Stochastic Maximum Principle for a Kind of Risk-sensitive Optimal Control Problem and Application to Portfolio Choice

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Abstract In this paper, we mainly study a kind of risk-sensitive optimal control problem motivated by a kind of portfolio choice problem in certain financial market. Using the classical convex variational technique, we obtain the maximum principle for this kind of problem. The form of the maximum principle is similar to its risk-neutral counterpart. But the adjoint equation and the variational inequality heavily depend on the risk-sensitive parameter γ. This is one of the main difference from the risk-neutral case. We use this result to solve a kind of optimal portfolio choice problem. The optimal portfolio strategy obtained by the Bellman dynamic programming principle is a special case of our result when the investor only invests the home bond and the stock. Computational results and figures explicitly illustrate the relationships between the maximum expected utility and the parameters of the model.

Key words Stochastic maximum principle, risk-sensitive control, convex variational technique, portfolio choice

1 Introduction

Since the publication of the deterministic maximum principle by Pontryagin *et al.*[1], much work has been done on its generalization to stochastic systems (See [2] $\sim$ [4]). But one of their assumptions is that the functions in the cost functional satisfies the usual linear growth or square growth conditions. Therefore, this assumption excludes at least one important case which rises from the portfolio choice problem in some financial market–the constant relative risk aversion (CRRA) case (See [5], for example).

For the sake of convenience, let us state the problem in detail below. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a complete filtered probability space with the natural filtration $\mathcal{F}_t = \sigma\{W(s), V(s); 0 \leq s \leq t\}$, where $(W(\cdot), V(\cdot))$ is a standard two-dimensional Brownian motion defined on this space with values in $\mathbf{R}^2$. We assume $\mathcal{F} = \mathcal{F}_T$, where $T > 0$ is a fixed time horizon. Throughout the paper, we only study the problem in the time interval $[0, T]$.

We consider a financial market in which two securities can be continuously traded. One of them is a foreign currency deposit, whose price $B(t)$ is assumed to satisfy

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t$$

where $r(t)$ is the interest rate of this kind of foreign currency deposit in bank at time t. The other asset is stock, and the price is described by

$$\mathrm{d}S(t) = \mu(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t)$$

where $\mu(t)$ is the instantaneous expected rate of return, and $\sigma(t)$ is the instantaneous volatility.

Now, let us consider an investor who invests in the foreign currency deposit and the stock and whose decisions cannot affect the prices in the financial market. The numeration is the domestic currency. There exists the real exchange risk for the currency deposit, and the currency exchange rate $e(t)$ satisfies

$$\mathrm{d}e(t) = \alpha(t)e(t)\mathrm{d}t + \beta(t)e(t)\mathrm{d}V(t)$$

where $\alpha(t)$ is the instantaneous expected rate in the currency exchange market, and $\beta(t)$ is the instantaneous volatility. We need to change the foreign currency deposit value into domestic currency and let $\theta(t) = e(t)B(t)$. Then Itô's formula implies that

$$\mathrm{d}\theta(t) = \theta(t)(r(t) + \alpha(t))\mathrm{d}t + \theta(t)\beta(t)\mathrm{d}V(t)$$

We assume that the trading of the investor is self-financed, *i.e.*, there is no infusion or withdrawal of funds over $[0, T]$. We denote by $x(t)$ the wealth of the investor with some initial endowment $x_0 > 0$, by $\pi(t)$ the amount that he invests in the stock. Then, the investor has $x(t) - \pi(t)$ savings in bank. Under the notations and interpretations, we have

$$\mathrm{d}x(t) = (x(t) - \pi(t))\frac{\mathrm{d}\theta(t)}{\theta(t)} + \pi(t)\frac{\mathrm{d}S(t)}{S(t)}$$

Obviously, all the wealth of the investor is modeled by

$$\begin{cases} \mathrm{d}x(t) = [(r(t) + \alpha(t))x(t) + (\mu(t) - r(t) - \alpha(t))\pi(t)]\mathrm{d}t + \\ \qquad \sigma(t)\pi(t)\mathrm{d}W(t) + (x(t) - \pi(t))\beta(t)\mathrm{d}V(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

Definition 1. An admissible portfolio strategy $\pi(t)$ is a $\mathcal{F}_t-$adapted square integrable progress with values in $\mathbf{R}$. The set of them is denoted by $\mathcal{A}_{ad}$.

