A Systematic Analysis Approach to Discrete-time Indirect Model Reference Adaptive Control

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Abstract This paper presents the design and analysis of indirect model reference adaptive control (MRAC) with normalized adaptive law for a class of discrete-time systems. The main work includes three parts. Firstly, the constructed plant parameter estimation algorithm not only possesses the same properties as those of traditional estimation algorithms but also avoids the possibility of division by zero. Secondly, by finding the relationship between the plant parameter estimate and controller parameter estimate and using the properties of plant parameter estimate, the similar properties of controller parameter estimate are also established. Thirdly, based on the relationship properties between the normalizing signal and all the signals in the closed-loop system and on some important mathematical tools on discrete-time systems, as in the continuous-time case, a systematic stability and convergence analysis approach to the discrete indirect MRAC scheme is developed rigorously.

Key words Discrete-time systems, indirect, model reference adaptive control (MRAC), normalized adaptive law

1 Introduction

During the last two decades, for linear continuous-time systems, the "certainty equivalence" adaptive controllers with normalized adaptive laws have dominated the literature of adaptive control due to the simplicity of the design as well as the robustness properties in the presence of modeling errors, see the widely cited in-depth monographes of [1∼5], and references therein. These controllers are obtained by independently designing a control law that meets the control objective assuming knowledge of all parameters, along with an adaptive law that generates on-line parameter estimates that are used to replace the unknowns in the control law. The normalized adaptive law could be a gradient or least squares algorithm. The control law is usually based on polynomial equalities resulting from a model reference or pole assignment objective based on linear systems theory. An important feature of this class of adaptive controllers is the use of error normalization, which allows complete separation of the adaptive and control law designs. Using the properties of $L_{2\delta}$-norm, the swapping lemmas and the Bellman-Gronwall Lemma, a more elaborate yet more systematic method is given in the analysis of adaptive control schemes.

To the best of our knowledge, the first analogous theoretical result for the discrete-time systems appears to have been given by [6], where the internal model control (IMC) implementation is used for the extended horizon adaptive control scheme. This work is somewhat successful in the sense that the ideal-case stability is established by exploiting the parameter convergence property (not to the true values) of the 'pure' least-squares algorithm. In [7], the adaptive IMC in the presence of modeling errors was further considered. Very recently, [8] studied the design and analysis of discrete direct model reference adaptive control (MRAC) with normalized adaptive laws in a systematic manner as in the continuous-time case.

It is well known that according to the difference of the estimated parameters, MRAC schemes can be characterized as direct or indirect and with normalized or unnormalized adaptive laws. As discussed in [3, 9], the indirect MRAC scheme has certain advantages over the corresponding direct scheme. Therefore, the purpose of this paper is to study the discrete indirect MRAC with normalized adaptive law in a systematic manner. Our main work consists of the following aspects: Firstly, the constructed plant parameter estimation algorithm not only possesses the same properties as those of traditional estimation algorithms but also avoids the possibility of division by zero. Secondly, by finding the relationship between the plant parameter estimate and controller parameter estimate and using the properties of plant parameter estimate, the similar properties of controller parameter estimate are also established. Thirdly, based on the relationship properties between the normalizing signal and all the signals in the closed-loop system, and some important mathematical tools on discrete-time systems, as in the continuous-time counterpart, a systematic stability and convergence analysis approach to the discrete indirect MRAC scheme is developed rigorously.

2 Problem statement

Consider the discrete-time linear time-invariant plant studied in [1]

$$y(t) = G_p(z) = \frac{\bar{Z}_p(z)}{R_p(z)}[u](t) = \frac{k_p Z_p(z)}{R_p(z)}[u](t) \tag{1}$$

where $u(t), y(t) \in \mathbf{R}$ are the plant input and output, respectively, $t \in \{0, 1, 2, \cdots\}$, $R_p(z) = z^n + \sum_{i=0}^{n-1} a_i^* z^i$, $\bar{Z}_p(z) = \sum_{j=0}^{m} b_j^* z^j$ with $b_m^* = k_p$, a_i^* and b_j^* being unknown constant parameters. The symbol z is used to denote the z-transform variable or time advance operator with the definition of $z[x](t) = x(t+1)$, *i.e.*, z^{-1} is the time delay operator $z^{-1}[x](t) = x(t-1)$.

The control objective is to develop an indirect adaptive control scheme such that all the signals in the closed-loop plant are uniformly bounded and the tracking error $e(t) = y(t) - y_m(t) \to 0$ as $t \to \infty$ for the following given reference output y_m,

$$y_m(t) = W_m(z)[r](t) = \frac{k_m Z_m(z)}{R_m(z)}[r](t) \tag{2}$$

where r is the reference input, which is assumed to be uniformly bounded.

We need the following assumptions.

Plant assumptions:

Assumption 1. $Z_p(z)$ is stable, *i.e.*, all zeros of $Z_p(z)$ are inside the unit circle of complex z-plant.

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Assumption 2. n, m, and the relative degree $n^* = n - m \geq 1$ are known.

Assumption 3. The sign of k_p is known, and there exists a known constant $\underline{k}_p > 0$ such that $|k_p| \geq \underline{k}_p$.

Reference model assumption:

M 1. $Z_m(z)$ and $R_m(z)$ are monic stable polynomials of degrees q_m and p_m, respectively, where $p_m \leq n$, and $W_m(z)$ has the same relative degree as that of (1).

Remark 1. By Assumption 1 and M 1, there exists a constant $\delta \in (0, 1]$ such that $G_p^{-1}(z)$ and $W_m(z)$ are analytic in $|z| \geq \sqrt{\delta}$ (i.e., $G_p^{-1}(\sqrt{\delta}z)$ and $W_m(\sqrt{\delta}z)$ are analytic in $|z| \geq 1$ using Lemma 3 in [6]). It is worth emphasizing that such a δ is only used in the stability analysis of closed-loop system but not in the practical applications due to the adaptive controller form (4), (5), (13)∼(16).

Notation 1. For any $t \in \{0, 1, 2, \cdots\}$, define the time increment of $\boldsymbol{x}(t)$ as $\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t+1) - \boldsymbol{x}(t)$, and the discrete-time L_2, L_{2e}, and $L_{2\delta}$ norms of $\boldsymbol{x}(t)$ as $\|\boldsymbol{x}\|_2 = (\sum_{i=0}^{\infty} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{x}(i))^{1/2}$, and $\|\boldsymbol{x}_t\|_{2e} = (\sum_{i=0}^{t} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{x}(i))^{1/2}$, $\|\boldsymbol{x}_t\|_{2\delta} = (\sum_{i=0}^{t} \delta^{t-i}\boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{x}(i))^{1/2}$, where δ is the same as that of Remark 1. The time advance operator vector $\boldsymbol{\alpha}_k(z) = [z^k, z^{k-1}, \cdots, z, 1]^{\mathrm{T}}$ for any $k = 0, 1, \cdots$. For simplicity, we sometimes denote any time function $\boldsymbol{x}(t)$ by $\boldsymbol{x}$, and z-transform operator polynomial $X(z)[\boldsymbol{x}](t)$ whose signification can be referred to Tao[1] by $X(z)[\boldsymbol{x}]$ or $X(z)\boldsymbol{x}$. c denotes any positive constant. Let $X(z,t) = \sum_{i=0}^{n} x_i(t)z^i$, $Y(z,t) = \sum_{j=0}^{m} y_j(t)z^j$ be any two left polynomial time advance operators polynomials with time-varying coefficients $x_i(t)$ and $y_j(t)$, define the algebraic product between them as $X(z,t) \cdot Y(z,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} x_i(t)y_j(t)z^{i+j}$. Obviously $X(z,t) \cdot Y(z,t) = Y(z,t) \cdot X(z,t)$.

