Dynamic Behaviors of Generalized Fractional Chaotic Systems

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Abstract In view of a new generalized fractional calculus proposed recently, this paper is devoted to applying the generalized fractional derivatives to study new generalized fractional chaotic systems. The chaotic properties depending on the new generalized fractional derivative are discussed and shown graphically. The generalized fractional derivative is described in the Caputo sense, and the finite difference approach for solving the generalized fractional chaotic system is presented. Since the generalized fractional derivative includes many existing fractional derivatives as special cases, we hope more attention will be brought into this field in the near future.

Key words Finite difference method, fractional calculus, fractional chaotic system, generalized fractional derivative, nonlinear dynamics

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1 Introduction

Fractional calculus and fractional differential equations have received considerable interest in the recent forty years. Fractional derivative means that the order of differentiation can be an arbitrary real number and even it can be a complex number. Fractional derivative modelling has been applied to many scientific and engineering fields, such as quantum mechanics [1], viscoelasticity and rheology [2], electrical engineering [3], electrochemistry [4], biology [5], biophysics and bioengineering [6], signal and image processing [7], mechatronics [8], and control theory [9]−[11]. Although few mathematical issues of fractional derivative remain unsolved, most of the difficulties have been overcome, and the applications of fractional calculus in above fields indicate that the fractional models can depict the property and behavior of a real-world problem more accurately. For a comprehensive review of fractional calculus, we refer readers to some monographs [12]−[14] and references therein. In contrast to integer order derivative, the way of identifying fractional derivative is not unique. There are several types of definitions, such as Riemann-Liouville derivative, Caputo derivative, Grünwald-Letnikov derivative, and so on. More details can be found in [13, Chapter 2]. In the recent years, the study of dynamical system with fractional order derivative becomes more and more popular [15]−[19]. Moreover, the dynamics in fractional dynamical system is more interesting.

Returning back to the fractional derivative, since it has several different definitions, how to develop a generalized form which can unify all the existing fractional derivatives becomes one important topic in fractional calculus [20]−[22]. Recently, a class of new generalized fractional integral and generalized fractional derivative is introduced in [22]. The new generalized fractional integral and generalized fractional derivative depend on a scale function and a weight function, which makes them more general. When the scale function and the weight function reduce to some specific cases, the generalized fractional operators will reduce to Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative and so on. However, the study of this new generalized fractional integral and generalized fractional derivative are in the very beginning stage now [23]−[26]. In [24], we show that in generalized fractional diffusion equation, the scale function allows the response domain to be scaled differently. It is required that the scale function should be strictly monotonically increasing or decreasing. A convex increasing scale function will compress the response domain towards to the initial time. A concave increasing scale function will stretch the response domain away from the initial time. The weight function allows the response to be assessed differently at different time, since in many applications, we may require an event to be weighed differently at different time point. For example, modeling of memory of a child may require a heavy weight at current time point, whereas the same for an older person may require more weight on the past. To be an initial attempt of application to chaotic dynamical systems, in this paper, we define a class of new generalized fractional chaotic systems by replacing the original derivatives with the new generalized fractional derivative, then apply a finite difference scheme to study the numerical solutions of two different generalized fractional chaotic systems, namely generalized fractional Lotka-Volterra system (GFLVS) and generalized fractional Lorenz system (GFLS). Their complex dynamics will be discussed, and the dynamic behavior depending on the weight and scale function will be shown graphically.

The rest of this paper is organized as follows: In Section 2, the preliminaries of fractional calculus are given. The new generalized fractional integral and generalized fractional derivative are shown. A finite difference approach for solving equations with generalized fractional derivative is carried out. In Section 3, we define the chaotic systems using the generalized fractional derivative of Caputo type, i.e., the GFLVS and GFLS. Some interesting dynamics of those two systems are shown graphically. Finally, the conclusions are drawn in Section 4.

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2 Mathematical Preliminaries

In this section, we introduce the preliminaries of generalized fractional derivatives, and show a proper numerical method for differential equations with such derivatives.

2.1 Generalized Fractional Calculus

Let us begin with the common fractional operators. In calculus, the n-fold integral of an integrable function $u(t)$ is defined as

$$I^n u(t) = \overbrace{\int_0^t \cdots \int_0^t}^{n \text{ times}} u(s)ds \cdots ds = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} u(s)ds$$

where $t \geq 0$, and $u(0)$ is well-defined. Replacing the positive integer n by a real number $\alpha > 0$, we have the following definition.