The investor wants to maximize his expected utility

$$J(\pi(\cdot)) = \frac{K}{1-R}\mathrm{E}[x(T)]^{1-R} \tag{2}$$

by choosing an appropriate portfolio from the admissible set $\mathcal{A}_{ad}$, where $K > 0$ is a fixed constant and $R \in (0, 1)$ is so called the Arrow-Pratt index of risk aversion[6].

In next section, we study the following optimal control problem which is a generalization of the above problem (1)

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and (2), *i.e.*, the control system has the non-linear form

$$\begin{cases} \mathrm{d}x(t) = b(t,x(t),\pi(t))\mathrm{d}t + f(t,x(t),\pi(t))\mathrm{d}W(t) + \\ \qquad g(t,x(t),\pi(t))\mathrm{d}V(t) \\ x(0) = x_0 \end{cases} \tag{3}$$

where x_0 is given and deterministic. The cost functional is defined by (4) in next section. Using the classic convex variational technique, we derive the maximum principle. In Section 3, we obtain the explicit optimal portfolio of the problem (1) and (2) by combining the maximum principle obtained in Section 2 with a direct formulation method. We also study the sensitivities of the investor′s optimal portfolio strategy and the maximum expected utility on the parameters of the model in Section 4. Computational results and figures in this section also support our viewpoints.

2 Optimal control problem and maximum principle

In this section, we consider the one-dimensional optimal control problem mentioned in Section 1.

Let $n = \max\{2, \lceil 2\gamma \rceil - 1\}$, where $\gamma > 0$ is a constant and $\lceil x \rceil$ denotes the integer part of x. We denote by $L^n(0,T)$ the space of $\mathcal{F}_t-$adapted processes with values in $\mathbf{R}$ such that $\mathrm{E}\int_0^T [x(t)]^n \mathrm{d}t < +\infty$. Let U be a non-empty convex subset of $\mathbf{R}$. We set $\mathcal{U}_{ad} = \{\pi(\cdot) \in L^n(0,T) : \pi(t) \in U, a.s., a.e.\}$. An element of $\mathcal{U}_{ad}$ is called admissible.

We assume that

H1. The functions b, f, and $g : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuously differential with respect to (x,π) and their partial derivatives are uniformly bounded.

Our problem is to maximize the following cost functional

$$J(\pi(\cdot)) = \frac{1}{\gamma}\mathrm{E}[\Phi(x(T))]^\gamma, \qquad \gamma > 0 \tag{4}$$

subject to the stochastic control system (3) and the admissible set $\mathcal{U}_{ad}$. Our task is to seek the necessary condition, so called the maximum principle, of the optimal control.

The cost functional (4) subject to (3) formulates a kind of risk-sensitive optimal control problem[7]. $\gamma > 0$, a fixed constant, is called the risk-sensitive parameter. If $\gamma = 1$, then (4) reduces to the usual risk-neutral case. See [2∼4], for example. Obviously, the function $1/\gamma[\Phi(x)]^\gamma$ in (4) does not satisfy the conditions of [3∼5]. Under our framework, we use the classical convex variational technique to obtain the maximum principle for this kind of problem.

For our aim, we need the following hypotheses on Φ:

H2. The function $\Phi : \mathbf{R} \to [0,+\infty)$ is continuously differential in x. Φ is bounded by $C(1+x)$ and its derivative Φ_x is also bounded.

H3. If $0 < \gamma \leq 1$, we assume $\mathrm{E}[\Phi(x(T))]^{(2\gamma-2)} < +\infty$; if $\gamma > 1$ then $\mathrm{E}[x(T)]^{(2\gamma-2)} < +\infty$.

