1} & A_i+B_iK_i\\ * & -(1-\alpha)P_i\end{bmatrix}x(k)\end{aligned}\tag{15}$$

$$\begin{aligned}&\Delta\ V_i(k)+\alpha V_i(k)+z^T(k)z(k)-\gamma^2\omega^T(s)\omega(s)\\ &=((A_i+B_iK_i)x(k)+D_{1i}\omega(k))^TP_i((A_i+B_iK_i)x(k)\\ &+D_{1i}\omega(k))-\gamma^2\omega^T(s)\omega(s)\\ &+(C_ix(k)+D_{2i}\omega(k))^T(C_ix(k)\\ &+D_{2i}\omega(k))-(1-\alpha)x^T(k)P_ix(k)\\ &=\eta^T(k)\Theta_i\eta(k)\end{aligned}\tag{16}$$

where $\eta(k)=(x^T(k),\omega^T(k))^T$.

$$\Theta_i=\begin{bmatrix}-I & 0 & C_i & D_{2i}\\ * & -P_i^{-1} & A_i+B_iK_i & D_{1i}\\ * & * & -(1-\alpha)P_i & 0\\ * & * & * & -\gamma^2I\end{bmatrix}$$

similarly, when $t\in(t_k,t_k+\Delta_k)$,we can get

$$\begin{aligned}\Delta V_i(k)-\beta V_i(k)\ &=V_i(k+1)-(1+\beta)V_i(k)\\ &=x^T(k)\begin{bmatrix}-P_i^{-1} & A_i+B_iK_j\\ * & -(1+\beta)P_i\end{bmatrix}x(k)\end{aligned}\tag{17}$$

$$\begin{aligned}&\Delta V_i(k)-\beta V_i(k)+z^T(k)z(k)-\gamma^2\omega^T(s)\omega(s)\\ &=((A_i+B_iK_j)x(k)+D_{1i}\omega(k))^TP_i((A_i+B_iK_j)x(k)\\ &+D_{1i}\omega(k))-\gamma^2\omega^T(s)\omega(s)\\ &+(C_ix(k)+D_{2i}\omega(k))^T(C_ix(k)+D_{2i}\omega(k))\\ &-(1-\alpha)x^T(k)P_ix(k)\\ &=\eta^T(k)\Theta_{ij}\eta(k)\end{aligned}$$

$$\Theta_{ij}=\begin{bmatrix}-I & 0 & C_i & D_{2i}\\ * & -P_i^{-1} & A_i+B_iK_j & D_{1i}\\ * & * & -(1+\beta)P_i & 0\\ * & * & * & -\gamma^2I\end{bmatrix}<0$$

Setting $S_i\triangleq P_i^{-1}$ and $U_i=K_iV_i$, from the fact $(S_i-V_i)S_i^{-1}(S_i-V_i)^T>0$ and $(S_i-V_j)S_i^{-1}(S_i-V_j)^T>0$, we can get the following inequalities: $-V_iS_i^{-1}V_i^T<S_i-V_i-V_i^T$ and $-V_jS_i^{-1}V_j^T<S_i-V_j-V_j^T$. Then, performing a congruent transformation to the inequality $\Theta_i\leq 0$ and $\Theta_{ij}\leq 0$ via diag$\{\ I\quad I\quad V_i^T\quad I\ \}$ and diag$\{\ I\quad I\quad V_j^T\quad I\ \}$, we can obtain (13) and (14). From Theorem 1 and Theorem 2, we can get that the closed-loop system is stable with an $l_2$-gain. ■

*Remark 4:* In Theorem 3, the conditions of $H_\infty$ stability are got for asynchronous switched linear system. We can get the same result for asynchronous switched system with nonlinear from (19).

$$x(k+1) = A_\sigma x(k) + B_\sigma u(k) + D_{1\sigma}\omega(k) + E_i f(x(k))$$
$$z(k) = C_\sigma x(k) + D_{2\sigma}\omega(k). \tag{18}$$

In this paper, without loss of generality, we always assume that $f(0) = 0$. For vector-valued functions $f$, we assume:

$$[f(x)-f(y)-W_1(x-y)]^T[f(x)-f(y)-W_2(x-y)]<0 \quad \forall x,y\in\mathbb{R}^n \tag{19}$$

where $W_1, W_2 \in \mathbb{R}^{n\times n}$ are known real constant matrices, and $W_1 + W_2$ is a positive definite matrix [27].