Definition 1 [13]: The left Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t)$ is defined as

$$\left(I_{0+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds \tag{1}$$

provided the integral is finite, where $\Gamma(\alpha)$ is the Gamma function.

The Riemann-Liouville fractional integral plays an important role in defining fractional derivatives. There are two basic approaches to define the fractional derivative, i.e., "first integration then differentiation" and "first differentiation then integration". The corresponding fractional derivatives are called Riemann-Liouville fractional derivative and Caputo fractional derivative, and the definitions are given as follows.

Definition 2 [13]: The left Riemann-Liouville fractional derivative of order $n-1 < \alpha < n$ of a function $u(t)$ is defined as

$$\left(D_{0+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dt^n}\right) \int_0^t (t-s)^{n-\alpha-1} u(s)ds \tag{2}$$

provided the right side of the identity is finite.

Definition 3 [13]: The left Caputo fractional derivative of order $n-1 < \alpha < n$ of a function $u(t)$ is defined as

$$\left({}^{c}D_{0+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds \tag{3}$$

provided the right side of the identity is finite.

Besides above, there also exist right Riemann-Liouville integral and derivative, and right Caputo fractional derivative [13]. Mathematically, the Riemann-Liouville and Caputo fractional operators are used in applications frequently. In most real-world models, we always employ the left Caputo fractional derivative. One reason is that we will study generalized fractional dynamical system later, and the derivative is taken with respect to time variable. In physical models, time is always running forward. The other reason is that in the differential equations with Caputo fractional derivative, the initial conditions are taken in the same form as for integer-order differential equations which have clear physical meanings in the practical application and can be easily measured [14]. In what follows, we will introduce the generalized fractional integral and derivative proposed in [22]. They extend nearly all the existing fractional operators. Now we list the generalized fractional integral and derivative defined on positive half axis. They will be used to define the generalized fractional chaotic systems in next section.

Definition 4 [22]: The left generalized fractional integral of order $\alpha > 0$ of a function $u(t)$ with respect to a scale function $\sigma(t)$ and a weight function $w(t)$ is defined as

$$\left(I_{0+;[\sigma,w]}^{\alpha} u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_0^t \frac{w(s)\sigma'(s)u(s)}{[\sigma(t)-\sigma(s)]^{1-\alpha}} ds \tag{4}$$

provided the integral exists, where $\sigma'(s)$ indicates the first derivative of the scale function σ.

Definition 5 [22]: The left generalized derivative of order m of a function $u(t)$ with respect to a scale function $\sigma(t)$ and a weight function $w(t)$ is defined as

$$\left(D_{[\sigma,w;L]}^{m} u\right)(t) = [w(t)]^{-1} \left[\left(\frac{1}{\sigma'(t)} D_t\right)^m (w(t)u(t))\right] \tag{5}$$

provided the right-side of equation is finite, where m is a positive integer.

Definition 6 [22]: The Caputo type left generalized fractional derivative of order $\alpha > 0$ of a function $u(t)$ with respect to a scale function $\sigma(t)$ and a weight function $w(t)$ is defined as

$$\left(D_{0+;[\sigma,w]}^{\alpha} u\right)(t) = \left(I_{0+;[\sigma,w]}^{m-\alpha} D_{[\sigma,w;L]}^{m} u\right)(t) \tag{6}$$

provided the right-side of equation is finite, where $m-1 \leq \alpha < m$, and m is a positive integer. Particularly, when $0 < \alpha < 1$, we have

$$\left(D_{0+;[\sigma,w]}^{\alpha} u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)} \int_0^t \frac{[w(s)u(s)]'}{[\sigma(t)-\sigma(s)]^{\alpha}} ds. \tag{7}$$