3 Discrete indirect MRAC with normalized adaptive law

In this section, the design of discrete indirect MRAC with normalized adaptive law is presented. It is easy to denote (1) as the following linear parametric model

$$\xi(t) = \boldsymbol{\theta}_p^{*\mathrm{T}}\boldsymbol{\phi}(t) \tag{3}$$

where $\xi = (z^n/\Lambda_p(z))y$, $\boldsymbol{\theta}_p^* = [b_m^*, \cdots, b_0^*, a_{n-1}^*, \cdots, a_0^*]^{\mathrm{T}}$, $\boldsymbol{\phi} = [(\boldsymbol{\alpha}_m^{\mathrm{T}}(z)/\Lambda_p(z))u, -(\boldsymbol{\alpha}_{n-1}^{\mathrm{T}}(z)/\Lambda_p(z))y]^{\mathrm{T}}$, $\Lambda_p(z)$ is an arbitrary monic stable polynomial of degree n and $1/\Lambda_p(z)$ is analytic in $|z| \geq \sqrt{\delta}$ for the same δ as in Remark 1. Since $\boldsymbol{\theta}_p^*$ is unknown, we first give an adaptive estimation algorithm for $\boldsymbol{\theta}_p^*$. Let $\boldsymbol{\theta}_p(t) = [b_m(t), \cdots, b_0(t), a_{n-1}(t), \cdots, a_0(t)]^{\mathrm{T}}$ be the estimate of $\boldsymbol{\theta}_p^*$ at time t, $t \in \{0, 1, 2, \cdots\}$, define the normalized estimation error ε as

$$\begin{aligned} \varepsilon(t) &= \frac{\xi(t) - \boldsymbol{\theta}_p^{\mathrm{T}}(t)\boldsymbol{\phi}(t)}{m^2(t)} = \frac{-\tilde{\boldsymbol{\theta}}_p(t)^{\mathrm{T}}\boldsymbol{\phi}(t)}{m^2(t)} \\ m^2(t) &= 1 + \boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\phi}(t) \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{\theta}}_p(t) = \boldsymbol{\theta}_p(t) - \boldsymbol{\theta}_p^*$. The estimation algorithm for $\boldsymbol{\theta}_p(t)$ is given by

$$\begin{aligned} &\boldsymbol{\theta}_p(t+1) = \bar{\boldsymbol{\theta}}_p(t+1) + \boldsymbol{\Delta}(t+1) \\ &\bar{\boldsymbol{\theta}}_p(t+1) = \boldsymbol{\theta}_p(t) + \Gamma\boldsymbol{\phi}(t)\varepsilon(t) \\ &\boldsymbol{\Delta}(t+1) = \begin{cases} 0, & \bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) \geq \underline{k}_p \\ \dfrac{\boldsymbol{\tau}_1}{\tau_2}\left(\underline{k}_p\mathrm{sgn}(k_p) - \bar{\theta}_{p1}(t+1)\right), & \text{otherwise} \end{cases} \end{aligned} \tag{5}$$

where $\Gamma = \mathrm{diag}\{\lambda_1, \cdots, \lambda_{n+m+1}\}$ is an adaptive gain matrix with $0 < \lambda_i < 2$, $i = 1, 2, \cdots, n+m+1$, $\bar{\theta}_{p1}(t+1)$ is the first element of $\bar{\boldsymbol{\theta}}_p(t+1)$, $\boldsymbol{\tau}_1$ is the first column of Γ, τ_2 is the first element of $\boldsymbol{\tau}_1$. The estimation algorithm (5) has the following properties.

Lemma 1. The estimation algorithm (5) guarantees that for all $t \in \{0, 1, \cdots\}$,

1) If $|\theta_{p1}(0)| \geq \underline{k}_p$ and the sign of $\theta_{p1}(0)$ is the same as that of k_p, then $|\theta_{p1}(t)| = |\hat{k}_p(t)| \geq \underline{k}_p$

2) $\boldsymbol{\theta}_p(t)$, $\varepsilon(t)$, $\varepsilon(t)m(t) \in L_\infty$

3) $\Delta\boldsymbol{\theta}_p(t)$, $\varepsilon(t)$, $\varepsilon(t)m(t) \in L_2$

where $\theta_{p1}(0)$ is the initial estimate of $\{\theta_{p1}(t)\}$, $\theta_{p1}(t)$ is the first element of $\boldsymbol{\theta}_p(t)$, and $\hat{k}_p(t)$ is the estimate of k_p. Obviously, $\theta_{p1}(t) = b_m(t) = \hat{k}_p(t)$.

Proof. See the appendix. □

Remark 2. By carefully constructing (5), the plant parameter estimation not only possesses the same properties 2) ∼ 3) as those of traditional adaptive estimation algorithms, but also guarantees $|\hat{k}_p(t)| \geq \underline{k}_p$, which is essential to avoid the possibility of division by zero in (14) ∼ (16).

By (5), one can obtain the estimation polynomials $\hat{\bar{Z}}_p(z,t)$ and $\hat{R}_p(z,t)$ for $\bar{Z}_p(s)$ and $R_p(s)$, respectively,

$$\begin{aligned} \hat{\bar{Z}}_p(z,t) &= b_m(t)z^m + \cdots + b_0(t) \\ \hat{R}_p(z,t) &= z^n + a_{n-1}(t)z^{n-1} + \cdots + a_0(t) \end{aligned} \tag{6}$$

As one does in the continuous-time case, assuming that $\boldsymbol{\theta}_p^*$ is known, the controller is chosen as

$$u = \boldsymbol{\theta}_c^{*\mathrm{T}}\boldsymbol{w} \tag{7}$$

where $\boldsymbol{\theta}_c^* = [\boldsymbol{\theta}_{c1}^{*\mathrm{T}}, \boldsymbol{\theta}_{c2}^{*\mathrm{T}}, \theta_{c3}^*, \theta_{c4}^*]^{\mathrm{T}}$, $\boldsymbol{\omega} = [\boldsymbol{\omega}_1^{\mathrm{T}}, \boldsymbol{\omega}_2^{\mathrm{T}}, y, r]^{\mathrm{T}}$, $\boldsymbol{\theta}_1^*, \boldsymbol{\theta}_2^* \in R^{n-1}$, $\theta_3, \theta_4 \in R$, $\boldsymbol{\omega}_1 = (\boldsymbol{\alpha}_{n-2}^{\mathrm{T}}(z)/\Lambda(z))u$, $\boldsymbol{\omega}_2 = (\boldsymbol{\alpha}_{n-2}^{\mathrm{T}}(z)/\Lambda(z))y$, $\Lambda(z)$ is an arbitrary monic stable polynomial of degree $n-1$ with $\Lambda(z) = \Lambda_0(z)Z_m(z)$, and $1/\Lambda(z)$ is analytic in $|z| \geq \sqrt{\delta}$ for the same δ as that of Remark 1. Without loss of generality, choose $\Lambda_p(z) = \Lambda(z)(z + \lambda_0)$ with $|\lambda_0| < \sqrt{\delta}$. Using the matching equations

$$\begin{aligned} &\theta_{c4}^* = \frac{k_m}{k_p} \\ &\boldsymbol{\theta}_{c1}^{*\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z)R_p(z) + (\boldsymbol{\theta}_{c2}^{*\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z) + \theta_{c3}^*\Lambda(z))k_pZ_p(z) = \\ &\Lambda(z)R_p(z) - R_m(z)\Lambda_0(z)Z_p(z) \end{aligned} \tag{8}$$

