Stabilizability May Be Sufficient for Robustly Stabilizing an Interval Plant

WU Qing-He[1]

Abstract This paper deals with the robust stabilization problem for an interval plant family $\mathcal{P}(s,\boldsymbol{\delta})$. It is shown that an interval plant may be robustly stabilized by a single controller $C(s)$ if every member plant of $\mathcal{P}(s,\boldsymbol{\delta})$ is stabilizable.

Key words Interval plant family, robust stabilization, stabilizability radius, stability radius

1 Introduction

Consider the plant family

$$\mathcal{P}(s,\boldsymbol{\delta}) := \left\{ P(s) \ : \ P(s) = \frac{N_0(s)+\Delta_N(s)}{D_0(s)+\Delta_D(s)} \right\}$$

where $N_0(s)=\sum_{j=0}^{n} b_j s^j$ and $D_0(s)=\sum_{i=0}^{n} a_i s^i$ are polynomials with constant coefficients, and

$$\begin{aligned}\Delta_N(s) &= w_{N,0}\delta_{N,0}+w_{N,1}\delta_{N,1}s+\cdots+\\ &\quad w_{N,n-1}\delta_{N,n-1}s^{n-1}+w_{N,n}\delta_{N,n}s^n\\ \Delta_D(s) &= w_{D,0}\delta_{D,0}+w_{D,1}\delta_{D,1}s+\cdots+\\ &\quad w_{D,n-1}\delta_{D,n-1}s^{n-1}+w_{D,n}\delta_{D,n}s^n\end{aligned}$$

are polynomials with uncertain coefficients. It is assumed that the uncertain parameter vector $\boldsymbol{\delta}=[\boldsymbol{\delta}_N^{\mathrm{T}} \ \ \boldsymbol{\delta}_D^{\mathrm{T}}]^{\mathrm{T}}$ with $\boldsymbol{\delta}_\alpha=[\,\delta_{\alpha,n} \ \ \delta_{\alpha,n-1} \ \ \cdots \ \ \delta_{\alpha,1} \ \ \delta_{\alpha,0}]^{\mathrm{T}}$ $(\alpha=N,D)$ is ∞-norm bounded, *i.e.* $\|\boldsymbol{\delta}\|_\infty=\max_i\{|\delta_{\alpha,i}|\}\le\delta$ for some $\delta>0$. We denote by $\bar{\Omega}_\delta$ the set of all $\boldsymbol{\delta}$ such that $\|\boldsymbol{\delta}\|_\infty\le\delta$. It is clear that $\mathcal{P}(s,\boldsymbol{\delta})$ with $\boldsymbol{\delta}\in\bar{\Omega}_\delta$ is the interval plant family

$$\mathcal{P}(s,\boldsymbol{\delta}) = \frac{\sum\limits_{j=0}^{n}[b_j^- \ , \ b_j^+]s^j}{\sum\limits_{i=0}^{n}[a_i^- \ , \ a_i^+]s^i}$$

where for $j,i=1,2,\ldots,n$, $b_j^-=b_j-w_{N,j}\delta$, $b_j^+=b_j+w_{N,j}\delta$, $a_i^-=a_i-w_{D,i}\delta$, and $a_i^+=a_i+w_{D,i}\delta$. The robust stabilization problem (RSP) associated with $\mathcal{P}(s,\boldsymbol{\delta})$ is to find a single proper controller $C(s)$, whose structure and parameters are invariant, so that the negative feedback system composed of $\mathcal{P}(s,\boldsymbol{\delta})$ and $C(s)$ is internally stable for all $P(s)\in\mathcal{P}(s,\boldsymbol{\delta})$ such that $\boldsymbol{\delta}\in\bar{\Omega}_\delta$.

For a given norm bound δ, Chapellat and Bhattacharyya[1] showed that a given controller stabilizes the whole interval plant family if it stabilizes its 32 edge plants, which can be viewed as a special case of the result of [2]. Barmish *et al.* further showed that when the controller is of first order, to verify if the controller stabilizes the interval plant family, one only needs to check if it stabilizes at most 16 vertex plants of the plant family[3]. However, all these approaches have not provided any solution to the substantial question if a given RSP is solvable, which requires that some necessary solvability conditions should be debuced for the RSP.

