Robust Approximations to Joint Chance-constrained Problems

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Abstract Two new approximate formulations to joint chance-constrained optimization problems are proposed in this paper. The relationships of CVaR (conditional-value-at-risk), chance constrains and robust optimization are reviewed. Firstly, two new upper bounds on $\mathrm{E}((\cdot)^+)$ are proposed, where E stands for the expectation and $x^+ = \max(0, x)$, based on which two approximate formulations for individual chance-constrained problems are derived. The approximations are proved to be the robust optimization with the corresponding uncertain sets. Then the approximations are extrapolated to joint chance-constrained problem. Finally numerical studies are performed to compare the solutions of individual and joint chance constraints approximations and the results demonstrate the validity of our method.

Key words Joint chance constraints, robust optimization, uncertain set, approximation

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Data uncertainty is very common in real-world optimization problems. For convenience, we usually use the "nominal value" in the model to search the optimal solution. However, ignoring the data uncertainty may lead to the obtained solution to be suboptimal or even infeasible for practical applications.

There are many methods to deal with optimization problems with uncertain data. Chance-constrained optimization seems to be the most natural one to restrict the violation probability, which was introduced by Charnes et al.[1]. Usually, the uncertain parameters in the optimization model are assumed to be independent, and we deal with the uncertain constraints separately. Calasfoire and Ghaoui[2] demonstrated that the individual chance-constrained problem is a second order cone constraint problem which is computationally tractable if the uncertain parameters are of radial distributions. But for most of the other distributions, chance-constrained problems are computationally intractable. If uncertain data are related and then constraints cannot be treated individually, the optimization problems become more difficult to handle with. In fact, Prekopa proved that, with only right hand side disturbances, a joint chance-constrained problem is convex only when the distributions are log-concave[3]. Difficulties in acquiring the distribution information and computation spurred researchers to find other effective methods.

Robust optimization is another important way to deal with the uncertain optimization problems. In this method, the uncertain data is defined as a deterministic data set, and the goal is to search the optimal solution which remains feasible for all values in the data set. Usually the data set is called uncertain space or uncertain set. One of the earliest endeavors in robust optimization was Soyster′s work in 1973[4]. Soyster proposed a worst-cases model that ensured feasibility of its solution for all realization of the uncertain data. There is no doubt that the solution is safe but over conservative. Then in robust optimization, "safety" becomes "relative", and the purpose is to obtain the trade-off between robustness and performance.

Ben-Tal and Nemirovski proposed ellipsoidal-set based robust optimization formulation, and then showed that it could be turned to a conic quadratic problem[5−6]. Bertsimas and Sim considered robust linear programming with coefficient uncertainty using an uncertainty set with budgets which could be used to control the conservative degree[7], and the uncertain set was alternatively described by an arbitrary norm[8]. Li et al. discussed different uncertain sets and their geometric relationship, derived the corresponding robust formulations[9], and then analyzed the probabilistic guarantees on constraint satisfaction[10].

CVaR (Conditional-value-at-risk), introduced by Ben-Tal and Teboulle, is a special class of optimized certainty equivalent risk measures[11]. And it is also known that CVaR is the tightest convex approximation to the individual chance constraint. But the difficulty lies in the evaluating of the expectation $\mathrm{E}((\cdot)^+)$, where E stands for the expectation and $x^+ = \max(0, x)$. Chen and Sim et al. provided several bounds on $\mathrm{E}((\cdot)^+)$[12], and showed different approximations to individual chance-constrained problems used in robust optimization are the consequences of applying different bounds on $\mathrm{E}((\cdot)^+)$[13]. The recent applications of robust optimization and the approximation to chance-constrained problems are reviewed in [14−15].