Remark 1. It appears as if H2 and H3 were rigorous, but it is not difficult to seek some functions satisfying the above two hypotheses. For example $\Phi(x) = \sin x + 1$, $\sin(\cos x) + 1$, $\cos(\sin x)$, *etc.* In particular, if $x \in [0,+\infty)$, then we let $\Phi(x) = x$, $\log(1+x)$, and $\arctan x$.

Let $\pi(\cdot)$ be an optimal control for the problem (3) and (4), and $x(\cdot)$ be the corresponding optimal trajectory. Let $\pi_1(\cdot) \in L^n(0,T)$ be given such that $\pi(\cdot) + \pi_1(\cdot) \in \mathcal{U}_{ad}$. We take $\pi^\varepsilon(\cdot) = \pi(\cdot) + \varepsilon\pi_1(\cdot)$, $0 \leq \varepsilon \leq 1$. Since $\mathcal{U}_{ad}$ is convex, $\pi^\varepsilon(\cdot) \in \mathcal{U}_{ad}$. We denote by $x^\varepsilon(\cdot)$ the trajectory of the control system (3) corresponding to $\pi^\varepsilon(\cdot)$.

Let us introduce the variational equation

$$\begin{cases} \mathrm{d}x_1(t) = [b_x(t,x(t),\pi(t))x_1(t) + b_\pi(t,x(t),\pi(t))\pi_1(t)]\mathrm{d}t + \\ \qquad [f_x(t,x(t),\pi(t))x_1(t) + f_\pi(t,x(t),\pi(t))\pi_1(t)]\mathrm{d}W(t) + \\ \qquad [g_x(t,x(t),\pi(t))x_1(t) + g_\pi(t,x(t),\pi(t))\pi_1(t)]\mathrm{d}V(t) \\ \quad x_1(0) = 0 \end{cases} \tag{5}$$

The following lemma can be proved similar to [2].

Lemma 1. Under the hypothesis H1, we have

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\mathrm{E}|\tilde{x}^\varepsilon(t)|^2 = 0$$

where we have already used the notation $\tilde{x}^\varepsilon(t) = \frac{1}{\varepsilon}[x^\varepsilon(t) - x(t)] - x_1(t)$.

Lemma 2. (**Variational inequality**) Let H1∼H3 hold. Then, we have

$$\mathrm{E}\{[\Phi(x(T))]^{\gamma-1}\Phi_x(x(T))x_1(T)\} \leq 0 \tag{6}$$

where $x_1(T)$ is given by the variational equation (5).

Proof. From the fact that $J(\pi^\varepsilon(\cdot)) - J(\pi(\cdot)) \leq 0$, it is easy to see

$$\frac{1}{\gamma}\mathrm{E}\{[\Phi(x^\varepsilon(T))]^\gamma - [\Phi(x(T))]^\gamma\} \leq 0$$

By the Taylor formula, we have

$$\begin{aligned} & J(\pi^\varepsilon(\cdot)) - J(\pi(\cdot)) = \\ & \varepsilon\mathrm{E}\{[\Phi(x(T))]^{\gamma-1}\Phi_x(x(T))(x_1(T) + \tilde{x}^\varepsilon(T))\} + \\ & \varepsilon\mathrm{E}\{[\Phi(x(T))]^{\gamma-1}[\Phi_x(x(T) + \theta\varepsilon(x_1(T) + \tilde{x}^\varepsilon(T))) - \\ & \Phi_x(x(T))](x_1(T) + \tilde{x}^\varepsilon(T))\} \leq 0 \end{aligned}$$

where $\theta \in (0,1)$. Obviously, for any $\gamma > 0$,

$$\begin{aligned} & [\Phi(x(T))]^{2\gamma-2}[\Phi_x(x(T) + \theta\varepsilon(x_1(T) + \tilde{x}^\varepsilon(T))) - \\ & \Phi_x(x(T))]^2 \to 0, \qquad \varepsilon \to 0 \end{aligned}$$