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1. Department of Automatic Control, Beijing Institute of Technology, Beijing 100081, P. R. China
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An apparent necessary solvability condition is that the plant family $\mathcal{P}(s,\boldsymbol{\delta})$ is stabilizable; that is, every member plant in $\mathcal{P}(s,\boldsymbol{\delta})$ is free of unstable pole-zero cancellation. A more restrictive necessary condition for the RSP is that every plant pair $\{\mathcal{P}(s,\boldsymbol{\delta}_1), \mathcal{P}(s,\boldsymbol{\delta}_2)\}$ formed by any two uncertainties $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \in \Omega_\delta$ must be simultaneously stabilizable because of the requirement that the controller is single. However, in a recent paper Wu *et al.* showed that the simultaneous stabilizability of every plant pair is equivalent to the stabilizability of $\mathcal{P}(s,\boldsymbol{\delta})$[4]. Because necessary and sufficient solvability condition for the simultaneous stabilization of n plants cannot be obtained as long as $n \geq 3$, the stabilizability of $\mathcal{P}(s,\boldsymbol{\delta})$ is the most unrestrictive necessary solvability condition for RSP that can be obtained up to now.

The purpose of this paper is to further investigate if the available necessary solvability condition for RSP, *i.e.* the stabilizability, is possibly also sufficient for an RSP to be solvable. The investigation will be accomplished in three steps. In the first step, the maximum norm bound of the uncertainty for the corresponding $\mathcal{P}(s,\boldsymbol{\delta})$ to be stabilizable, called the stabilizability radius of $\mathcal{P}(s,\boldsymbol{\delta})$, will be calculated. In the second step, a controller $C(s)$ is designed to stabilize the nominal plant $P_0(s)$, and the stability radius of the closed-loop system will be calculated, which is a function of the controller parameters with the stabilizability radius being its upper bound. In the third step, the controller parameters will be adjusted to see if the upper bound can be achieved. For the sake of simplicity, all the steps will be illustrated using a numerical example. The investigation shows that the stability radius of the closed-loop system and the stabilizability radius can be equal even for the simplest P-controller.

2 Main results

Consider the plant family $\mathcal{P}(s,\boldsymbol{\delta}) = \dfrac{N_0(s)+\Delta_N(s)}{D_0(s)}$ with

$$\begin{aligned} &N_0(s) = s^5+7s^4+19s^3+27s^2+20s+6 \\ &D_0(s) = (s^2+3.7690^2)(s^3-3s^2+3s-2) \\ &\Delta_N(s) = w_{N,5}\delta_{N,5}s^5 + w_{N,4}\delta_{N,4}s^4 + w_{N,3}\delta_{N,3}s^3 + \\ &\qquad w_{N,2}\delta_{N,2}s^2 + w_{N,1}\delta_{N,1}s + w_{N,0}\delta_{N,0} \end{aligned} \tag{1}$$

where $w_{N,5}=0.1$, $w_{N,4}=5$, $w_{N,3}=2$, $w_{N,2}=4$, $w_{N,1}=1$, and $w_{N,0}=1$ are the uncertainty weightings, $\boldsymbol{\delta} = [\delta_{N,5}\ \ \delta_{N,4}\ \ \delta_{N,3}\ \ \delta_{N,2}\ \ \delta_{N,1}\ \ \delta_{N,0}]^{\mathrm{T}}$ with $\|\boldsymbol{\delta}\|_\infty = \max_j\{|\delta_{N,j}|,\ j=0,1,2,3,4,5\} \leq \delta$ for some positive number δ is the uncertain parameter vector. We are interested in the problem to find a controller $C(s)$, as simple as possible, to robustly stabilize the interval plant $\mathcal{P}(s,\boldsymbol{\delta})$ for some positive number δ such that $\mathcal{P}(s,\boldsymbol{\delta})$ is stabilizable.