It is showed that robust optimization in approximating individual chance-constrained has been paid an extensive attention on. However, for joint chance-constrained problems, there are only a few efforts that have been made. A direct way to deal with joint chance-constrained problem is to decompose it into an individual chance-constrained problem, and Bonferroni′s inequality can be used as a sufficient condition, but in many cases the results are even more conservative. Chen et al.[13] proposed a novel smart formulation for approximating joint chance-constrained problems that improved the standard approach using Bonferroni′s inequality. In their method, a very important step is to calculate the tightest bound on $\mathrm{E}((\cdot)^+)$, which needs to deal with several intractable parameters such as forward and backward deviations. But sometimes, we can only obtain limited information about the uncertain data such as bounds, etc. In this paper, we propose two new upper bounds on $\mathrm{E}((\cdot)^+)$ which only need the bounds of the uncertain data, and then give two new approximations for joint chance constraints.

The rest of this paper is organized as follows. In Section 1, we give the problem statement, review the relation-

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ship among $\mathrm{E}((\cdot)^+)$, CVaR, and individual chance constraints, then introduce Chen's approximate approach to joint chance constraints. In Section 2, we propose two new bounds of $\mathrm{E}((\cdot)^+)$, derive two new approximating formulations for joint chance constraints, and analyze the relationship between the violation degree and the controlling parameters of the uncertain sets in robust optimization. In Section 3, a numerical example is presented. In Section 4, conclusions are presented.

1 Approximation of chance constraints

In this paper, we denote random variables with tilde sign, such as $\tilde{z}$. Boldface lower-case letters represent vectors such as $\boldsymbol{x}$, and boldface upper-case letters represent matrices such as $\boldsymbol{A}$. We denote $x^+ = \max(x, 0)$, and use $\mathrm{E}(\cdot)$ to represent expectation.

1.1 Individual and joint chance constraints

Consider the following linear programming (LP) optimization problem:

$$\begin{aligned} \max\quad & \boldsymbol{cx} \\ \text{s.t.}\quad & \sum_j \tilde{a}_{ij} x_j \le \tilde{b}_j, \quad \forall i \end{aligned} \tag{1}$$

where $\tilde{a}_{ij}$ and $\tilde{b}_j$ represent the true value of the parameters which are subjected to uncertainty. If the uncertain parameters are bounded, the perturbation ranges can be expressed as

$$\tilde{a}_{ij} \in [a^0_{ij} - \hat{a}_{ij}, a^0_{ij} + \hat{a}_{ij}], \tilde{b}_j \in [b^0_j - \hat{b}_j, b^0_j + \hat{b}_j], \\ i \in 1, 2, \cdots, N, j \in 1, 2, \cdots, J$$

where a^0_{ij} and b^0_j represent the nominal value of the parameters, $\hat{a}_{ij}$ and $\hat{b}_j$ represent constant perturbations (which are positive).

Assume the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_j$ are linear dependent, and can be expressed as

$$\tilde{a}_{ij} = a^0_{ij} + \sum_{k=1}^{K} a^k_{ij} \tilde{z}_k$$

$$\tilde{b}_j = b^0_j + \sum_{k=1}^{K} b^k_j \tilde{z}_k$$

$$\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_K)^{\mathrm{T}} \in \mathcal{W}$$

where $\tilde{\boldsymbol{z}}$ is an independent random vector. Suppose set $\mathcal{W}$ is a second-order conic representable set proposed by Ben-Tal and Nemirovski[5], which includes box, polyhedral and ellipsoidal sets. We describe the box set as

$$\mathcal{W} = \{\tilde{\boldsymbol{z}} : -\boldsymbol{z} \le \tilde{\boldsymbol{z}} \le \boldsymbol{z}\}$$

By reformulating equation (1), we have

$$(\sum_j a^0_{ij} x_j - b^0_j) + (\sum_k \sum_j a^k_{ij} x_j \tilde{z}_k - \sum_k b^k_j \tilde{z}_k) \le 0$$

$$\underbrace{(\sum_j a^0_{ij} x_j - b^0_j)} + \sum_k \underbrace{(\sum_j a^k_{ij} x_j - b^k_j)}\, \tilde{z}_k \le 0$$

Let $y^0_i = \sum_j a^0_{ij} x_j - b^0_j$ and $y^k_i = \sum_j a^k_{ij} x_j - b^k_j$. Then

$$y^0_i + \sum_K y^k_i \tilde{z}_k \le 0$$

which can be expressed as

$$y^0_i + \boldsymbol{y}_i^{\mathrm{T}} \tilde{\boldsymbol{z}}_k \le 0, \quad \boldsymbol{y}_i^{\mathrm{T}} = \{y^1_i, y^2_i, \cdots, y^K_i\}$$