$y = y_m$ can be easily achieved. The existence of θ_c^* can be guaranteed by [3]. Similar to equation (6.6.24) in [3], the controller parameters are calculated by

$$\theta_{c4}^* = \frac{k_m}{k_p} \tag{9}$$

$$\boldsymbol{\theta}_{c1}^{*\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z) = \Lambda(z) - \frac{1}{k_p}\bar{Z}_p(z)Q(z) \tag{10}$$

$$\boldsymbol{\theta}_{c2}^{*\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z) + \theta_{c3}^*\Lambda(z) = \frac{1}{k_p}\Big(Q(z)R_p(z) - \Lambda_0(z)R_m(z)\Big) \tag{11}$$

where $Q(z)$ is the quotient of $\Lambda_0(z)R_m(z)/R(z)$. Applying (8) to the signal $u/\Lambda(z)R_p(z)$, it is easy to obtain that $u - \boldsymbol{\theta}_{c1}^{*\mathrm{T}}\boldsymbol{\omega}_1 - \boldsymbol{\theta}_{c2}^{*\mathrm{T}}\boldsymbol{\omega}_2 - \theta_{c3}^*y = \theta_{c4}^*W_m^{-1}(z)y$. Subtracting both sides of this equation by θ_{c4}^*r with $r = W_m^{-1}(z)y_m$ and $e = y - y_m$, one obtains the parametric model on $\boldsymbol{\theta}_c^*$

$$e = W_m(z)\frac{1}{\theta_{c4}^*}\Big(u - \boldsymbol{\theta}_c^{*\mathrm{T}}\boldsymbol{\omega}\Big) \tag{12}$$

Since $\boldsymbol{\theta}_p^*$ is unknown, obviously from (9) ∼ (11), the controller parameter $\boldsymbol{\theta}_c^*$ is also unknown. Hence, by (12), the certainly equivalence adaptive control law is chosen as

$$u(t) = \boldsymbol{\theta}_c^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \tag{13}$$

and the estimate $\boldsymbol{\theta}_c = [\boldsymbol{\theta}_{c1}^{\mathrm{T}}, \boldsymbol{\theta}_{c2}^{\mathrm{T}}, \theta_{c3}, \theta_{c4}]^{\mathrm{T}}$ is calculated by

$$\theta_{c4}(t) = \frac{k_m}{\hat{k}_p(t)} \tag{14}$$

$$\boldsymbol{\theta}_{c1}^{\mathrm{T}}(t)\boldsymbol{\alpha}_{n-2}(z) = \Lambda(z) - \frac{1}{\hat{k}_p(t)}\hat{\bar{Z}}_p(z,t) \cdot \hat{Q}(z,t) \tag{15}$$

$$\boldsymbol{\theta}_{c2}^{\mathrm{T}}(t)\boldsymbol{\alpha}_{n-2}(z) + \theta_{c3}(t)\Lambda(z) = \frac{1}{\hat{k}_p(t)}\Big(\hat{Q}(z,t) \cdot \hat{R}_p(z,t) - \Lambda_0(z)R_m(z)\Big) \tag{16}$$

where $|\hat{k}_p(t)| > 0$ is guaranteed by Lemma 1 1), $\hat{Q}(z,t)$ is the quotient of $\Lambda_0(z)R_m(z)/\hat{R}(z,t)$. Noting $\Lambda(z) = \Lambda_0(z)Z_m(z)$, obviously,

$$\hat{Q}(z,t) = \boldsymbol{q}^{\mathrm{T}}(t)\boldsymbol{\alpha}_{n^*-1}(z) \tag{17}$$

where $\boldsymbol{q}(t) = [q_{n^*-1}(t), \cdots, q_1(t), q_0(t)]^{\mathrm{T}}$ and $q_{n^*-1} = 1$. By Lemma 1, one gets the following properties on $\boldsymbol{\theta}_c(t)$.

Lemma 2. $\boldsymbol{\theta}_c(t)$ and $\boldsymbol{q}(t)$ have the following properties for all $t \in \{0, 1, 2, \cdots\}$,

1) $\boldsymbol{q}(t) \in L_\infty$, $\Delta\boldsymbol{q}(t) \in L_2$;

2) $\boldsymbol{\theta}_c(t) \in L_\infty$, $\Delta\boldsymbol{\theta}_c(t) \in L_2$.

Proof. See the appendix. □

4 Main results

Define a fictitious normalizing signal m_f as

$$m_f^2(t) = 1 + \|u_{t-1}\|_{2\delta}^2 + \|y_{t-1}\|_{2\delta}^2 \tag{18}$$

where $\|(\cdot)_t\|_{2\delta}$ is the same as defined in Notation 1. The relationship properties between m_f and all the signals in the closed-loop plant are established by the following lemma.

Lemma 3. For the discrete indirect MRAC scheme consisting of plant (1), the reference model (2), the adaptive law (5), and the control law (13) with $\boldsymbol{\theta}_c$ satisfying (14) ∼ (16), if Assumptions 1 ∼ 3 and M 1 hold, then

1) $\boldsymbol{\omega}_1(t)/m_f(t)$, $\boldsymbol{\omega}_2(t)/m_f(t)$, $\|(\boldsymbol{\omega}_1)_{t-1}\|_{2\delta}/m_f(t)$, $\|(\boldsymbol{\omega}_2)_{t-1}\|_{2\delta}/m_f(t)$, $\|\boldsymbol{\omega}_{t-1}\|_{2\delta}/m_f(t) \in L_\infty$;

2) $u(t)/m_f(t)$, $y(t)/m_f(t)$, $\boldsymbol{\omega}(t)/m_f(t)$, $\boldsymbol{\omega}_p(t)/m_f(t)$, $\|(\boldsymbol{\omega}_p)_{t-1}\|_{2\delta}/m_f(t)$, $W(z)\boldsymbol{\omega}(t)/m_f(t)$, $W(z)\boldsymbol{\omega}_p(t)/m_f(t)$, $W(z)\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}(t)\boldsymbol{\omega}_p(t)/m_f(t)$, $\boldsymbol{\phi}(t)/m_f(t)$, $W(z)\boldsymbol{\phi}(t)/m_f(t)$, $m(t)/m_f(t) \in L_\infty$, where $W(z)$ is any proper function that is analytic in $|z| \geq \sqrt{\delta}$ for the same δ as above, $\boldsymbol{\omega}_p = [\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, y, W_m^{-1}(z)y]^{\mathrm{T}}$, $\tilde{\boldsymbol{\theta}}_c = \boldsymbol{\theta}_c - \boldsymbol{\theta}_c^*$.

Proof. By Lemma 2, this lemma can be proved in a similar way of Lemma 6 in [8]. □

We state the main results in this paper.