For $\mathcal{P}(s,\boldsymbol{\delta})$ to be stabilizable, its nominal value $P_0(s) = \mathcal{P}(s,\mathbf{0}) = \dfrac{N_0(s)}{D_0(s)}$ must be stabilizable, which is true for the given $P_0(s)$. Because the roots of a polynomial are continuous functions of its coefficients, $\mathcal{P}(s,\boldsymbol{\delta})$ remains stabilizable for all $\boldsymbol{\delta}$ such that $\|\boldsymbol{\delta}\|_\infty$ is sufficiently small. However, as $\|\boldsymbol{\delta}\|_\infty$ increases, the numerator and the denominator of $\mathcal{P}(s,\boldsymbol{\delta})$ may share some common root s^* in the closed right-half complex plane $\bar{\mathcal{C}}_+$. The maximum norm bound δ_s for $\boldsymbol{\delta}$ such that $\mathcal{P}(s,\boldsymbol{\delta})$ avoids pole-zero cancellation in $\bar{\mathcal{C}}_+$ for all $\boldsymbol{\delta}$ s.t. $\|\boldsymbol{\delta}\|_\infty < \delta_s$ is called the stabilizability radius. Hence, $\|\boldsymbol{\delta}\|_\infty < \delta_s$ forms a necessary solvability condition for RSP. In the first step, we calculate the stabilizability radius. We shall use the example plant family given in (1) to illustrate the procedure for calculating δ_s and the associated worst case uncertain parameter vector $\boldsymbol{\delta}_w$. Because the denominator is a fixed polynomial $D_0(s)$, and $s^3-3s^2+3s-2 = (s-2)(s^2-s+1)$, unstable cancellations can occur only at the isolated unstable poles: $s_1 = 2$, $s_2 = j3.7690 \triangleq j\omega_0$, $s_3 = 1/2 + j\sqrt{3}/2$, $s_4 = \bar{s}_2$, and $s_5 = \bar{s}_3$. Furthermore, because $N_0(s)+\Delta_N(s)$ is a real polynomial, for complex s,

$$\begin{aligned} &N_0(s)+\Delta_N(s) = 0 \\ \Longleftrightarrow\quad &\overline{N_0(s)+\Delta_N(s)} = N_0(\bar{s})+\Delta_N(\bar{s}) = 0 \end{aligned}$$

Hence, we need only consider the cancellations at s_1, s_2, and s_3. Denote by $\rho_N(s_i)$ the norm of the uncertain parameter vector $\boldsymbol{\delta}_{N,w,i}$ which is the minimum norm solution to $N_0(s_i)+\Delta_N(s_i) = N_0(s_i)+\boldsymbol{w}_N^{\mathrm{T}}(s_i)\boldsymbol{\delta} = 0$, where $\boldsymbol{w}_N^{\mathrm{T}}(s) = [w_{N,5}s^5\ \ w_{N,4}s^4\ \ w_{N,3}s^3\ \ w_{N,2}s^2\ \ w_{N,1}s\ \ w_{N,0}]$, that is

$$\|\boldsymbol{\delta}_{N,w,i}\|_\infty = \inf\left\{ \|\boldsymbol{\delta}\|_\infty\ :\ N_0(s_i)+\boldsymbol{w}_N^{\mathrm{T}}(s_i)\boldsymbol{\delta} = 0 \right\}$$

Then, the stabilizability radius of $\mathcal{P}(s,\boldsymbol{\delta})$ is given by

$$\delta_s = \min\{\rho_N(s_1),\ \rho_N(s_2),\ \rho_N(s_3)\}$$

For s_i is real, we have $\rho_N(s_i) = |N_0(s_i)|/\|\boldsymbol{w}_N(s_i)\|_1$,

$$\boldsymbol{\delta}_{N,w,i} = [-1\ \ -1\ \ \ldots\ \ -1]^{\mathrm{T}}\cdot \mathrm{sign}\,[N_0(s_i)]\,\rho_N(s_i)$$

For s_i is complex, using the same technique developed in [5] for the calculation of the ∞-norm stability radius of control system with interval plant, we can develop the procedure for calculating $\rho_N(s_i)$ and the corresponding worst case uncertain parameter $\boldsymbol{\delta}_{N,w,i}$. We first define

$$\boldsymbol{n}_0(s_i) \triangleq \begin{bmatrix} \mathrm{Im}N_0(s_i) \\ \mathrm{Re}N_0(s_i) \end{bmatrix},\quad W_N^{\mathrm{T}}(s_i) \triangleq \begin{bmatrix} \mathrm{Im}\boldsymbol{w}_N^{\mathrm{T}}(s_i) \\ \mathrm{Re}\boldsymbol{w}_N^{\mathrm{T}}(s_i) \end{bmatrix}$$