The individual chance constraints can be represented as

$$P(y^0_i + \boldsymbol{y}_i^{\mathrm{T}} \tilde{\boldsymbol{z}}_k \le 0) \ge 1 - \varepsilon_i \tag{2}$$

Then the original optimization problem with uncertain parameters (1) can be represented as follows:

$$\begin{aligned} \max\quad & \boldsymbol{cx} \\ \text{s.t.}\quad & P(y^0_i + \boldsymbol{y}_i^{\mathrm{T}} \tilde{\boldsymbol{z}}_k \le 0) \ge 1 - \varepsilon_i, \quad \forall i \end{aligned}$$

And the joint chance constraint is defined as

$$P(y^0_i + \boldsymbol{y}_i^{\mathrm{T}} \tilde{\boldsymbol{z}}_k \le 0, i \in M) \ge 1 - \varepsilon \tag{3}$$

Equation (3) requires all the linear constraints to be joint feasible with the probability of at least $1-\varepsilon$, where $\varepsilon \in (0, 1)$ is a desired safety factor. Then the original optimization problem with uncertain parameters (1) can be represented as follows:

$$\begin{aligned} \max\quad & \boldsymbol{cx} \\ \text{s.t.}\quad & P(y^0_i + \boldsymbol{y}_i^{\mathrm{T}} \tilde{\boldsymbol{z}}_k \le 0, i \in M) \ge 1 - \varepsilon \end{aligned}$$

1.2 Approximation from CVaR measure

From the work of [11] and [12], CVaR function of $y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}}$ can be defined as

$$\rho_{1-\varepsilon}(y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}}) := \min_\beta \{\beta + \frac{1}{\varepsilon} \mathrm{E}(y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}} - \beta)^+\}$$

and the upper bound of the CVaR function can be used as an approximation of the individual chance constraints.

Chen et al.[13] defined the upper bound of $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}})^+)$ as $\pi(y_0, \boldsymbol{y})$, and defined

$$\eta_{1-\varepsilon}(y_0, \boldsymbol{y}) := \min_\beta \{\beta + \frac{1}{\varepsilon} \pi(y_0 - \beta, \boldsymbol{y})\}$$

then

$$\begin{aligned} &\rho_{1-\varepsilon}(y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}}) = \\ &\quad \min_\beta \{\beta + \frac{1}{\varepsilon} \mathrm{E}(y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}} - \beta)^+\} \le \\ &\quad \eta_{1-\varepsilon}(y_0, \boldsymbol{y}) \end{aligned}$$

So a sufficient condition for satisfing the individual chance constraint (2) is

$$\eta_{1-\varepsilon}(y_0, \boldsymbol{y}_i) \le 0$$

Then the approximation to individual chance constraint can be presented as follows:

$$\begin{aligned} &\eta_{1-\varepsilon}(y_0 + \boldsymbol{y}^{\mathrm{T}} \tilde{\boldsymbol{z}}) = \\ &\quad \min_\beta \{\beta + \frac{1}{\pi}(y_0 - \beta, \boldsymbol{y}_i)\} \le 0 \end{aligned} \tag{4}$$

Chen et al. also gave the approximation to joint chance constraints (3) represented as follows:

$$\begin{aligned} &\gamma_{1-\varepsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, M) := \\ &\quad \min_{w_0, \boldsymbol{w}} \{\min_\beta [\beta + \frac{1}{\varepsilon} \pi(w_0 - \beta, \boldsymbol{w})] + \\ &\quad \frac{1}{\varepsilon} (\sum_{i \in M} \pi(\alpha_i y^0_i - w_0, \alpha_i \boldsymbol{y}_i - \boldsymbol{w}))\} \le 0 \end{aligned} \tag{5}$$

where $\boldsymbol{\alpha} \in \mathbf{R}^M$ (set of M dimensional real vector), $\boldsymbol{\alpha} > 0$, is a given vector of positive constants. And the difficulty to deal with it lies in the evaluation of expectation $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$.