Since for $\gamma > 1$ there exists

$$\begin{aligned} & [\Phi(x(T))]^{2\gamma-2}[\Phi_x(x(T) + \theta\varepsilon(x_1(T) + \tilde{x}^\varepsilon(T))) - \\ & \Phi_x(x(T))]^2 \leq 4^\gamma C_1^2 C^{2\gamma-2}(1 + |x(T)|^{2\gamma-2}) \end{aligned}$$

where we have already used the inequality $|m_1 + m_2|^n \leq 2^n(|m_1|^n + |m_2|^n)$, $n > 0$; and for $0 < \gamma \leq 1$, it follows that

$$\begin{aligned} & [\Phi(x(T))]^{2\gamma-2}[\Phi_x(x(T) + \theta\varepsilon(x_1(T) + \tilde{x}^\varepsilon(T))) - \\ & \Phi_x(x(T))]^2 \leq 4C_1^2[\Phi(x(T))]^{2\gamma-2} \end{aligned}$$

Then, from the Lebesgue controlled convergence theorem we derive

$$\begin{aligned} & \mathrm{E}\{[\Phi(x(T))]^{\gamma-1}[\Phi_x(x(T) + \theta\varepsilon(x_1(T) + \tilde{x}^\varepsilon(T)))] \\ & (x_1(T) + \tilde{x}^\varepsilon(T))\} \to 0, \qquad \varepsilon \to 0 \end{aligned}$$

By Lemma 1, it follows that

$$\varepsilon\mathrm{E}\{[\Phi(x(T))]^{\gamma-1}\Phi_x(x(T))x_1(T)\} + o(\varepsilon) \leq 0$$

The above inequality is divided by ε and let $\varepsilon \to 0$, then we obtain the desired result. □

For deriving the maximum principle, we introduce the following adjoint equation

$$\left\{\begin{array}{l} -\mathrm{d}p(t) = [b_x(t,x(t),\pi(t))p(t) + f_x(t,x(t),\pi(t))q(t)+ \\ \qquad g_x(t,x(t),\pi(t))k(t)]\mathrm{d}t - q(t)\mathrm{d}W(t) - k(t)\mathrm{d}V(t) \\ p(T) = [\Phi(x(T))]^{\gamma-1}\Phi_x(x(T)) \end{array}\right. \tag{7}$$

H1∼H3 imply that (7) has a pair of unique solution $(p(\cdot),q(\cdot),k(\cdot)) \in L^2(0,T)\times L^2(0,T)\times L^2(0,T)$.

By applying Itô's formula to $\langle p(t), x_1(t)\rangle$, it can be checked from Lemma 2 that

$$\mathrm{E}\int_0^T \langle H_\pi(t,x(t),\pi(t),p(t),q(t),k(t)),\pi_1(t)\rangle \mathrm{d}t \le 0 \tag{8}$$

where the Hamiltonian function $H:[0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is defined as

$$\begin{array}{l} H(t,x(t),p(t),q(t),k(t),\pi(t)) = \\ \qquad \langle b(t,x(t),\pi(t)),p(t)\rangle + \langle f(t,x(t),\pi(t)),q(t)\rangle + \\ \qquad \langle g(t,x(t),\pi(t)),k(t)\rangle \end{array}$$

So for any $\bar\pi \in U$, we have, a.s., a.e.,

$$\langle H_\pi(t,x(t),p(t),q(t),k(t),\pi(t)),\bar\pi - \pi(t)\rangle \le 0 \tag{9}$$

Therefore, we get the following theorem.

Theorem 1. **(Maximum principle)** Assume H1∼H3 hold. Let $\pi(\cdot)$ be the optimal control to the risk-sensitive optimal control problem (3) and (4), and $x(\cdot)$ be the corresponding optimal trajectory. Then, for any $\bar\pi \in U$, the maximum condition (9) holds.

Remark 2. Although the form of the maximum condition (9) is similar to its risk-neutral counterpart, it is worth pointing out that both (6) and (7) explicitly depend on γ. To the best of our knowledge, this is not seen in the existing literature of the risk-neutral case.

3 Application to portfolio choice problem

In this section, we study the optimal portfolio choice problem (1) and (2) mentioned in Section 1.