2.2 Finite Difference Method

Now we introduce a finite difference method for solving differential equations with generalized fractional derivative. Consider the following generalized fractional differential equation:

$$\begin{cases} \left(D_{0+;[\sigma,w]}^{\alpha} u\right)(t) = f(t, u(t)), & 0 < t \leq T \\ u(0) = u_0 \end{cases} \tag{8}$$

where $0 < \alpha < 1$ and T is the final time. Without loss of generality, on a uniform mesh $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots < t_N = T$, the Caputo type generalized fractional derivative of $u(t)$ can be approximated as

$$\begin{aligned} &(D_{0+;[\sigma,w]}^{\alpha} u)(t_{j+1}) \\ &\quad= \frac{[w(t_{j+1})]^{-1}}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} \frac{[w(s)u(s)]'}{[\sigma(t_{j+1})-\sigma(s)]^{\alpha}} ds \\ &\quad= \frac{w_{j+1}^{-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \frac{[w(s)u(s)]'}{[\sigma(t_{j+1})-\sigma(s)]^{\alpha}} ds \\ &\quad\approx \frac{w_{j+1}^{-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \frac{\frac{w_{k+1}u_{k+1}-w_k u_k}{t_{k+1}-t_k}}{[\sigma_{j+1}-\sigma(s)]^{\alpha}} ds \\ &\quad\approx \sum_{k=0}^{j} \left(A_k^j u_{k+1} - B_k^j u_k\right) \end{aligned} \tag{9}$$

where

$$A_k^j = \frac{w_{j+1}^{-1} w_{k+1}}{\Gamma(2-\alpha)(\sigma_{k+1}-\sigma_k)} \times \left[(\sigma_{j+1}-\sigma_k)^{1-\alpha} - (\sigma_{j+1}-\sigma_{k+1})^{1-\alpha}\right]$$

$$B_k^j = \frac{w_{j+1}^{-1} w_k}{\Gamma(2-\alpha)(\sigma_{k+1}-\sigma_k)} \times \left[(\sigma_{j+1}-\sigma_k)^{1-\alpha} - (\sigma_{j+1}-\sigma_{k+1})^{1-\alpha}\right]$$

$k = 0, 1, 2, \ldots, j$, $u_j = u(t_j)$, $w_j = w(t_j)$, and $\sigma_j = \sigma(t_j)$.

Therefore, we obtain the finite difference scheme:

$$\sum_{k=0}^{j} \left(A_k^j u_{k+1} - B_k^j u_k\right) = f(t_{j+1}, u_{j+1}) \tag{10}$$

and the corresponding iteration scheme as

$$u_{j+1} = \begin{cases} \frac{1}{A_j^j}\left[f_j - \sum\limits_{k=0}^{j-1}\left(A_k^j u_{k+1} - B_k^j u_k\right) + B_j^j u_j\right], & j = 1, 2, \ldots, N-1 \\ \frac{1}{A_0^0}\left(f_0 + B_0^0 u_0\right), & j = 0 \end{cases} \tag{11}$$

where $f_j = f(t_j, u_j)$.

In what follows, we will apply this method to solve the generalized fractional chaotic systems. The numerical analysis of the above scheme can be found in [26].

3 Dynamic Behavior of Generalized Fractional Chaotic Systems

In this section, we introduce two nonlinear dynamical systems but redefine them with Caputo type generalized fractional derivative. The classical and fractional senses are special cases of the new generalized fractional system below.

3.1 Generalized Fractional Lotka-Volterra and Generalized Fractional Lorenz System

Replacing the derivative with the generalized fractional derivative defined by (7), we define the generalized fractional Lotka-Volterra system (GFLVS) as

$$\begin{cases} D_{0+;[\sigma,w]}^{\alpha_1} x = ax - bxy + mx^2 - sx^2 z \\ D_{0+;[\sigma,w]}^{\alpha_2} y = -cy + dxy \\ D_{0+;[\sigma,w]}^{\alpha_3} z = -pz + sx^2 z \end{cases} \tag{12}$$

where $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ (α_1, α_2, α_3 can be the equal or different) are the orders of the derivative and parameters a, b, c, d are positive. a represents the natural growth rate of the prey in the absence of predators, b represents the effect of predator on the prey, c represents the natural death rate of the predator in the absence of prey, d represents the efficiency and propagation rate of the predator in the presence of prey, and m, p, s are positive constants.

By selecting the parameters $a = 1$, $b = 1$, $c = 1$, $d = 1$, $m = 2$, $s = 2.7$, $p = 3$ and the initial condition $[x_0, y_0, z_0] = [1.5, 1.5, 1.5]$, when $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$, (12) represents the generalized fractional Lotka-Volterra chaotic system and the phase portraits of the system (12) are described through Figs. 1 (a) and 1 (b). In Fig. 1 (a), the chaotic phenomenon is shown. Moreover, the GFLVS reduces to the fractional Lotka-Volterra system as $\sigma(t) = t$ and $w(t) = 1$. In Fig. 1 (b), we see that when the scale function is specified as a power function, and the weight function is taken as an exponential function, the chaotic attractor vanishes and then a stable equilibrium point appears.