2 Novel upper bounds of $\mathrm{E}((y_0+\boldsymbol{y}^{\mathrm{T}}\tilde{z})^+)$ and joint chance constraints approximations

In this section, we give two new bounds on $\mathrm{E}((\cdot)^+)$, analyze the relationship between the approximation of individual chance constraints and the robust optimization, and then derive two new approximating formulations for joint chance constraints.

2.1 The upper bounds of $\mathrm{E}((y_0+\boldsymbol{y}^{\mathrm{T}}\tilde{z})^+)$

Theorem 1. Suppose the primitive uncertainties $\{\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_K\}$ have zero means. The following functions, (6) and (7), are the upper bounds of $\mathrm{E}((y_0+\boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$, where $\boldsymbol{y} = (y_1, y_2, \cdots, y_K)^{\mathrm{T}}, \tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_K)^{\mathrm{T}}$

$$\begin{aligned}&\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+) \leq \pi_e(y_i^0, \boldsymbol{y}_i) = \\ &\quad \frac{1}{2}(y_i^0 + \sqrt{(y_i^0)^2 + \sum_k (z_k)^2 (y_i^k)^2}) \end{aligned} \tag{6}$$

$$\begin{aligned}&\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+) \leq \pi_p(y_i^0, \boldsymbol{y}_i) = \\ &\quad \frac{1}{2}(y_i^0 + |y_i^0|) + \frac{1}{2}\sum_k |y_i^k z_k| \end{aligned} \tag{7}$$

Proof. For the first bound $\pi_e(y_i^0, \boldsymbol{y}_i))$

$$\begin{aligned}&\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+) = \\ &\quad \mathrm{E}((y_i^0 + \sum_k y_i^k \tilde{z}_k)^+) \overset{\text{(a)}}{=} \\ &\quad \frac{1}{2}(y_i^0 + \mathrm{E}(|y_i^0 + \sum_k y_i^k \tilde{z}_k|)) \overset{\text{(b)}}{=} \\ &\quad \frac{1}{2}(y_i^0 + \sqrt{(y_i^0)^2 + \sum_k (\sigma_k^2 (y_i^k)^2)}) \overset{\text{(c)}}{\leq} \\ &\quad \frac{1}{2}(y_i^0 + \sqrt{(y_i^0)^2 + \sum_k (z_k^2 (y_i^k)^2)}) = \\ &\quad \pi_e(y_i^0, \boldsymbol{y}_i)\end{aligned}$$

where σ_k^2 is the variance of $\tilde{z}_k$, (a) and (b) uses Jensen′s inequality and the relation, $w^+ = (w + |w|)/2$, and formula $(\frac{1}{2}(y_i^0 + \sqrt{(y_i^0)^2 + \sum_k (\sigma_k^2 (y_i^k)^2)}))$ is the third bound in paper [12], (c) is because $\sigma_k^2 \leq z_k^2$.

So $\pi_e(y_i^0, \boldsymbol{y}_i)$ is an upper bound of $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$.

For the second bound $\pi_p(y_i^0, \boldsymbol{y}_i)$,

$$\begin{aligned}&\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+) = \\ &\quad \mathrm{E}((y_i^0 + \sum_k y_i^k \tilde{z}_k)^+) \overset{\text{(a)}}{=} \\ &\quad \frac{1}{2}(y_i^0 + \mathrm{E}(|y_i^0 + \sum_k y_i^k \tilde{z}_k|)) \leq \\ &\quad \frac{1}{2}(y_i^0 + \mathrm{E}(|y_i^0| + \sum_k |y_i^k \tilde{z}_k|)) \leq \\ &\quad \frac{1}{2}(y_i^0 + |y_i^0| + \mathrm{E}(\sum_k |y_i^k \tilde{z}_k|)) \leq \\ &\quad \frac{1}{2}(y_i^0 + |y_i^0|) + \frac{1}{2}\sum_k |y_i^k \tilde{z}_k| = \\ &\quad \pi_p(y_i^0, \boldsymbol{y}_i)\end{aligned}$$

where (a) uses relationship $w^+ = (w + |w|)/2$.