Theorem 1. Consider the indirect MRAC scheme with the normalized adaptive law consisting of the discrete-time plant (1), the reference model (2), the adaptive law (5), and the control law (13) with $\boldsymbol{\theta}_c$ satisfying (14) ∼ (16). If Assumptions 1 ∼ 3 and M 1 hold, then

1) all the signals of the closed-loop plant are uniformly bounded;

2) $\lim_{t\to\infty} e(t) = 0$.

Proof. This theorem is proved by four steps.

Step 1. Express the input and output of the closed-loop plant in terms of $\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}$.

From (2), (12), and (13), it follows that

$$y = y_m + e = W_m(z)\Big(r + \frac{1}{\theta_{c4}^*}\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}\Big) \tag{19}$$

where $\tilde{\boldsymbol{\theta}}_c = \boldsymbol{\theta}_c - \boldsymbol{\theta}_c^*$. Using (1), (19), and Assumptions 1, 2, and M 1, and Remark 1, the input of the closed-loop plant is given by

$$u = G_p^{-1}(z)W_m(z)\Big(r + \frac{1}{\theta_{c4}^*}\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}\Big) \tag{20}$$

and $G_p^{-1}(z)W_m(z)$ is proper and analytic in $|z| \geq \sqrt{\delta}$. Therefore, applying Lemma 3 in [8] to (19) and (20), it follows that $\|y_{t-1}\|_{2\delta} \leq c + c\|(\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega})_{t-1}\|_{2\delta}$ and $\|u_{t-1}\|_{2\delta} \leq c + c\|(\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega})_{t-1}\|_{2\delta}$, which one substitutes in (18) to obtain

$$m_f^2(t) \leq c + c\|(\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega})_{t-1}\|_{2\delta}^2 \tag{21}$$

Step 2. Use Lemmas 1 ∼ 3, Lemmas 3 ∼ 5 in [8] to bound $\|\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}\|$ from above.

One applies (15) to the signal $\big(W_m(z)/\Lambda(z)\big)u$, and (16) to $\big(W_m(z)/\Lambda(z)\big)y$ to obtain

$$\boldsymbol{\theta}_{c1}^{\mathrm{T}}W_m(z)\boldsymbol{\omega}_1 = W_m(z)u - \frac{1}{\hat{k}_p}\big(\hat{\bar{Z}}_p(z,t) \cdot \hat{Q}(z,t)\big)\frac{W_m(z)}{\Lambda(z)}u \tag{22}$$

$$\boldsymbol{\theta}_{c2}^{\mathrm{T}}W_m(z)\boldsymbol{\omega}_2 + \theta_{c3}W_m(z)y = \frac{1}{\hat{k}_p}\big(\hat{Q}(z,t) \cdot \hat{R}_p(z,t) - \Lambda_0(z)R_m(z)\big)\frac{W_m(z)}{\Lambda(z)}y \tag{23}$$

where $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are defined in (7) below. Defining $\boldsymbol{\theta}_{c0} = [\boldsymbol{\theta}_{c1}^{\mathrm{T}}, \boldsymbol{\theta}_{c2}^{\mathrm{T}}, \theta_{c3}]^{\mathrm{T}}$ and $\boldsymbol{\omega}_0 = [\boldsymbol{\omega}_1^{\mathrm{T}}, \boldsymbol{\omega}_2^{\mathrm{T}}, y]^{\mathrm{T}}$, combining (22) and (23) leads to

$$\boldsymbol{\theta}_{c0}^{\mathrm{T}}W_m(z)\boldsymbol{\omega}_0 = W_m(z)u - \frac{1}{\hat{k}_p}\big(\hat{\bar{Z}}_p(z,t) \cdot \hat{Q}(z,t)\big)\frac{W_m(z)}{\Lambda(z)}u + \frac{1}{\hat{k}_p}\big(\hat{Q}(z,t) \cdot \hat{R}_p(z,t) - \Lambda_0(z)R_m(z)\big)\frac{W_m(z)}{\Lambda(z)}y \tag{24}$$

Repeating the same manipulation to (10) and (11), one has

$$\boldsymbol{\theta}_{c0}^{*\mathrm{T}}W_m(z)\boldsymbol{\omega}_0 = W_m(z)u - \frac{1}{k_p}\big(\bar{Z}_p(z)Q(z)\big)\frac{W_m(z)}{\Lambda(z)}u + \frac{1}{k_p} \cdot \big(Q(z)R_p(z) - \Lambda_0(z)R_m(z)\big)\frac{W_m(z)}{\Lambda(z)}y \tag{25}$$

where $\boldsymbol{\theta}_{c0}^* = [\boldsymbol{\theta}_{c1}^{*\mathrm{T}}, \boldsymbol{\theta}_{c2}^{*\mathrm{T}}, \theta_{c3}^*]^{\mathrm{T}}$. Setting $\tilde{\boldsymbol{\theta}}_{c0} = \boldsymbol{\theta}_{c0} - \boldsymbol{\theta}_{c0}^*$ and subtracting (25) from (24), it follows that

$$\tilde{\boldsymbol{\theta}}_{c0}^{\mathrm{T}}W_m(z)\boldsymbol{\omega}_0 = e_1 + e_2 + e_3 \tag{26}$$

obviously,

$$e_1 = -\frac{1}{\hat{k}_p}\big(\hat{\bar{Z}}_p(z,t) \cdot \hat{Q}(z,t)\big)\frac{W_m(z)}{\Lambda(z)}u + \frac{1}{\hat{k}_p}\big(\hat{Q}(z,t) \cdot \hat{R}_p(z,t)\big)\frac{W_m(z)}{\Lambda(z)}y \tag{27}$$

$$e_2 = -\frac{1}{\hat{k}_p}\frac{\Lambda_0(z)R_m(z)}{\Lambda(z)}W_m(z)y + \frac{1}{k_p}\frac{\Lambda_0(z)R_m(z)}{\Lambda(z)}W_m(z)y = -\tilde{\theta}_{c4}y \tag{28}$$

$$e_3 = \frac{\bar{Z}_p(z)Q(z)}{k_p\Lambda(z)}W_m(z)u - \frac{Q(z)R_p(z)}{k_p\Lambda(z)}W_m(z)y = 0 \tag{29}$$

by (1), (2), (9), (14), and $\Lambda(z)=\Lambda_0(z)Z_m(z)$, $\tilde{\theta}_{c4}=\theta_{c4}-\theta^*_{c4}$. Substituting (27)~(29) into (26) leads to

$$e_1=\tilde{\boldsymbol{\theta}}_c^{\rm T}W_m(z)\boldsymbol{\omega}_p \tag{30}$$

where $\boldsymbol{\omega}_p=[\boldsymbol{\omega}_0^{\rm T},W_m^{-1}(z)y]$. Obviously, $\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}=\tilde{\boldsymbol{\theta}}_{c0}^{\rm T}\boldsymbol{\omega}_0+\tilde{\theta}_{c4}r$, $\tilde{\boldsymbol{\theta}}_{c0}=\boldsymbol{\theta}_{c0}-\boldsymbol{\theta}^*_{c0}$, and from (19), $r=W_m^{-1}(z)y-(1/\theta^*_{c4})\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}$ is obtained, therefore

$$\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}=\frac{\theta^*_{c4}}{\theta_{c4}}\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p \tag{31}$$