*Corollary 1:* Consider switched nonlinear system (19) and let $0 < \alpha < 1, \beta > 0$ be given constants and $W_1$, $W_2$ be given matrices. If there exist matrices $S_i > 0$, $V_i > 0$ and $U_i, \forall i \in \mathcal{I}$, such that for $\forall i, j \in \mathcal{I}, i \neq j$, the following inequalities hold

$$\Psi_i = \begin{bmatrix} -I & 0 & C_iV_i & 0 & D_{2i} & 0 \\ * & -S_i & A_iV_i+B_iU_i & E_i & D_{1i} & 0 \\ * & * & (1-\alpha)(S_i-V_i-V_i^T) & V_i\breve{W}_2 & 0 & V_i\breve{W}_1 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{20}$$

$$\Psi_{ij} = \begin{bmatrix} -I & 0 & C_iV_j & 0 & D_{2i} & 0 \\ * & -S_i & A_iV_j+B_iU_j & E_i & D_{1i} & 0 \\ * & * & (1+\beta)(S_i-V_j-V_j^T) & V_i\breve{W}_2 & 0 & V_i\breve{W}_1 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{21}$$

where $\breve{W}_1 = (W_1^T - W_2^T)/\sqrt{2}, \breve{W}_2 = (W_1^T + W_2^T)/2$, then there is a set of mode-dependent stabilizing controllers with asynchronous delay such that system (19) is stable with an $H_\infty$ performance index $\gamma_s$ for the switching signal (5). Moreover, if (21) and (22) have a feasible solution, then the admissible controller can be given by

$$K_i = U_iV_i^{-1}. \tag{22}$$

*Proof:* From (20), we can get

$$\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} \breve{W} & -\breve{W}_2 \\ * & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \leq 0 \tag{23}$$

where $\breve{W} = (W_1^TW_2 + W_2^TW_1)/2$.

Follow the same line, together with (24), when $t \in (t_k + \Delta_k, t_{k+1})$, we can get

$$\Delta V_i(k) + \alpha V_i(k) \leq \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} -P_i^{-1} & A_i+B_iK_i & E_i \\ * & -(1-\alpha)P_i - \breve{W} & \breve{W}_2 \\ * & * & -I \end{bmatrix} \times \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}$$

$$\Delta V_i(k) - \beta V_i(k) + z^T(k)z(k) - \gamma^2\omega^T(s)\omega(s) \leq \begin{bmatrix} x(k) & f(x(k)) & \omega(k) \end{bmatrix}^T \Omega_{ij} \begin{bmatrix} x(k) & f(x(k)) & \omega(k) \end{bmatrix}$$

where

$$\Omega_i = \begin{bmatrix} -I & 0 & C_i & 0 & D_{2i} \\ * & -P_i^{-1} & A_i+B_iK_i & E_i & D_{1i} \\ * & * & -(1-\alpha)P_i - \breve{W} & \breve{W}_2 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}.$$

When $t \in (t_k, t_k + \Delta_k)$ , we can get

$$\Delta V_i(k) - \beta V_i(k) \leq \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} -P_i^{-1} & A_i+B_iK_j & E_i \\ * & -(1+\beta)P_i - \breve{W} & \breve{W}_2 \\ * & * & -I \end{bmatrix} \times \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}$$

$$\Delta V_i(k) - \beta V_i(k) + z^T(k)z(k) - \gamma^2\omega^T(s)\omega(s) \leq \begin{bmatrix} x(k) & f(x(k)) & \omega(k) \end{bmatrix}^T \Omega_{ij} \begin{bmatrix} x(k) & f(x(k)) & \omega(k) \end{bmatrix}$$

where

$$\Omega_i = \begin{bmatrix} -I & 0 & C_i & 0 & D_{2i} \\ * & -P_i^{-1} & A_i+B_iK_j & E_i & D_{1i} \\ * & * & -(1+\beta)P_i - \breve{W} & \breve{W}_2 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}.$$

From the fact

$$(W_1 - W_2)^T(W_1 - W_2) = W_1^TW_1 - W_1^TW_2 - W_2^TW_1 + W_2^TW_2$$

we can get that

$$-W_1^TW_2 - W_2^TW_1 \leq (W_1 - W_2)^T(W_1 - W_2).$$

Then performing a congruent transformation to the inequality $\Theta_i \leq 0$ and $\Theta_{ij} \leq 0$ via $\text{diag}\{ I \quad I \quad V_i \quad I \}$ and $\text{diag}\{ I \quad I \quad V_j \quad I \}$ , we can obtain (21) and (22). From Theorem 1 and Theorem 2, we can get that the closed-loop system is stable with an $l_2$-gain. ■

## 4 Numerical Example

In this section, we give an example to demonstrate the effectiveness of the proposed method.