So $\pi_p(y_i^0, \boldsymbol{y}_i)$ is also an upper bound of $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$. □

2.2 Relationship with robust optimization

Using our proposed bounds of $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$ and equation (4), we can get two novel approximations of individual chance constraints. We all know the robust optimization can be used to approximate the individual chance constraints, and different bounds of $\mathrm{E}((\cdot)^+)$ correspond to different uncertain sets[13]. In this section, we study the relationship between our approximations and the robust optimization with the corresponding uncertain sets.

Corollary 1. By defining

$$\eta_{1-\varepsilon}(y_i^0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}}) := \min_{\beta}\{\beta + \frac{1}{\varepsilon}\pi(y_0 - \beta, \boldsymbol{y}_i)\}$$

then the following equations (8) and (9) hold,

$$\begin{aligned}&\eta_{1-\varepsilon}^e(y_i^0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}}) = \\ &\quad \min_{\beta}\{\beta + \frac{1}{\varepsilon}\pi(y_i^0 - \beta, \boldsymbol{y}_i)\} = \\ &\quad y_i^0 + \max_{z \in u_{ellipsoidal}}\{\sum_k y_i^k \tilde{z}_k\}\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\eta_{1-\varepsilon}^p(y_i^0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}}) = \\ &\quad \min_{\beta}\{\beta + \frac{1}{\varepsilon}\pi(y_i^0 - \beta, \boldsymbol{y}_i)\} = \\ &\quad y_i^0 + \max_{z \in u_{polyhedral}}\{\sum_k y_i^k \tilde{z}_k\}\end{aligned} \tag{9}$$

where

$$u_{ellipsoidal} = \{\tilde{\boldsymbol{z}} : \|\boldsymbol{Z}^{-1}\tilde{\boldsymbol{z}}\|_2 \leq \sqrt{\frac{1-\varepsilon}{\varepsilon}}\}$$

$$\boldsymbol{Z} = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_k \end{pmatrix}$$

and

$$u_{polyhedral} = \{\tilde{\boldsymbol{z}} : \|\boldsymbol{Z}^{-1}\tilde{\boldsymbol{z}}\|_1 \leq \frac{k}{2\varepsilon}\}$$

Proof. For the first bound $\pi_e(y_i^0, \boldsymbol{y}_i)$,

$$\begin{aligned}&\eta_{1-\varepsilon}^e(y_i^0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}}) = \\ &\quad \min_{\beta}\{\beta + \frac{1}{\varepsilon}\pi(y_i^0 - \beta, \boldsymbol{y}_i)\} =\end{aligned}$$

$$\min_{\beta}\{\beta + \frac{1}{2\varepsilon}(y_i^0 - \beta) + \frac{1}{2\varepsilon}\sqrt{(y_i^0-\beta)^2 + \sum_k z_k^2 (y_i^k)^2}\} =$$
$$y_i^0 + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{\sum_k z_k^2 (y_i^k)^2}$$

where the last equality comes from choosing the optimum β^*,

$$\beta^* = y_i^0 + \frac{\sqrt{\sum_k (z_k)^2 (y_i^k)^2}(1-2\beta)}{2\sqrt{\beta(1-\beta)}}$$

And the last formula has been proved to be equivalent to the robust counterpart under ellipsoidal uncertainty set $u_{ellipsoidal}$ by Zukui[9]. The adjustable parameter controlling the size of the uncertain set is

$$\sqrt{\frac{1-\varepsilon}{\varepsilon}}$$

which is denoted as $\boldsymbol{\Omega}$.