From (31), Lemma 5 in [8] and assumption M 1, by choosing a_0 to satisfy $|a_0|\le\sqrt{\delta}/2$, one concludes that

$$[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}](t-1)=\frac{\theta^*_{c4}}{\theta_{c4}(t-1)}[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](t-1)=\frac{\theta^*_{c4}}{\theta_{c4}(t-1)}\cdot\Big(F_1(z,a_0)\frac{1}{z}[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](t-1)+F(z,a_0)[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](t-1)\Big) \tag{32}$$

and $\|F_1(z,a_0)\|_{\infty\delta}\le ca_0$, $\|F(z)W_m^{-1}(z)\|_{\infty\delta}\le ca_0^{n^*}$, where c is a constant independent of a_0, $F(z,a_0)=a_0^{n^*}/(z+a_0)^{n^*}$, $F_1(z,a_0)=(1-F(z,a_0))z$. Applying Lemma 4 in [8] and (30), it leads to

$$[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](t-1)=W_m^{-1}(z)(\tilde{\boldsymbol{\theta}}_c^{\rm T}(t-1)W_m(z)[\boldsymbol{\omega}_p](t-1)+W_c(z)[W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](z-1)[\tilde{\boldsymbol{\theta}}_c]](t-1))=W_m^{-1}(z)\cdot(e_1(t-1)+W_c(z)[W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](z-1)[\tilde{\boldsymbol{\theta}}_c]](t-1)) \tag{33}$$

for any $t\ge1$, where $W_c(z)$ and $W_b(z)$ are strictly proper and have the same poles as those of $W_m(z)$. Substituting (33) in (32), one gets

$$[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}](t-1)=\frac{\theta^*_{c4}}{\theta_{c4}(t-1)}\Big(F_1(z,a_0)\frac{1}{z}[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](t-1)+F(z,a_0)W_m^{-1}(z)\big(e_1(t-1)+W_c(z)\cdot\big[W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](z-1)[\tilde{\boldsymbol{\theta}}_c^{\rm T}]\big](t-1)\big)\Big) \tag{34}$$

It can be noted that (9), (14), and $\hat{k}_p(t-1)$ is the first element of $\boldsymbol{\theta}_p(t-1)$; therefore, $|\theta^*_{c4}/\theta_{c4}(t-1)|=|\hat{k}_p(t-1)/k_p|<c$ by Lemma 1 2). It follows from Lemmas 3 and 5 in [8], $\|F_1(z,a_0)\|_{\infty\delta}\le ca_0$ and $\|FW_m^{-1}\|_{\infty\delta}\le ca_0^{n^*}$ that $\|(\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega})_{t-1}\|_{2\delta}\le ca_0\|(\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p)_{t-1}\|_{2\delta}+ca_0^{n^*}\big(\|(e_1)_{t-1}\|_{2\delta}+\|(W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](z-1)[\tilde{\boldsymbol{\theta}}_c])_{t-1}\|_{2\delta}\big)$, where c is a constant independent of a_0. By the definition of $W_b(z)$ and Lemma 3 2), $W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](t-1)/m_f(t-1)\in L_\infty$, therefore, $\|(W_b(z)z[\boldsymbol{\omega}_p^{\rm T}](z-1)[\tilde{\boldsymbol{\theta}}_c])_{t-1}\|_{2\delta}\le c\|(((z-1)[\tilde{\boldsymbol{\theta}}_c])m_f)_{t-1}\|_{2\delta}$. Because $\tilde{\boldsymbol{\theta}}_c\in L_\infty$ by Lemma 2 and Lemma 3, it follows that $\|(\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p)_{t-1}\|_{2\delta}\le c\|(\boldsymbol{\omega}_p)_{t-1}\|_{2\delta}\le cm_f(t)$. Hence,

$$\|(\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega})_{t-1}\|_{2\delta}\le ca_0m_f(t)+ca_0^{n^*}\big(\|(e_1)_{t-1}\|_{2\delta}+\|(((z-1)[\tilde{\boldsymbol{\theta}}_c])m_f)_{t-1}\|_{2\delta}\big) \tag{35}$$

Now, let us consider $\|(e_1)_{t-1}\|_{2\delta}$ in (35). Using the definition of algebraic product, (6), $\Lambda_p(z)=\Lambda(z)(z+\lambda_0)$ in (8) above and (17), one can get

$$(\hat{\bar{Z}}_p(z,t)\cdot\hat{Q}(z,t))\frac{W_m(z)}{\Lambda(z)}u=\sum_{j=0}^{n^*-1}q_j(t)\hat{\bar{Z}}_p(z,t)\frac{W_m(z)z^j(z+\lambda_0)}{\Lambda_p(z)}u \tag{36}$$

$$(\hat{Q}(z,t)\cdot\hat{R}_p(z,t))\frac{W_m(z)}{\Lambda(z)}y=\sum_{j=0}^{n^*-1}q_j(t)\hat{R}_p(z,t)\frac{W_m(z)z^j(z+\lambda_0)}{\Lambda_p(z)}y \tag{37}$$

Noting that

$$\hat{R}_p(z,t)\frac{W_m(z)z^j(z+\lambda_0)}{\Lambda_p(z)}y-\hat{\bar{Z}}_p(z,t)\frac{W_m(z)z^j(z+\lambda_0)}{\Lambda_p(z)}u=\tilde{\boldsymbol{\theta}}_p^{\rm T}W_m(z)z^j(z+\lambda_0)\boldsymbol{\phi} \tag{38}$$

by (1) and $\boldsymbol{\phi}$ in (3) below. With (4), $-\varepsilon m^2=\tilde{\boldsymbol{\theta}}_p^{\rm T}\boldsymbol{\phi}$. Noting that $W_m(z)z^j(z+\lambda_0)$ is at least proper for any $j=0,1,\cdots,n^*-1$, by Lemma 4 in [8], one gets

$$\tilde{\boldsymbol{\theta}}_p^{\rm T}(t)W_m(z)z^j(z+\lambda_0)\boldsymbol{\phi}=-W_m(z)z^j(z+\lambda_0)\varepsilon m^2-W_{mcj}(z)[(W_{mbj}(z)z\boldsymbol{\phi}^{\rm T})(z-1)\tilde{\boldsymbol{\theta}}_p] \tag{39}$$

where $W_{mcj}(z)$ and $W_{mbj}(z)$ are strictly proper and have the same poles as those of $W_m(z)z^j(z+\lambda_0)$. Substituting (36) ~ (39) into (27), one obtains

$$e_1=\frac{1}{\hat{k}_p}\left(\sum_{j=0}^{n^*-1}q_j\Big(-W_m(z)z^j(z+\lambda_0)\varepsilon m^2-W_{mcj}(z)[(W_{mbj}(z)z\boldsymbol{\phi}^{\rm T})(z-1)\tilde{\boldsymbol{\theta}}_p]\Big)\right) \tag{40}$$

Using Lemma 3 in [8] and Lemmas 1 ~ 3, and taking the $L_{2\delta}$ norm on both sides of (40), it follows that

$$\|(e_1)_{t-1}\|_{2\delta}\le c(\|(\varepsilon mm_f)_{t-1}\|_{2\delta}+\|(m_f(z-1)\tilde{\boldsymbol{\theta}}_p)_{t-1}\|_{2\delta}) \tag{41}$$

Combining (35) and (41) leads to

$$\|(\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega})_{t-1}\|_{2\delta}\le ca_0m_f(t)+c\|(\tilde{g}m_f)_{t-1}\|_{2\delta} \tag{42}$$

and $\tilde{g}\in L_2$ by Lemma 1 and Lemma 2, where $\tilde{g}^2=a_0^{2n^*}(\varepsilon^2m^2+|(z-1)\tilde{\boldsymbol{\theta}}_p|^2+|(z-1)\tilde{\boldsymbol{\theta}}_c|^2)$.