Similarly, we define the generalized fractional Lorenz system (GFLS) as

$$\begin{cases} D_{0+;[\sigma,w]}^{\alpha_1} x = r(y - x) \\ D_{0+;[\sigma,w]}^{\alpha_2} y = x(\rho - z) - y \\ D_{0+;[\sigma,w]}^{\alpha_3} z = xy - \beta z \end{cases} \tag{13}$$

where r is the Prandtl number, ρ is the Rayleigh number and β is the size of the region approximated by the system. The fractional order $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ may take different values.

By taking the parameters $r = 10$, $\rho = 28$, $\beta = 8/3$, and the initial condition $[x_0, y_0, z_0] = [0.5, 0.5, 0.5]$, when $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$, (13) represents the generalized fractional Lorenz chaotic system and the phase portraits of the system (13) are described through Figs. 1 (c) and 1 (d). In Fig. 1 (c), the chaotic attractor of fractional Lorenz system is presented. When we take scale function as a power function, and weight function as exponential function, the GFLS remains chaotic. However, the shape of the attractor changes, which is shown in Fig. 1 (d).

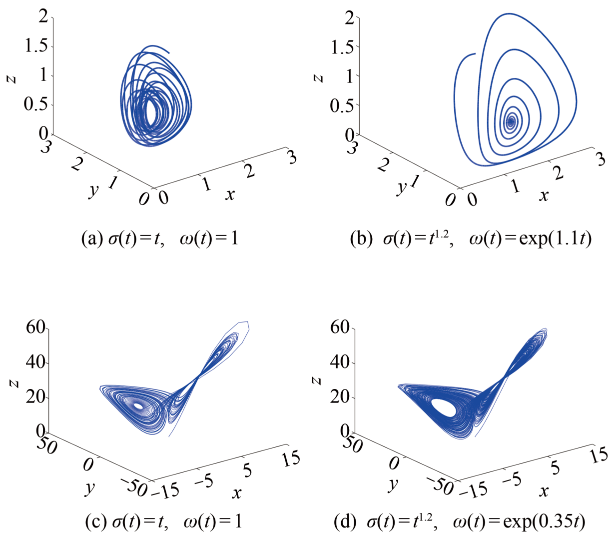


Fig. 1. Phase portraits of GFLVS (top row, (a) and (b)) and GFLS (bottom row, (c) and (d)).

3.2 Analysis of the Influence of Scale and Weight Functions

Now we analyze the influence of the scale and weight functions on the responses of generalized fractional differential equation. For simplicity, we consider

$$D_{0+;[\sigma,w]}^{\alpha} u(t) = Au(t) + f(t) \tag{14}$$

where $A \neq 0$ is a constant.

Equation (14) is equivalent to

$$\frac{[w(t)]^{-1}}{\Gamma(1-\alpha)} \int_0^t \frac{[w(s)u(s)]'}{[\sigma(t)-\sigma(s)]^{\alpha}} ds = Au(t) + f(t). \tag{15}$$

Let $v(t) = w(t)u(t)$, we have

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{v(s)'}{[\sigma(t)-\sigma(s)]^\alpha}ds = Av(t) + w(t)f(t). \quad (16)$$

According to [13], we deduce the solution of (16) as:

$$\begin{aligned} v(t) =\; & E_\alpha\left(A[\sigma(t)-\sigma(0)]^\alpha\right)v_0 \\ & + \int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} \\ & \times E_{\alpha,\alpha}[A(\sigma(t)-\sigma(s))^\alpha]w(s)f(s)ds \end{aligned} \quad (17)$$

which implies that

$$\begin{aligned} u(t) =\; & \frac{w(0)}{w(t)}E_\alpha\left(A[\sigma(t)-\sigma(0)]^\alpha\right)u_0 \\ & + \frac{1}{w(t)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} \\ & \times E_{\alpha,\alpha}[A(\sigma(t)-\sigma(s))^\alpha]w(s)f(s)ds \end{aligned} \quad (18)$$

where u_0 is the initial condition, and E is the Mittag-Leffler function.