We assume that

H4. $\sigma(\cdot)>0$, $\beta(\cdot)>0$, $r(\cdot)$, $\mu(\cdot)$ and $\alpha(\cdot)$ are deterministic and uniformly bounded, and $\sigma^{-1}(\cdot)$, $\beta^{-1}(\cdot)$ are also uniformly bounded.

Under the hypothesis H4, we can easily get from Theorem 1 that

$$(\mu(t)-r(t)-\alpha(t))p(t)+\sigma(t)q(t)-\beta(t)k(t)=0 \tag{10}$$

Let $\pi(\cdot)=m(\cdot)x(\cdot)$, here and below $m(\cdot)$ and $n(\cdot)$ are deterministic functions, which will be determined later on. Then, the corresponding optimal wealth equation and the adjoint equation are

$$\left\{\begin{array}{l} \mathrm{d}x(t) = [r(t)+\alpha(t)+(\mu(t)-r(t)-\alpha(t))m(t)]x(t)\mathrm{d}t+ \\ \qquad \sigma(t)m(t)x(t)\mathrm{d}W(t)+(1-m(t))\beta(t)x(t)\mathrm{d}V(t) \\ x(0)=x_0 \end{array}\right. \tag{11}$$

and

$$\left\{\begin{array}{l} -\mathrm{d}p(t) = [(r(t)+\alpha(t))p(t)+\beta(t)k(t)]\mathrm{d}t- \\ \qquad q(t)\mathrm{d}W(t)-k(t)\mathrm{d}V(t) \\ p(T)=K[x(T)]^{-R} \end{array}\right. \tag{12}$$

To get the explicit optimal portfolio $\pi(\cdot)$, the usual solving method is to use Feynman-Kac formula to derive a partial differential equation (PDE), then combine the maximum condition to obtain the desired result[8]. However, it is difficult to obtain an explicit solution of the PDE. But if we note the terminal condition of (12), then we can give a direct formulation method to avoid the complicated computation steps. The fact below shows that the direct method is indeed very convenient and very useful for us to treat the problem (1) and (2).

Let

$$p(t)=K[x(t)]^{-R}\mathrm{e}^{\int_t^T n(s)\mathrm{d}s} \tag{13}$$

Using Itô's formula to $p(t)$ defined by (13), then, we derive

$$\begin{array}{l} -\mathrm{d}p(t) = \{n(t)+R[r(t)+\alpha(t)+(\mu(t)-r(t))m(t)] - \\ \dfrac{1}{2}(R+1)[\sigma^2(t)m^2(t)+(1-m(t))^2\beta^2(t)]\}p(t)\mathrm{d}t + \\ Rm(t)\sigma(t)p(t)\mathrm{d}W(t)+R(1-m(t))\beta(t)p(t)\mathrm{d}V(t) \end{array}$$

Comparing the drift term and the diffusion term of the above expression with (12), we have

$$\begin{array}{rcl} n(t) &=& r(t)+\alpha(t)-R\{r(t)+\alpha(t)+ \\ && [(\mu(t)-r(t)-\alpha(t))m(t)+(1-m)\beta^2(t)] - \\ && \dfrac{1}{2}(R+1)[\sigma^2(t)m^2(t)+(1-m(t))^2\beta^2(t)]\} \\ q(t) &=& -Rm(t)\sigma(t)p(t) \\ k(t) &=& -R(1-m(t))\beta(t)p(t) \end{array} \tag{14}$$