Step 3. Use discrete-time Bellman-Gronwall Lemma in [1] to prove conclusion 1).

Using (42) in (41), one has $m_f^2(t)\le c+c\|(\tilde{g}m_f)_{t-1}\|_{2\delta}^2+ca_0^2m_f^2(t)$. Following the above proof, one can see that the coefficient c of the third term on the right-hand side of the inequality is independent of a_0. Thus, by choosing appropriately small a_0 such that $ca_0^2<1/2$, $m_f^2(t)\le c+c\sum_{i=0}^{t-1}\delta^{t-i-1}\tilde{g}^2(i)m_f^2(i)$ can be obtained. Using the discrete-time Bellman-Gronwall Lemma in [1] and $\tilde{g}\in L_2$, following the similar discussion from (34) to $m_f\in L_\infty$ in [10], $m_f(t)\in L_\infty$ holds, then conclusion 1) holds by Lemma 3.

Step 4. Establish the convergence of the tracking error. By (19), (31), $e=y-y_m$ and Lemma 4 in [8], one has

$$e=W_m(z)\left[\frac{1}{\theta^*_{c4}}\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}\right]=W_m(z)\left[\frac{1}{\theta_{c4}}\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p\right]=\frac{1}{\theta_{c4}}W_m(z)\cdot[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p]+W_c(z)\Big[W_b(z)z[\tilde{\boldsymbol{\theta}}_c^{\rm T}\boldsymbol{\omega}_p](z-1)\Big[\frac{1}{\theta_{c4}}\Big]\Big] \tag{43}$$

Substituting (33) at time t in (43) leads to

$$e = \frac{1}{\theta_{c4}}\Big(e_1 + W_c(z)[(W_b(z)z[\boldsymbol{\omega}_p^{\mathrm{T}}])(z-1)[\tilde{\boldsymbol{\theta}}_c]]\Big) + W_c(z)\Big[(W_b(z)z[\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}_p])(z-1)\Big[\frac{1}{\theta_{c4}}\Big]\Big] \tag{44}$$

Since $\varepsilon m, (z-1)\tilde{\boldsymbol{\theta}}_p \in L_2$, $1/\hat{k}_p, q_j, m, W_{mbj}z\boldsymbol{\phi}^{\mathrm{T}} \in L_\infty$ for $j = 0, 1, \cdots, n^*-1$, $W_m(z)z^j(z+\lambda_0)$ and W_{mcj} are stable and at least proper for $j = 0, 1, \cdots, n^*-1$, by (40), $e_1 \in L_2$ by equation (2.249) in [1]. Because $1/\theta_{c4}(t) = \hat{k}_p(t)/k_m \in L_\infty$ and $\Delta\theta_{c4}(t) \in L_2$ for any $t \geq 0$, one has $(z-1)[\frac{1}{\theta_{c4}}](t) = \frac{1}{\theta_{c4}(t+1)} - \frac{1}{\theta_{c4}(t)} = \frac{-(z-1)\theta_{c4}(t)}{\theta_{c4}(t+1)\theta_{c4}(t)} \in L_2$, which, together with e_1, $(z-1)\tilde{\boldsymbol{\theta}}_c \in L_2$, $W_b(z)z[\boldsymbol{\omega}_p^{\mathrm{T}}]$, $W_b(z)z[\tilde{\boldsymbol{\theta}}_c^{\mathrm{T}}\boldsymbol{\omega}_p] \in L_\infty$ by Lemma 3 2), and $W_c(z)$ being stable, means that $e \in L_2$. Hence, $\lim_{t\to\infty} e(t) = 0$. □

5 Conclusions

In this paper, as its continuous counterpart in [3], we consider a systematic stability and convergence analysis approach to the discrete indirect MRAC scheme with normalized adaptive law for a class of discrete-time systems. Future work will be directed at the application of this methodology to discrete-time systems with unmodeled dynamics and multivariable discrete-time systems.

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Appendix

Proof of Lemma 1.

1) We prove the conclusion by considering two cases of $\mathrm{sgn}(k_p)$.

Case 1.1 If $k_p > 0$, we consider two cases.

a) When $\bar{\theta}_{p1}(t)\mathrm{sgn}(k_p) \geq \underline{k}_p$, by (5), then $\boldsymbol{\Delta}(t) = 0$, and then $\theta_{p1}(t) = \bar{\theta}_{p1}(t) \geq \underline{k}_p$.

b) When $\bar{\theta}_{p1}(t)\mathrm{sgn}(k_p) < \underline{k}_p$, from the definition of $\boldsymbol{\tau}_1$ and τ_2, (5) and $k_p > 0$, $\theta_{p1}(t) = \bar{\theta}_{p1}(t) + (\underline{k}_p\mathrm{sgn}(k_p) - \bar{\theta}_{p1}(t)) = \underline{k}_p\mathrm{sgn}(k_p) = \underline{k}_p$ follows.

a) and b) prove that $\theta_{p1}(t) \geq \underline{k}_p$ for all $t \in \{1, 2, \cdots\}$

Case 1.2 If $k_p < 0$, we consider the same cases.

a) When $\bar{\theta}_{p1}(t)\mathrm{sgn}(k_p) \geq \underline{k}_p$, then $\boldsymbol{\Delta}(t) = 0$, and then $\theta_{p1}(t) = \bar{\theta}_{p1}(t) \leq -\underline{k}_p$.

b) When $\bar{\theta}_{p1}(t)\mathrm{sgn}(k_p) < \underline{k}_p$, similarly, $\theta_{p1}(t) = \underline{k}_p\mathrm{sgn}(k_p) = -\underline{k}_p$.

a) and b) imply that $\theta_{p1}(t) \leq -\underline{k}_p$ for all $t \in \{1, 2, \cdots\}$.

Combining Case 1.1 and Case 1.2, conclusion 1) can be proved.

2) From the definition of $\boldsymbol{\Delta}(t+1)$ in (5), we prove that

$$I = 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\boldsymbol{\phi}(t)\varepsilon(t) + 2\tilde{\boldsymbol{\theta}}_p^{\mathrm{T}}(t)\Gamma^{-1}\boldsymbol{\Delta}(t+1) + 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\Gamma^{-1}\boldsymbol{\Delta}(t+1) \leq 0 \tag{A1}$$

holds for all $t \in \{1, 2, \cdots\}$ from two cases.

Case 2.1 If $\bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) \geq \underline{k}_p$, then $\boldsymbol{\Delta}(t+1) = 0$, and then $I = 0$.