That is,

$$\eta_{1-\varepsilon}^e (y_i^0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}}) = y_i^0 + \max_{z\in u_{ellipsoidal}} \{\sum_k y_i^k \tilde{z}_k\} \tag{10}$$
$$u_{ellipsoidal} = \{\tilde{\boldsymbol{z}} : \|\boldsymbol{Z}^{-1}\tilde{\boldsymbol{z}}\|_2 \le \sqrt{\frac{1-\varepsilon}{\varepsilon}}\}$$

where

$$\boldsymbol{Z} = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_k \end{pmatrix}$$

Applying the first bound $\pi_e(y_i^0, \boldsymbol{y}_i)$, we can also give the first approximation for the individual chance constrains (2) as following:

$$y_i^0 + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{\sum_k z_k^2 (y_i^k)^2} \le 0 \tag{11}$$

For the second bound,

$$\eta_{1-\varepsilon}^p (y_i^0 + \sum_k y_i^k \tilde{z}_k) =$$
$$\min_{\beta}\{\beta + \frac{1}{\varepsilon}\mathrm{E}((y_i^0 + \sum_k y_i^k \tilde{z}_k - \beta)^+)\} \le$$
$$\min_{\beta}\{\beta + \frac{1}{\varepsilon}\pi_p(y_i^0 - \beta, \boldsymbol{y}_i)\} =$$
$$\min_{\beta}\{\beta + \frac{1}{2\varepsilon}(y_i^0 - \beta) + \frac{1}{2\varepsilon}|y_i^0 - \beta| + \frac{1}{2\varepsilon}\sum_k |y_i^k z_k|\} \overset{(a)}{=}$$
$$y_i^0 + \frac{1}{2\varepsilon}\sum_k |y_i^k z_k| \le$$
$$y_i^0 + \frac{k}{2\varepsilon}\max_k |y_i^k z_k| \overset{(t=\max|y_i^k z_k|, \forall k)}{=}$$
$$y_i^0 + \frac{k}{2\varepsilon}t \overset{(t\ge|y_i^k z_k|, \forall k)}{=}$$
$$y_i^0 + \frac{k}{2\varepsilon}t$$

Equation (a) comes from choosing the optimum β^*, $\beta^* = y_i^0$. And the last formula has also been proved to be equivalent to the robust counterpart under polyhedral uncertainty set $u_{polyhedral}$ by Zukui[9]. The adjustable parameter controlling the size of the uncertain set is $k/2\varepsilon$, which we denote as $\boldsymbol{\Gamma}$.

That is

$$\eta_{1-\varepsilon}^p (y_i^0 + \sum_k y_i^k \tilde{z}_k) = y_i^0 + \max_{z\in u_{polyhedral}} \sum_k y_i^k \tilde{z}_k \tag{12}$$
$$u_{polyhedral} = \{\tilde{\boldsymbol{z}} : \|\boldsymbol{Z}^{-1}\tilde{\boldsymbol{z}}\|_1 \le \frac{k}{2\varepsilon}\}$$

Applying the second bound $\pi_p(y_i^0, \boldsymbol{y}_i)$, we can also give the second approximation for the individual chance constrains (2) as following:

$$\begin{aligned} & y_i^0 + \frac{k}{2\varepsilon}t \le 0 \\ & t \ge |y_i^k z_k|, \quad \forall k \end{aligned} \tag{13}$$

where t is an auxiliary variable.

2.3 Approximations to joint constraints

Applying the upper bounds $\pi_e(y_i^0, \boldsymbol{y}_i)$ and $\pi_p(y_i^0, \boldsymbol{y}_i)$ of $\mathrm{E}((y_0 + \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{z}})^+)$ to formula (5), we can obtain two new approximations to joint chance constrains.

For the first bound $\pi_e(y_i^0, \boldsymbol{y}_i)$, given a vector of positive constants, $\boldsymbol{\alpha} \in \mathbf{R}^L$ (set of L dimensional real vector), $\boldsymbol{\alpha} \ge 0$, the joint chance constrains can be treated as following,

$$\begin{aligned} & \min_{w_0, \boldsymbol{w}} \{\min_{\beta}(\beta + \frac{1}{2\varepsilon}(w_0 - \beta + \sqrt{(w_0-\beta)^2 + \boldsymbol{w}^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{w}})) + \\ & \frac{1}{2\varepsilon}\sum_{i\in J}(\alpha_i y_i^0 - w_0 + \\ & \sqrt{(\alpha_i y_i^0)^2 + (\alpha_i \boldsymbol{y}_i - \boldsymbol{w})^{\mathrm{T}}\boldsymbol{Z}^2(\alpha_i \boldsymbol{y}_i - \boldsymbol{w})})\} \le 0 \end{aligned} \tag{14}$$