*Example:* Considering the system (1) with two subsystems, and the parameters of each subsystem are given as follows:

$$A_1 = \begin{bmatrix} 0.3 & 0.3 \\ 0.9 & -0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -0.4 \\ -0.3 & 0.4 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.3 \\ -0.6 \end{bmatrix}, B_2 = \begin{bmatrix} -1.4 \\ -0.1 \end{bmatrix}$$
$$D_{11} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, D_{12} = \begin{bmatrix} -0.3 \\ 0.3 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix}$$
$$D_{21} = 0.5,\ D_{22} = 0.1$$

Let $\alpha = 0.8$, $\beta = 0.3$, and $\Delta_{\max} = 1$, we consider the asynchronous switching in the design phase and turn to Theorem 2, by utilizing LMI Toolbox, we can get $\gamma = 2.5615$, $\gamma_s = 4.5614$ and the controller parameters are obtained as follow: $K_1 = \begin{bmatrix} 1.2853 & -0.2038 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.8182 & -0.4078 \end{bmatrix}$.

The simulation results are given in Figs. 1 and 2. The switching signals are shown in Fig. 1, the system state response under the switching signal is shown in Fig. 2. From the simulation results, we can get that the switching signal of the controller is delayed by the switching signal of the subsystem, and the system state is stable under the design of switching signals.

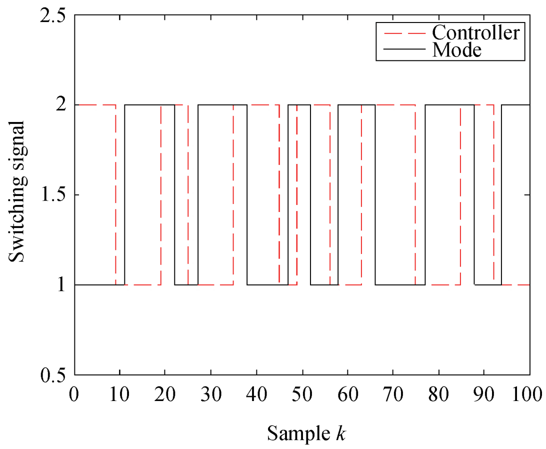


Fig. 1. The switching signal.

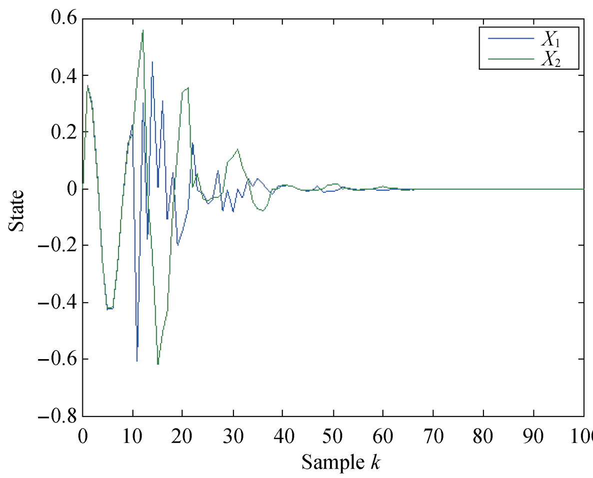


Fig. 2. The state.

Integrating the whole simulation results, we can get that the desired switched controller is feasible and effective for systems (1) under the admissible switching signals.

## 5 Conclusion

A new method for the stability and $l_2$-gain of state-dependent switching law under dwell time constraint is introduced. By allowing the subsystems to be unstable within a bounded time of the interval $[k_l, k_{l+1})$, $\forall l \in \mathbb{N}$, the more general conditions for $H_\infty$ controller have been derived and formulated. Then the corresponding controller is obtained. An example is given to illustrate the validity of the obtained theoretical results.

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