Case 2.2 If $\bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) < \underline{k}_p$, defining $f_1(t+1) = \underline{k}_p\mathrm{sgn}(k_p) - \bar{\theta}_{p1}(t+1)$, from the definition of $\boldsymbol{\tau}_1$, τ_2 and Γ, and (5), one has

$$\begin{aligned} I = \; & 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\Gamma^{-1}\left[\bar{\boldsymbol{\theta}}_p(t+1) - \boldsymbol{\theta}_p(t)\right] + 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\cdot \\ & \Gamma^{-1}\tilde{\boldsymbol{\theta}}_p(t) + 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\Gamma^{-1}\boldsymbol{\Delta}(t+1) = \\ & \frac{2}{\lambda_1}f_1(t+1)\left[\bar{\theta}_{p1}(t+1) - \theta_{p1}(t)\right] + \\ & \frac{2}{\lambda_1}f_1(t+1)\tilde{\theta}_{p1}(t) + \frac{2}{\lambda_1}f_1^2(t+1) = \\ & \frac{2}{\lambda_1}f_1(t+1)\left[\bar{\theta}_{p1}(t+1) - \theta_{p1}^* + f_1(t+1)\right] \end{aligned} \tag{A2}$$

where θ_{p1}^* is the first element of $\boldsymbol{\theta}_p^*$, $\tilde{\theta}_{p1}(t) = \theta_{p1}(t) - \theta_{p1}^*$. Let us consider the sign of k_p.

a) If $k_p > 0$, from $|k_p| \geq \underline{k}_p$, the definition of f_1 and $\theta_{p1}^* = b_m = k_p$, it follows that $f_1(t+1) > 0$, $\bar{\theta}_{p1}(t+1) - \theta_{p1}^* + f_1(t+1) = \underline{k}_p - \theta_{p1}^* = \underline{k}_p - k_p \leq 0$, which implies that $I \leq 0$ from (A2).

b) If $k_p < 0$, one can obtain that $f_1(t+1) = -\underline{k}_p - \bar{\theta}_{p1}(t+1) < 0$, $\bar{\theta}_{p1}(t+1) - \theta_{p1}^* + f_1(t+1) = -\underline{k}_p - \theta_{p1}^* = -\underline{k}_p - k_p \geq 0$ from $|k_p| \geq \underline{k}_p$ and the definition of $f_1(t+1)$, therefore, $I \leq 0$.

a) and b) prove that $I \leq 0$ for Case 2.2.

After proving (A1), by (4), the time increment of $V(\tilde{\boldsymbol{\theta}}_p(t)) = \tilde{\boldsymbol{\theta}}_p^{\mathrm{T}}(t)\Gamma^{-1}\tilde{\boldsymbol{\theta}}_p(t)$ along (5) satisfies

$$\begin{aligned} & V(\tilde{\boldsymbol{\theta}}_p(t+1)) - V(\tilde{\boldsymbol{\theta}}_p(t)) = \\ & (\tilde{\boldsymbol{\theta}}_p(t+1) + \tilde{\boldsymbol{\theta}}_p(t))^{\mathrm{T}}\Gamma^{-1}(\tilde{\boldsymbol{\theta}}_p(t+1) - \tilde{\boldsymbol{\theta}}_p(t)) \leq \\ & -\left(2 - \frac{\boldsymbol{\phi}^{\mathrm{T}}(t)\Gamma\boldsymbol{\phi}(t)}{m^2(t)}\right)\varepsilon^2(t)m^2(t) + I \leq \\ & -\alpha_1\varepsilon^2(t)m^2(t) \end{aligned} \tag{A3}$$

where $\alpha_1 = 2 - \max_{i=1,2,\cdots,n+m+1}(\lambda_i) > 0$ by $0 < \lambda_i < 2$, which implies that $\tilde{\boldsymbol{\theta}}_p(t)$, $\boldsymbol{\theta}_p(t) \in L_\infty$. From (4) and $\boldsymbol{\theta}_p(t) \in L_\infty$, one has $\varepsilon(t), \varepsilon(t)m(t) \in L_\infty$.

3) By (A3), one has $\varepsilon(t)m(t) \in L_2$. Since $|m(t)| \geq 1$, then $\varepsilon(t) \in L_2$. Next, we prove that $\Delta\boldsymbol{\theta}_p(t) \in L_2$ holds. From the definition of $\boldsymbol{\Delta}(t+1)$ in (5), we first prove that

$$J = 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1)\left[(\bar{\boldsymbol{\theta}}_p(t+1) - \boldsymbol{\theta}_p(t)) + \boldsymbol{\Delta}(t+1)\right] \leq 0 \tag{A4}$$

for any $t \in \{1, 2, \cdots\}$ from two cases.

Case 3.1 If $\bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) \geq \underline{k}_p$, then $\boldsymbol{\Delta}(t+1) = 0$, and then $J = 0$.

Case 3.2 If $\bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) < \underline{k}_p$, similar to the proof of (A2), one has

$$\begin{aligned} J = {} & 2f_1(t+1)[(\bar{\theta}_{p1}(t+1) - \theta_{p1}(t)) + f_1(t+1)] = \\ & 2f_1(t+1)(k_p\mathrm{sgn}(k_p) - \theta_{p1}(t)). \end{aligned} \tag{A5}$$

Let us consider the sign of k_p. When $k_p > 0$, by conclusion of Case 1.1, one has $\theta_{p1}(t) \geq \underline{k}_p$, thus, $k_p\mathrm{sgn}(k_p) - \theta_{p1}(t) = \underline{k}_p - \theta_{p1}(t) \leq 0$, while $f_1(t+1) = \underline{k}_p - \bar{\theta}_p(t+1) > 0$ from $\bar{\theta}_{p1}(t+1)\mathrm{sgn}(k_p) < \underline{k}_p$ and $k_p > 0$, hence $J \leq 0$. When $k_p < 0$, by conclusion of Case 1.2, one has $k_p\mathrm{sgn}(k_p) - \theta_{p1}(t) = -\underline{k}_p - \theta_{p1}(t) \geq 0$, while $f_1(t+1) = -\underline{k}_p - \bar{\theta}_{p1}(t+1) < 0$, thus $J \leq 0$.

Case 3.1 and Case 3.2 prove (A4), which together with (4), (5), and $\varepsilon m \in L_2$, leads to

$$\begin{aligned} & \sum_{t=0}^{\infty} \Delta\boldsymbol{\theta}_p^{\mathrm{T}}(t)\Delta\boldsymbol{\theta}_p(t) = \\ & \sum_{t=0}^{\infty} (\Gamma\boldsymbol{\phi}(t)\varepsilon(t) + \boldsymbol{\Delta}(t+1))^{\mathrm{T}} (\Gamma\boldsymbol{\phi}(t)\varepsilon(t) + \boldsymbol{\Delta}(t+1)) \leq \\ & \sum_{t=0}^{\infty} \Bigg(\frac{\boldsymbol{\phi}^{\mathrm{T}}(t)\Gamma^2\boldsymbol{\phi}(t)}{m^2(t)}\varepsilon^2(t)m^2(t) + 2\boldsymbol{\Delta}^{\mathrm{T}}(t+1) \cdot \\ & \quad \big((\bar{\boldsymbol{\theta}}_p(t+1) - \boldsymbol{\theta}_p(t)) + \boldsymbol{\Delta}(t+1) \big) \Bigg) \leq \\ & \max_{i=1,2,\cdots,n+m+1} \{\lambda_i^2\} \sum_{t=0}^{\infty} \varepsilon^2(t)m^2(t), \end{aligned} \tag{A6}$$

which implies that $\Delta\boldsymbol{\theta}_p \in L_2$. □

Proof of Lemma 2.