Then $N_0(s_i)+\boldsymbol{w}_N^{\mathrm{T}}(s_i)\boldsymbol{\delta} = 0$, if and only if $\boldsymbol{n}_0(s_i) + W_N^{\mathrm{T}}(s_i)\boldsymbol{\delta} = \mathbf{0}$. Denote by $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_6$ the column vectors of $W_N^{\mathrm{T}}(s_i)$, that is, $W_N^{\mathrm{T}}(s_i) = [\boldsymbol{f}_1\ \ \boldsymbol{f}_2\ \ \boldsymbol{f}_3\ \ \boldsymbol{f}_4\ \ \boldsymbol{f}_5\ \ \boldsymbol{f}_6]$. If $0 < \angle\boldsymbol{f}_i \leq \pi$, let $\tilde{\boldsymbol{f}}_i = \boldsymbol{f}_i$, and if $\pi < \angle\boldsymbol{f}_i \leq 2\pi$, let $\tilde{\boldsymbol{f}}_i = -\boldsymbol{f}_i$. It is clear that there exists a $\tilde{\boldsymbol{\delta}} = [\tilde{\delta}_1\ \ \tilde{\delta}_2\ \ \tilde{\delta}_3\ \ \tilde{\delta}_4\ \ \tilde{\delta}_5\ \ \tilde{\delta}_6]^{\mathrm{T}}$ having elements $\tilde{\delta}_i = 1$ or $\tilde{\delta}_i = -1$ such that

$$\begin{aligned} &W_N^{\mathrm{T}}(s_i)\mathrm{diag}\left\{ \tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \tilde{\delta}_4, \tilde{\delta}_5 \right\} = \\ &\left[\tilde{\boldsymbol{f}}_1\ \ \tilde{\boldsymbol{f}}_2\ \ \tilde{\boldsymbol{f}}_3\ \ \tilde{\boldsymbol{f}}_4\ \ \tilde{\boldsymbol{f}}_5\ \ \tilde{\boldsymbol{f}}_6 \right] \triangleq \tilde{W}_N^{\mathrm{T}}(s_i) \end{aligned}$$

with all the column vectors of $\tilde{W}_N^{\mathrm{T}}(s_i)$ being in the upper-half plane. Furthermore, if the vectors $\tilde{\boldsymbol{f}}_{i_1}, \tilde{\boldsymbol{f}}_{i_2}, \ldots, \tilde{\boldsymbol{f}}_{i_k}$ are collinear and all the other $\tilde{\boldsymbol{f}}_j$ with $j \notin \{i_1, i_2, \ldots, i_k\}$ are not collinear with $\tilde{\boldsymbol{f}}_{i_1}$, we define

$$\check{\boldsymbol{f}}_1 = \tilde{\boldsymbol{f}}_{i_1} + \tilde{\boldsymbol{f}}_{i_2} + \ldots + \tilde{\boldsymbol{f}}_{i_k}$$

It is clear that

$$\check{\boldsymbol{f}}_1 = \tilde{W}_N^{\mathrm{T}}(s_i)\check{\boldsymbol{\delta}}_1 \tag{2}$$

where the elements of $\check{\boldsymbol{\delta}}_1$, denoted by $\check{\delta}_{1,j}$, are as follows

$$\check{\delta}_{1,j} = \begin{cases} 1 & \text{if } j \in \{i_1, i_2, \ldots, i_k\} \\ 0 & \text{if } j \notin \{i_1, i_2, \ldots, i_k\} \end{cases} \tag{3}$$

Suppose that there are q non-collinear vectors in $\tilde{W}_N^{\mathrm{T}}(s_i)$. Then we can define q non-collinear vectors $\check{\boldsymbol{f}}_1, \check{\boldsymbol{f}}_2, \ldots, \check{\boldsymbol{f}}_q$ using (2). Without loss of generality, we assume that

$$0 < \angle\check{\boldsymbol{f}}_1 < \angle\check{\boldsymbol{f}}_2 < \ldots < \angle\check{\boldsymbol{f}}_q \leq \pi$$

and $\left[\, \check{\boldsymbol f}_1 \ \ \check{\boldsymbol f}_2 \ \ \ldots \ \ \check{\boldsymbol f}_q \,\right] = \tilde W_N^{\rm T}(s_i) \left[\, \check{\boldsymbol\delta}_1 \ \ \check{\boldsymbol\delta}_2 \ \ \ldots \ \ \check{\boldsymbol\delta}_q \,\right]$. Finally, we define the $2q$ column vectors $\hat{\boldsymbol f}_1, \hat{\boldsymbol f}_2, \ldots, \hat{\boldsymbol f}_q, \hat{\boldsymbol f}_{q+1}, \hat{\boldsymbol f}_{q+2}, \ldots, \hat{\boldsymbol f}_{2q}$ with