For the second bound $\pi_p(y_i^0, \boldsymbol{y}_i)$, the joint chance constrains can be treated as following,

$$\begin{aligned} & \min_{w_0, \boldsymbol{w}} \{\min_{\beta}(\beta + \frac{1}{2\varepsilon}(w_0 - \beta + |w_0 - \beta| + \sum_k |w_k z_k|)) + \\ & \frac{1}{2\varepsilon}\sum_{i\in J}(\alpha_i y_i^0 - w_0 + |\alpha_i y_i^0 - w_0| + \\ & \sum_k |(\alpha_i y_i^k - w_k) z_k|)\} \le 0 \end{aligned} \tag{15}$$

3 Computational studies

Example. Consider the following LP problems

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ \text{s.t.} \quad & \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \le 140 + \tilde{b}_1 \\ & \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \le 72 + \tilde{b}_2 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{16}$$

The optimal result of the nominal problem is 100 ($x_1^* = 8, x_2^* = 3$). We assume the coefficients $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{b}_1, \tilde{b}_2$ as follows:

$$\tilde{a}_{11} = 10 + 0.5\tilde{z}_1 + 0.5\tilde{z}_2, \quad \tilde{a}_{12} = 20 + \tilde{z}_1 + \tilde{z}_2$$
$$\tilde{a}_{21} = 6 + 0.2\tilde{z}_1 + 0.4\tilde{z}_2, \quad \tilde{a}_{22} = 8 + 0.5\tilde{z}_1 + 0.3\tilde{z}_2$$
$$\tilde{b}_1 = 140 + 10\tilde{z}_1 + 4\tilde{z}_2, \quad \tilde{b}_2 = 72 + 3\tilde{z}_1 + 4.2\tilde{z}_2$$

where $\tilde{z}_1, \tilde{z}_2$ are independent zero mean variables with unknown distributions and $\tilde{z}_1, \tilde{z}_2 \in [-1, 1]$.

We can know the maximum perturbation ranges of the coefficients $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{b}_1, \tilde{b}_2$ are $\pm 10\,\%$ of their nominal values which are consistent with the individual constraints and independent coefficients examples. The LP problem (16) can be reformulated as (17):

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ \text{s.t.} \quad & y_1^0 + y_1^1\tilde{z}_1 + y_1^2\tilde{z}_2 \le 0 \\ & y_2^0 + y_2^1\tilde{z}_1 + y_2^2\tilde{z}_2 \le 0 \\ & y_1^0 = 10x_1 + 20x_2 - 140 \\ & y_2^0 = 6x_1 + 8x_2 - 72 \\ & y_1^1 = 0.5x_1 + x_2 - 10 \\ & y_1^2 = 0.5x_1 + x_2 - 4 \\ & y_2^1 = 0.2x_1 + 0.5x_2 - 3 \\ & y_2^2 = 0.4x_1 + 0.3x_2 - 4.2 \end{aligned} \tag{17}$$

In the LP problem (17), the first two constraints have the same uncertain parameters. The individual chance constraints can be expressed respectively as (18) and the joint chance constraints can be expressed respectively as (19).

$$\begin{array}{l} P(y_1^0 + y_1^1\tilde{z}_1 + y_1^2\tilde{z}_2 \le 0) \ge 1 - \varepsilon_1 \\ P(y_2^0 + y_2^1\tilde{z}_1 + y_2^2\tilde{z}_2 \le 0) \ge 1 - \varepsilon_2 \end{array} \tag{18}$$

$$P\begin{pmatrix} y_1^0 + y_1^1\tilde{z}_1 + y_1^2\tilde{z}_2 \le 0) \\ y_2^0 + y_2^1\tilde{z}_1 + y_2^2\tilde{z}_2 \le 0) \end{pmatrix} \ge 1 - \varepsilon \tag{19}$$

Applying our approximations in equations (11) and (13) for individual chance constraints (18), and in equations (14) and (15) for joint chance constraint (19). For simplicity, we assume $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$. The other constraints can be processed normally. The formulation is solved by Gams. In this example, parameter $k = 2$. For the same ε, the relationship between Ω and Γ can be showed in Fig. 1. The solution of the example is shown in Fig. 2.