To further simplify (15) and (16), express

$$\Lambda(z) = z^{n-1} + \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z) \tag{A7}$$

$$\begin{aligned} \Lambda_0(z)R_m(z) = {} & z^{n+n^*-1} + \bar{\bar{\boldsymbol{r}}}_1^{\mathrm{T}}\bar{\boldsymbol{\alpha}}(z) + \bar{\bar{r}}_2 z^{n-1} + \\ & \bar{\bar{\boldsymbol{r}}}_3^{\mathrm{T}}\boldsymbol{\alpha}_{n-2}(z) \end{aligned} \tag{A8}$$

$$\hat{\bar{Z}}_p(z,t) \cdot \hat{Q}(z,t) = \hat{k}_p(t)z^{n-1} + \boldsymbol{\alpha}^{\mathrm{T}}(t)\boldsymbol{\alpha}_{n-2}(z) \tag{A9}$$

$$\begin{aligned} \hat{Q}(z,t) \cdot \hat{R}_p(z,t) = {} & z^{n+n^*-1} + \bar{\bar{\boldsymbol{\beta}}}_1^{\mathrm{T}}(t)\bar{\boldsymbol{\alpha}}(z) + \bar{\bar{\beta}}_2(t)z^{n-1} + \\ & \bar{\bar{\boldsymbol{\beta}}}_3^{\mathrm{T}}(t)\boldsymbol{\alpha}_{n-2}(z) \end{aligned} \tag{A10}$$

where $\bar{\boldsymbol{\alpha}}(z) = [z^{n+n^*-2}, \cdots, z^n]^{\mathrm{T}}$, $\bar{\bar{\boldsymbol{r}}}_1 = [r_{n+n^*-2}, \cdots, r_n]^{\mathrm{T}}$, $\bar{\bar{r}}_2 \in \mathbf{R}^1$, $\bar{\bar{\boldsymbol{r}}}_3 = [r_{n-2}, \cdots, r_0]^{\mathrm{T}}$, $\boldsymbol{\alpha}(t) = [\alpha_{n-2}(t), \cdots, \alpha_0(t)]^{\mathrm{T}}$, $\bar{\bar{\boldsymbol{\beta}}}_1(t) = [\beta_{n+n^*-2}(t), \cdots, \beta_n(t)]^{\mathrm{T}}$, $\bar{\bar{\beta}}_2(t) \in \mathbf{R}^1$, $\bar{\bar{\boldsymbol{\beta}}}_3(t) = [\beta_{n-2}(t), \cdots, \beta_0(t)]^{\mathrm{T}}$. Since $\hat{Q}(z,t)$ is the quotient of $\Lambda_0(z)R_m(z)/\hat{R}_p(z,t)$, by (6), (17), (A9) and the polynomial′s division, for each fixed t, $q_{n^*-1}(t) = 1$, $q_i(t) = r_{n+i} - \sum_{j=i+1}^{n^*-1} \alpha_{n+i-j}(t)q_j(t)$, $i = 0, 1, \cdots, n^* - 1$. When $i = n^* - 2$, for any $t \in \{0, 1, 2, \cdots\}$, $q_{n^*-2}(t) = r_{n+n^*-2} - a_{n-1}(t)q_{n^*-1}(t) = r_{n+n^*-2} - a_{n-1}(t)$, $\Delta q_{n^*-2}(t) = -\Delta a_{n-1}(t)$, which implies that $q_{n^*-2}(t) \in L_\infty$ and $\Delta q_{n^*-2}(t) \in L_2$ using Lemma 1. Repeating the similar arguments as above, it follows that for any $t \in \{0, 1, 2 \cdots\}$, $q_i(t) \in L_\infty$, $\Delta q_i(t) \in L_2$, $i = 0, 1, \cdots, n^* - 1$. By (6), (17), and (53), one has $\alpha_i(t) = \sum_{l+j=i} q_l(t)b_j(t)$, $i = 0, 1, \cdots, n-2$, which together with $q_l(t)$, $b_j(t) \in L_\infty$ and $\Delta q_l(t)$, $\Delta b_j(t) \in L_2$ implies that $\alpha_i(t) \in L_\infty$, $\Delta\alpha_i(t) = \alpha_i(t+1) - \alpha_i(t) = \sum_{l+j=i}(q_l(t+1)\Delta b_j(t) + \Delta q_l(t) \cdot b_j(t)) \in L_2$, and thus $\boldsymbol{\alpha}(t) \in L_\infty$, $\Delta\boldsymbol{\alpha}(t) \in L_2$. Similarly, from (6), (17) and (A10) it follows that $\bar{\bar{\boldsymbol{\beta}}}_i(t) \in L_\infty$, $\Delta\bar{\bar{\boldsymbol{\beta}}}_i(t) \in L_2$, $i = 1, 3$, $\bar{\bar{\beta}}_2(t) \in L_\infty$, $\Delta\bar{\bar{\beta}}_2(t) \in L_2$.

Substituting (A7) ~ (A10) into (15) and (16) and equating the coefficients of the powers of z on both sides of these two equations, respectively, it is easy to obtain that for any $t \in \{0, 1, 2 \cdots\}$, $\boldsymbol{\theta}_{c1}(t) = \boldsymbol{\lambda} - \dfrac{\boldsymbol{\alpha}(t)}{\hat{k}_p(t)}$, $\boldsymbol{\theta}_{c2}(t) = \dfrac{\bar{\bar{\boldsymbol{\beta}}}_3(t) - \bar{\bar{\boldsymbol{r}}}_3 + \boldsymbol{\lambda}(\bar{\bar{r}}_2 - \bar{\bar{\beta}}_2(t))}{\hat{k}_p(t)}$, $\theta_{c3}(t) = \dfrac{\bar{\bar{\beta}}_2(t) - \bar{\bar{r}}_2}{\hat{k}_p(t)}$, $\theta_{c4}(t) = \dfrac{k_m}{\hat{k}_p(t)}$. From 1) of Lemma 1, one knows that $|\hat{k}_p(t)| > \underline{k}_p$ for any $t \geq 0$, which together with $\boldsymbol{\alpha}(t) \in L_\infty$ implies that $\boldsymbol{\theta}_{c1}(t) \in L_\infty$. Because $\Delta\hat{k}_p(t) = \Delta\theta_{p_1}(t)$ and $\Delta\boldsymbol{\alpha}(t) \in L_2$, $\Delta\boldsymbol{\theta}_{c1}(t) = \dfrac{\boldsymbol{\alpha}(t)\Delta\hat{k}_p(t)}{\hat{k}_p(t+1) \cdot \hat{k}_p(t)} - \dfrac{\Delta\boldsymbol{\alpha}(t)}{\hat{k}_p(t+1)} \in L_2$. Using the same arguments as above, one can show that $\boldsymbol{\theta}_{c2}(t) \in L_\infty$, $\Delta\boldsymbol{\theta}_{c2}(t) \in L_2$, $\theta_{ci}(t) \in L_\infty$, $\Delta\theta_{ci}(t) \in L_2$, $i = 3, 4$, that is, $\boldsymbol{\theta}_c(t) \in L_\infty$, $\Delta\boldsymbol{\theta}_c(t) \in L_2$. □

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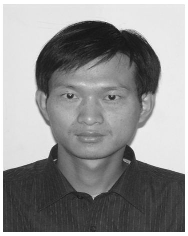

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