$$\begin{aligned}
\hat{\boldsymbol f}_1 &= -\left(\check{\boldsymbol f}_1 + \check{\boldsymbol f}_2 + \cdots + \check{\boldsymbol f}_q\right) \\
&= \left[\check{\boldsymbol f}_1 \ \ \check{\boldsymbol f}_2 \ \ \ldots \ \ \check{\boldsymbol f}_q\right] \underbrace{\left[-1 \ \ -1 \ \ \ldots \ \ -1\right]^{\rm T}}_{\check{\boldsymbol\delta}_1} \\
&= W_N^{\rm T}(s_i) {\rm diag}\left\{ \tilde\delta_1\,,\, \tilde\delta_2\,,\, \tilde\delta_3\,,\, \tilde\delta_4\,,\, \tilde\delta_5\,,\, \tilde\delta_6 \right\} \cdot \\
&\qquad \left[\check{\boldsymbol\delta}_1 \ \ \check{\boldsymbol\delta}_2 \ \ \ldots \ \ \check{\boldsymbol\delta}_q\right] \check{\boldsymbol\delta}_1 \triangleq W_N^{\rm T}(s_i)\hat{\boldsymbol\delta}_1 \\
\hat{\boldsymbol f}_j &= \hat{\boldsymbol f}_{j-1} + 2\check{\boldsymbol f}_{j-1} = W_N^{\rm T}(s_i)\hat{\boldsymbol\delta}_j, \quad j = 2, 3, \ldots, q \\
\hat{\boldsymbol f}_{q+j} &= -\hat{\boldsymbol f}_j = W_N^{\rm T}(s_i)\hat{\boldsymbol\delta}_j, \quad j = q+1, q+2, \ldots, 2q
\end{aligned} \tag{4}$$

Using the vectors $\hat{\boldsymbol f}_j$ we define the sectors

$$\mathcal{S}_j = \left\{ \boldsymbol z \ : \ \angle\hat{\boldsymbol f}_j \le \angle\boldsymbol z < \angle\hat{\boldsymbol f}_{j+1} \right\} \tag{5}$$

where $j = 1, 2, \ldots, q, q+1, q+2, \ldots, 2q$. Then we have

Theorem 1. $\hat{\boldsymbol f}_j$ and $\mathcal{S}_j$, $j = 1, 2, \ldots, q, q+1, q+2, \ldots, 2q$, are defined in (4) and (5).

1) $\rho_N(s_i)$ is the unique solution to the linear equations

$$\rho_N^{-1}(s_i)\boldsymbol n_0(s_i) = \hat{\boldsymbol f}_j + l_j\left[\hat{\boldsymbol f}_{j+1} - \hat{\boldsymbol f}_j\right] \tag{6}$$

where $\hat{\boldsymbol f}_j$ and $\hat{\boldsymbol f}_{j+1}$ are chosen so that $\boldsymbol n_0(s_i) \in \mathcal{S}_j$.

2) With $\rho_N(s_i)$ and l_j given in (6), $\hat{\boldsymbol\delta}_j$ and $\hat{\boldsymbol\delta}_{j+1}$ corresponding to $\hat{\boldsymbol f}_j$ and $\hat{\boldsymbol f}_{j+1}$ in (4), the worst case uncertain parameter vector $\boldsymbol\delta_{N,w,i}$ is given by

$$\boldsymbol\delta_{N,w,i} = -\left(l_j\hat{\boldsymbol\delta}_{j+1} + (1-l_j)\hat{\boldsymbol\delta}_j\right)\rho_N(s_i)$$

From the above procedure, for the numerical example we obtain

$\rho_N(s_1) = \dfrac{N_0(s_1)}{\sum\limits_{j=0}^{5} |w_{N,j}s_1^j|} = 3.8071$

$\boldsymbol\delta_{N,w,1} = -[1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1]^{\rm T} \cdot 3.8071$

$\mathcal{P}(s, \boldsymbol\delta_{N,w,1}) = \dfrac{(s-2)N_1(s)}{(s-2)(s^2+3.7690^2)(s^2-s+1)}$, with $N_1(s) = 0.6193s^4 - 10.7970s^3 - 10.2081s^2 - 8.6447s - 1.0964$;