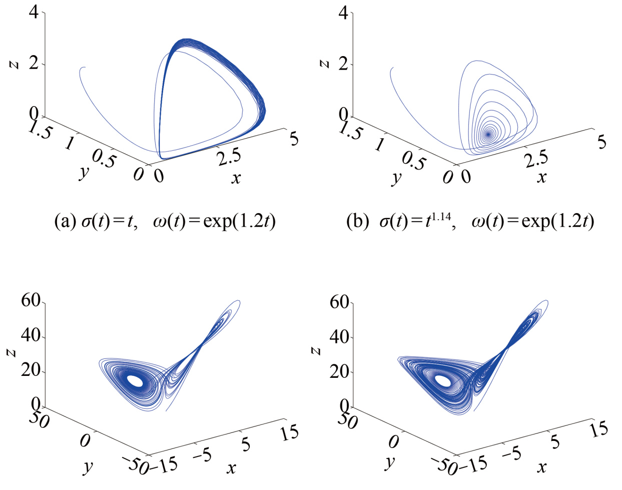
In (18), we observe that how the weight and scale functions influence the behavior of (14). First of all, the weight function cannot be zero in the domain, otherwise solution $u(t)$ will go to infinity. Second, the scale function cannot be periodic, and if it is, the generalized fractional derivative will be infinity at $t = s$. For an intuitive comprehension, we present some numerical simulations in the following.

3.3 Dynamics of GFLVS and GFLS Depend on Scale and Weight Functions

The fractional chaotic systems are sufficiently generalized by using the generalized fractional derivative, since many existing fractional derivatives, as well as integer order derivatives, are special cases of the generalized fractional derivative. In our numerical experiments, we find many interesting dynamical behaviors of generalized fractional chaotic systems which are never found in common fractional or integer order chaotic systems. Here we present some particular simulation results. However, our discussion depends on Figs. 2 and 3, and others figures are not shown here.

First, we simulate the influence of scale function on dynamics of chaotic systems. In GFLVS, we take fractional order $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$, weight function $w(t) = \exp(1.2t)$, and other parameters are the same as before. In GFLS, we select fractional order $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$, weight function $w(t) = \exp(0.1t)$, and other parameters are the same as before. The dynamic behaviors of GFLVS and GFLS with scale function $\sigma(t) = t$ and $t^{1.14}$ are individually presented in Fig. 2.

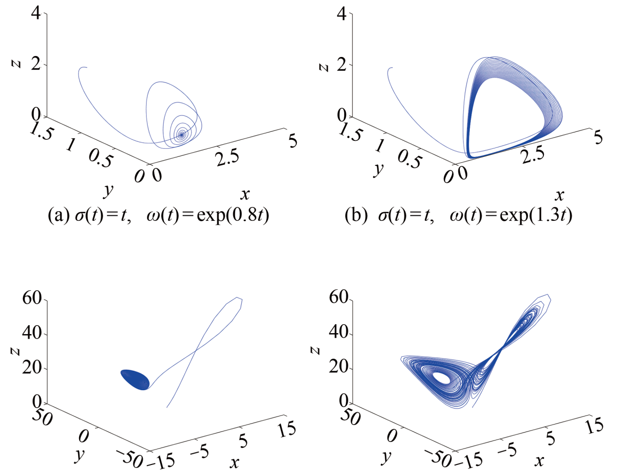
Second, we simulate the influence of weight function on dynamics of chaotic systems. In GFLVS, we take fractional order $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$, scale function $\sigma(t) = t$, and other parameters are the same as before. In GFLS, we select fractional order $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$, scale function $\sigma(t) = t$, and other parameters are the same as before. The dynamic behaviors of GFLVS with weight function $w(t) = \exp(0.8t)$, $\exp(1.3t)$, and GFLS with weight function $w(t) = \exp(2+0.5t)$ and $\exp(2+0.2t)$ are presented in Fig. 3.



(a) $\sigma(t)=t$, $\omega(t)=\exp(1.2t)$ (b) $\sigma(t)=t^{1.14}$, $\omega(t)=\exp(1.2t)$

(c) $\sigma(t)=t$, $\omega(t)=\exp(0.1t)$ (d) $\sigma(t)=t^{1.14}$, $\omega(t)=\exp(0.1t)$

Fig. 2. Influence of scale function $\sigma(t)$ on GFLVS (top row, (a) and (b)) and GFLS (bottom row, (c) and (d)).