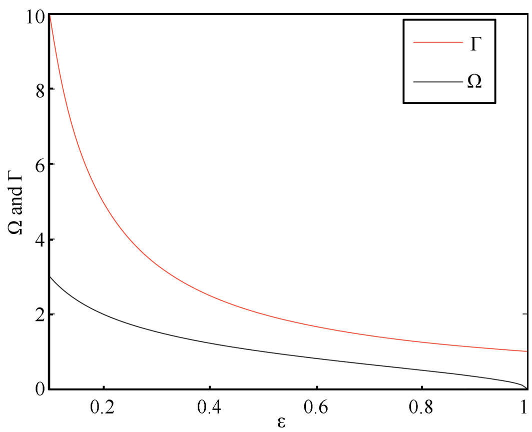

Fig. 1 Comparison of Ω and Γ in different values of ε

Based on the solution, the following remark can be made.

1) From the illustration in Fig. 1, it can be observed that for the same ε, $\Omega > \Gamma$, and the ellipsoidal uncertain set is entirely covered by the polyhedral uncertain set. Because the ellipsoidal uncertain set corresponds to the first bound $\pi_e(y_i^0, \boldsymbol{y}_i)$ and the first approximation for individual chance constrain (11), and the polyhedral uncertain set corresponds to the second bound $\pi_p(y_i^0, \boldsymbol{y}_i)$ and the second approximation for individual chance constrains (13), so our approximation formulation (13) should be more conservative than formulation (11).

2) From Fig. 2, it can be observed that, for the same ε, the solution of the first approximation model for individual chance constrains problem is always better than the second approximation model.

3) From Fig. 2, it can also be observed that, for the same ε, the solution of the first approximation model for joint chance constrains is always better than the second approximation model.

4) From Fig. 2, it can also be observed that, for the same uncertain set, ellipsoidal or polyhedral, which corresponding to the upper bound $\pi_e(y_i^0, \boldsymbol{y}_i)$ and $\pi_p(y_i^0, \boldsymbol{y}_i)$, the approximation model for joint chance constrains problem is always better than the approximation model for individual chance constrains problem.

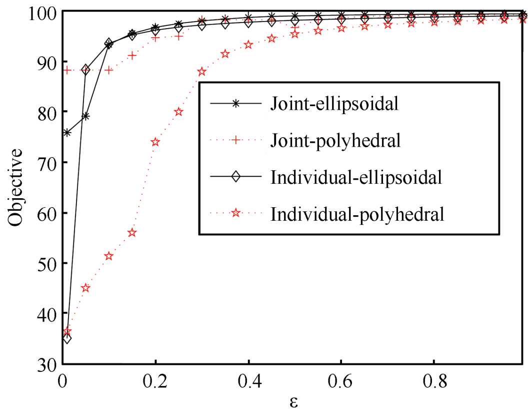

Fig. 2 Solutions to individual and joint chance-constrained problem approximations with different $\mathrm{E}((\cdot)^+)$ upper bounds

4 Conclusion

In this paper, we propose two approximate formulations for joint chance constrains problem. The key of this methodology is to obtain the upper bound of $\mathrm{E}((\cdot)^+)$. After reviewing the relationship among $\mathrm{E}((\cdot)^+)$, CVaR, individual chance constraints and joint chance constraints, two new $\mathrm{E}((\cdot)^+)$ upper bound are proposed, and the approximations of individual chance constrains are developed, which are then shown to be the robust optimization with corresponding uncertain sets. Then this methodology are extended to joint chance constrains problems. The different approximation formulations for individual and joint chance constrains are compared through a numerical study. The results show that our approximation for joint chance-constrained problem can decrease the conservation and give better solution than approaches using Bonferronis inequality. This approximation technology can also be used in many practical optimization problems such as resource allocation, supply chains, and production planning.

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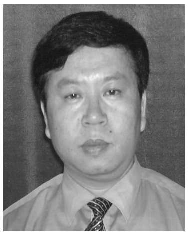

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