$\rho_N(s_2) = 1/1.0307 = 0.9702$

$\boldsymbol\delta_{N,w,2} = [1 \ \ -1 \ \ -1 \ \ 1 \ \ 1 \ \ -1]^{\rm T} \cdot 0.9702$

$\mathcal{P}(s, \boldsymbol\delta_{N,w,2}) = \dfrac{1.0970(s^2+3.7690^2)N_2(s)}{(s^2+3.7690^2)(s-2)(s^2-s+1)}$, with $N_2(s) = s^3 + 1.9589s^2 + 1.3457s + 0.3228$;

$\rho_N(s_3) = 5.4930$

$\boldsymbol\delta_{N,w,3} = [5.4930 \ 2.7465 \ -5.4930 \ -5.4929 \ -2.7465 \ 5.4930]^{\rm T}$

$\mathcal{P}(s, \boldsymbol\delta_{N,w,3}) = \dfrac{(s^2-s+1)N_3(s)}{(s^2-s+1)(s-2)(s^2+3.7690^2)}$, with $N_3(s) = 1.5493s^3 + 22.2817s^2 + 28.7465s + 11.4930$

The stabilizability radius is thus

$$\delta_{\rm s} = \min\left\{ 0.9702\,,\, 3.807\,,\, 5.4930 \right\} = 0.9702$$

$\mathcal{P}(s, \boldsymbol\delta)$ is stabilizable as long as $\|\boldsymbol\delta\|_\infty < \delta_s$.

In the second step, we stabilize the nominal plant $P_0(s)$ and determine the stability radius $\delta_{\max}$ of the closed-loop system which is the maximum norm bound of the uncertain parameter vector $\boldsymbol\delta$ s.t. the closed-loop system is stable for all $\boldsymbol\delta$ s.t. $\|\boldsymbol\delta\|_\infty < \delta_{\max}$. It can be readily checked that $P_0(s)$ can be stabilized by a proportional controller $C(s) = K$ as long as $K > K_{\rm cr} = 4.7351$. For the proportional controller, the characteristic polynomial of the closed-loop system composed of $\mathcal{P}(s, \boldsymbol\delta)$ and $C(s) = K$ is $\mathcal{F}(s, \boldsymbol\delta) = D_0(s) + KN_0(s) + K\Delta_N(s)$. Because $D_0(j\omega_0) = 0$ and $N_0(j\omega_0) + \Delta_{N,w,2}(j\omega_0) = 0$, where $\Delta_{N,w,2}(s) = \boldsymbol w_N^{\rm T}(s)\boldsymbol\delta_{N,w,2}$ with $\|\boldsymbol\delta_{N,w,2}\|_\infty = \delta_s$, $\mathcal{F}(s, \boldsymbol\delta)$ has a root pair $\pm j\omega_0$ for all K as long as the norm bound of $\boldsymbol\delta$ reaches δ_s. Hence, the stabilizability radius δ_s is an upper bound for the stability radius $\delta_{\max}$. Because $N_0(s) + \Delta_N(s)$ is an interval polynomial, $\mathcal{F}(s, \boldsymbol\delta)$ is an interval polynomial as well. In this case, the stability radius $\delta_{\max}$ is given by

$$\min\left\{ \rho_1\,,\, \rho_2\,,\, \rho_{\mathcal R}^{-1}(H_0^{-1}H_{\delta,1})\,,\, \rho_{\mathcal R}^{-1}(H_0^{-1}H_{\delta,2}) \right\} \tag{7}$$

where $\rho_1 = |\alpha_5|/(Kw_{N,5}) = 10 + 10/K$, $\rho_2 = |\alpha_0|/(Kw_{N,0}) = 6 - 28.4103/K$, with $\alpha_i = a_i + Kb_i$ being the coefficient of the term s^i in the nominal polynomial $D_0(s) + KN_0(s)$, $\rho_{\mathcal R}(\cdot)$ is the real spectral radius of a matrix, H_0 is the Hurwitz matrix of the nominal polynomial $f_0(s) \triangleq D_0(s) + KN_0(s)$, $H_{\delta,1}$ and $H_{\delta,2}$ are the Hurwitz matrix defined by the uncertainty weighting $Kw_{N,i}$ (see [6] for details). It is clear that $\delta_{\max}$ is a function of the controller parameters.