(a) $\sigma(t)=t$, $\omega(t)=\exp(0.8t)$ (b) $\sigma(t)=t$, $\omega(t)=\exp(1.3t)$

(c) $\sigma(t)=t$, $\omega(t)=\exp(2+0.5t)$ (d) $\sigma(t)=t$, $\omega(t)=\exp(2+0.2t)$

Fig. 3. Influence of weight function $w(t)$ on GFLVS (top row, (a) and (b)) and GFLS (bottom row, (c) and (d)).

Finally, to end this section, we make some remarks based on the numerical experiments above. Some other figures are not listed here for shortening the length of paper.

1) The GFLVS is chaotic with scale function $\sigma(t) = t$, weight function $w(t)$ is a nonzero constant, and fractional order $\alpha_i = 0.95$, $i = 1, 2, 3$ [27]. However, From Fig. 1 (a), Fig. 2 (a) and Fig. 3 (a), we may see that as the weight function varies, the chaotic attractor vanishes and then a limit cycle emerges or the system converges to a stable equilibrium point. Furthermore, from Fig. 2 (a) and Fig. 2 (b), we observe that as the scale function varies, the limit cycle tends to be a stable equilibrium point. From Fig. 3 (a) and Fig. 3 (b), it is shown that as the weight function varies, the limit cycle can be generated from a stable equilibrium point.

2) The GFLS is chaotic with scale function $\sigma(t) = t$, weight function $w(t)$ is a nonzero constant, and fractional order $\sum_{i=1}^{3}\alpha_i > 2.91$ [28]. In simulation, on one hand, Figs. 1 (c) and 1 (d), indicate that with suitable scale and weight functions, the GFLS also has a chaotic attractor. On

the other hand, Fig. 1 (c), Fig. 1 (d), Fig. 2 (c), Fig. 2 (d), and Fig. 3 (d) imply that the scale and weight functions can influence the shape and position of chaotic attractor. From Figs. 3 (c) and 3 (d), we observe that with some suitable weight function, the chaotic attractor tends to be an asymptotically stable equilibrium point.

3) Our previous work [23]−[26] verified that in generalized fractional integral and generalized fractional derivative, the basic property of scale function $\sigma(t)$ is that it changes the time axis, which means that if the time domain is specified as $[0,T]$, then the response of the dynamical system is obtained over $[\sigma(0),\sigma(T)]$, provided the scale function is monotone increasing. Since the chaotic dynamical systems are sensitive to the initial conditions, when we take different scale functions in generalized fractional chaotic system, many different dynamical behaviors will be drawn.

4) A similar observation to weight function can be found in [23]−[26], which shows that in generalized fractional integral and generalized fractional derivative, the basic property of weight function $w(t)$ is that it puts different weights for function in different positions of domain. The classical fractional operators have memory property which makes them excellent tools to model the diffusion process with heredity. Generally, in left Caputo type generalized fractional derivative, the monotonic increasing weight function is coincident with the inner memory property of fractional operator, while the monotonic decreasing weight function can destroy this inner property. One can also follow our numerical method and try other scale and weight functions in numerical experiments.

5) In Figs. 2 and 3, one can observe that both changing the scale and weight functions make the systems change between different dynamical behavior (e.g., limit cycle and stable equilibrium point). These phenomena can be regarded as general cases for generalized fractional chaotic systems. We shall guess that either scale function or weight function would influence the dynamics of generalized fractional chaotic systems. In Fig. 2, the weight function is fixed so that the influence of scale function on GFLVS and GFLS is presented. Similarly, in Fig. 3, the scale function is fixed so that the influence of weight function on GFLVS and GFLS is shown. From (18), we clearly see that the scale function plays an important role in scaling the long time behavior of dynamics since it is located in the generalized exponential function, and the weight function provides a different average since it lies inside the integral, and it is a variable coefficient simultaneously. Apparently, the behavior of function u depends on the changing of scale and weight functions.

4 Conclusions

In this paper, we presented a class of new generalized fractional chaotic system, using the new generalized fractional derivative proposed recently. Many dynamical systems with integer or fractional order derivatives can be extended by replacing the derivative with the generalized fractional derivative. Therefore, the new generalized fractional dynamical systems considered in this paper can exhibit more complex dynamic behaviors. In simulations, we show that the dynamical behaviors of such systems not only depend on fractional order, but also depend on the scale and weight functions.

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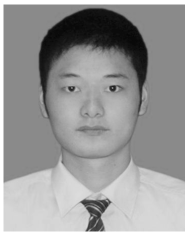

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