In the third step, we adjust the parameters of the controller (in our case there is only one adjustable parameter K) to check if the stability radius $\delta_{\max}$ is equal to the stabilizability radius δ_s. ρ_2, $\rho_{\mathcal R}^{-1}(H_0^{-1}H_{\delta,2})$ and $\rho_{\mathcal R}^{-1}(H_0^{-1}H_{\delta,2})$ are depicted in Fig. 1. $\rho_1 = 10 + 1/K > 10 >> \delta_s$ is omitted. From Fig. 1, it is clear that for $K > K_{\rm cr}$ but sufficiently small, $\delta_{\max} = \rho_2$, Because $\lim_{K\to\infty}\rho_2 = 6$ as K increases, $\delta_{\max}$ will be replaced by $\rho_{\mathcal R}^{-1}(H_0^{-1}H_{\delta,1})$; iteratively, it can be seen that when $K \ge K{\rm cr}, w = 16.19684966469630$, the stability radius $\delta_{\max}(K)$ is exactly the stabilizability radius δ_s. Therefore, the interval plant given in (1) can be robustly stabilized if and only if its each member plant is stabilizable.

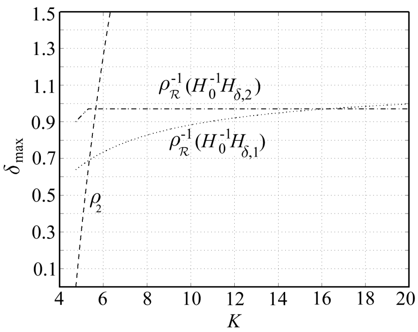

Fig. 1 $\delta_{\max}$ as a function of the controller parameter K

It should be noted here that the stability radius problem can be recast into a real structured singular value (SSV) problem proposed by Doyle[7]. This SSV problem is of rank one and can be reduced to the analytical form (7). In fact, calculating the stability radius is a minimum norm solution problem to linear equations, which always enjoys an analytical solution. Hence, recasting the stability radius problem into SSV without paying attention to its rank—one feature has no theoretical and numerical advantage for providing a solution to the problem solving. In the 7.0 and more advanced versions of Matlab, there is an SSV related function "robuststab", which can be used to get lower and upper bounds for the real stability radius. Using this function to estimate the stability radius of our illustrating example, we get the result UpperBound: Inf, LowerBound: 0.9796 when $K = 16$. The lower bound for the stability radius obtained by Matlab is clearly above the theoretically established upper bound δ_s even for the simplest controller.

We further note that the parametric RSP can be related to the $\mathcal{H}_\infty$ control problem. Because $P_0(s)$ is stabilizable, there exist $D_C(s)$ and $N_C(s)$ with $\deg D_C(s) = n$, $\deg N_C(s) \le n$, such that $D_C(s)D_0(s) + N_C(s)N_0(s) = G_1(s)G_2(s)$, where $G_1(s)$ and $G_2(s)$ are nth order stable polynomials. Using the Youla parametrization of the set of all the stabilizing controllers of $P_0(s)$[8], the RSP can be transformed into the problem of finding a stable transfer function $Q(s)$ such that $1+\underbrace{\dfrac{\Delta_N(s)}{G_1(s)}\left(\dfrac{N_C(s)}{G_2(s)} - Q(s)\dfrac{D_0(s)}{G_1(s)}\right)}_{=\Phi_{\Delta_N}(s)}$ is stable and has a stable inverse for all $\boldsymbol{\delta}_N$ such that $\|\boldsymbol{\delta}_N\|_\infty \le \delta$. Because $\Phi_{\Delta_N}(s)$ is stable for all $Q(s) \in \mathcal{RH}_\infty$, the latter condition is implied by $\|\Phi_{\Delta_N}(s)\|_\infty < 1$, which is equivalent to

$$\left\|\frac{\delta\Delta_{N,\max}(s)}{G_1(s)}\left(\frac{N_C(s)}{G_2(s)} - Q(s)\frac{D_0(s)}{G_1(s)}\right)\right\|_\infty = \|\Phi(s)\|_\infty < 1$$

where $\Delta_{N,\max}(s)$ can be any one of the four Kharitonov polynomials $K_{\Delta_N,i}(s)$, $i = 1, 2, 3, 4$, of $\Delta_N(s)$ when $\delta = 1$. Using the Nevanlinna-Pick interpolation theory[9], we can show that for the numerical example there exist $\mathcal{RH}_\infty$-functions $Q(s)$ such that $\|\Phi(s)\|_\infty < 1$ as long as $\delta < \delta_s$. Therefore, the RSP for the interval plant (1) can be non-conservatively reduced to an $\mathcal{H}_\infty$-control problem.

3 Concluding remarks

Using a numerical example, the stabilizability and the robust stabilization problems of an interval plant family have been investigated. For the sake of simplicity, the plant $\mathcal{P}(s, \boldsymbol{\delta})$ is assumed to have uncertainties only in the numerator. Hence, the calculation of δ_s is reduced to the evaluation of $\rho_N(s_i)$ at a few isolated points $s_i \in \bar{\mathcal{C}}_+$. $\mathcal{P}(s, \boldsymbol{\delta})$ has a pure imaginary pole pair, a complex conjugate pole pair with positive real part, and a positive pole, so that the three cases in the calculation of the stabilizability radius δ_s, that is, $\rho_N(j\omega)$, $\rho_N(\sigma + j\omega)$ with $\sigma > 0$ and $\omega > 0$ and $\rho_N(\sigma)$ with $\sigma \ge 0$ can be all covered. The key steps in the calculation of δ_s have been illustrated using this example. For the general case where uncertainties exist in the numerator and the denominator, one- or two-dimensional sweeping is needed for finding $\delta_s = \inf_{s^* \in \bar{\mathcal{C}}_+} \rho(s^*)$, where $\rho(s^*) = \max_{\alpha=N,D}\{\rho_\alpha(s^*)\}$, which can be completed by any one of the various one- or two-variable graphics routines (*e.g.* the function "mesh" of Matlab), as long as $\rho_\alpha(s^*)$ can be exactly determined, as has been done in this paper. Hence, the problem of calculating the stabilizability radius for an interval plant has been completely solved in this paper. It is clear that the results of the edge and the extreme point for checking the robust stability of interval systems can be applied only to a stabilizable interval plant, and the stabilizability radius provides an exact measure for checking the stabilizability of an interval plant. The algorithm for calculating δ_s can be easily adapted for the general case where the coefficient vectors of $N_0(s)+\Delta_N(s)$ and $D_0(s) + \Delta_D(s)$ are affine in the ∞-norm bounded (interval type) uncertain vectors $\boldsymbol{\delta}_N$ and $\boldsymbol{\delta}_D$, respectively, subject to suitably defined vectors $\boldsymbol{w}_N^{\mathrm{T}}(s)$ and $\boldsymbol{w}_D^{\mathrm{T}}(s)$. Moreover, the stabilizability radius in terms of any vector norm can be defined and calculated in a similar manner. It is clear that stability is the most fundamental requirement for any control system, and for a control system to be stable the plant must be stabilizable. The stabilizability radius provides a measure for the unavoidable parameter uncertainties in every control system so that the most fundamental requirement may be fulfilled. The choice of a stable $N_0(s)$ has also some insights. The locations of the poles and the zeros of the nominal plant play an important role in the two-dimensional sweeping. Indeed, if $N_0(s)$ and $D_0(s)$ share some common root $s^* \in \bar{\mathcal{C}}_+$, $\delta_s = 0$. It can be expected that when $N_0(s)$ and $D_0(s)$ have no common root in $\bar{\mathcal{C}}_+$, but $N_0(s)$ has a root $s_N \in \bar{\mathcal{C}}_+$ near a root $s_D \in \mathcal{C}$ of $D_0(s)$ and vice versa, $\delta_s = \inf_{s^* \in \bar{\mathcal{C}}_+} \rho(s^*)$ will be achieved at some $s^* \in \bar{\mathcal{C}}_+$ "between" s_N and s_D. For the example illustrated, it can be expected that $\delta_s = \rho_N(j\omega_0)$, because the pole $j\omega_0$ is nearest to the zeros of $P_0(s)$, as also has been confirmed by the calculation. This observation may be helpful in an effective and exact sweeping for finding δ_s. $\delta_s = \rho_N(j\omega_0)$ means that the calculation of δ_s is performed on the imaginary axis. However, recent research reveals that the stabilizability of a plant family may be also a necessary and sufficient solvability condition for the associated RSP when δ_s is attained in the open right half plane, and a robust stabilizer can be designed systematically.

We hope that research interests will be invoked by the results presented in this paper and that the parametric robust stabilization problem can be completely and analytically solved in the near future.

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WU Qing-He Professor at Beijing Institute of Technology. His research interest covers H_∞ control, robust control, and multidimensional systems. E-mail: qinghew@bit.edu.cn