Substituting (14) into (10), we easily get

$$m(t)=\frac{\mu(t)-r(t)-\alpha(t)+R\beta^2(t)}{(\sigma^2(t)+\beta^2(t))R}$$

Therefore,

$$\pi(t) = \frac{\mu(t)-r(t)-\alpha(t)+R\beta^2(t)}{(\sigma^2(t)+\beta^2(t))R}x(t) \tag{15}$$

where $x(t)$ is the solution of the following corresponding optimal wealth equation

$$\left\{\begin{array}{l} \mathrm{d}x(t) = x(t)\{[r(t)+\alpha(t)+ \\ \quad (\mu(t)-r(t)-\alpha(t))\dfrac{\mu(t)-r(t)-\alpha(t)+R\beta^2(t)}{(\sigma^2(t)+\beta^2(t))R}]\mathrm{d}t+ \\ \quad \dfrac{\mu(t)-r(t)-\alpha(t)+R\beta^2(t)}{(\sigma^2(t)+\beta^2(t))R}\sigma(t)\mathrm{d}W(t)+ \\ \quad \dfrac{r(t)+\alpha(t)+R\sigma^2(t)-\mu(t)}{(\sigma^2(t)+\beta^2(t))R}\beta(t)\mathrm{d}V(t)\} \\ x(0)=x_0 \end{array}\right. \tag{16}$$

Clearly, $x(t)>0$ and $\mathrm{E}[x(t)]^{-2R}<+\infty$. Therefore, the hypothesis H3 holds indeed.

Proposition 1. Assume that H4 holds. Then, the optimal solution to the optimal portfolio choice problem (1) and (2) is given by (15) and (16).

Remark 3. We must point out that the optimal investment proportion $\pi(\cdot)/x(\cdot)$ defined by (15) clearly depends on the Arrow-Pratt index of risk aversion of the investor R. $\pi(\cdot)/x(\cdot)$ is decreasing with respect to R, $r(\cdot)$, $\alpha(\cdot)$, $\sigma(\cdot)$, $\beta(\cdot)$ and increasing with respect to $\mu(\cdot)$. These phenomena coincide with our intuition. This is usually the so-called sensitivities on the parameters of the model.

4 An interesting example

In this section, we study an interesting example. For simplification, assume that $\alpha(\cdot) = \beta(\cdot) \equiv 0$, *i.e.*, the investor invests the bond in home market. Then, it is under the framework of Merton[5] and Xu[8]. From Proposition 1 we have

Corollary 1. Assume that H4 holds and $\alpha(\cdot) = \beta(\cdot) \equiv 0$. Then the optimal portfolio and the corresponding maximum utility of the problem (1) and (2) are given by

$$\begin{aligned} \pi(t) \quad = \quad & \frac{\mu(t) - r(t)}{\sigma^2(t)R} \mathrm{e}^{\int_0^t [r(s) + (1 - \frac{1}{2R}) \frac{(\mu(s) - r(s))^2}{\sigma^2(s)R}] \mathrm{d}s} \cdot \\ & \mathrm{e}^{\int_0^t \frac{\mu(s) - r(s)}{\sigma(s)R} \mathrm{d}W(s)} x_0 \end{aligned} \tag{17}$$

$$J_{\max}(\pi(\cdot)) = \frac{K}{1-R} x_0^{1-R} \mathrm{e}^{(1-R)\int_0^T [r(t) + \frac{(\mu(t) - r(t))^2}{2\sigma^2(t)R}] \mathrm{d}t}$$

We now study the relationships between the investor′s expected utility $J_{\max}(\pi(\cdot))$ and the parameters K, x_0, $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$. For simplification, hereinafter we assume that the condition $\mu(\cdot) \geq \sigma^2(\cdot)R$ holds. For the case $\mu(\cdot) < \sigma^2(\cdot)R$, it is easy to see. Clearly, $J_{\max}(\pi(\cdot))$ is increasing with respect to K, x_0, T, $\mu(\cdot)$ and decreasing with respect to $\sigma(\cdot)$. However, it is difficult for us to identify the influence of R on $J_{\max}(\pi(\cdot))$.

Generally speaking, the investor pays more attention to the influence of the parameters $\mu(\cdot)$ and $r(\cdot)$ on his maximum expected utility $J_{\max}(\pi(\cdot))$. Therefore, to do further research on the relationships between them, we assume that $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$ are all non-zero constants. Then the maximum expected utility can be rewritten as

$$J_{\max}(\pi(\cdot)) = \frac{K}{1-R} x_0^{1-R} \mathrm{e}^{(1-R)T[r + \frac{(\mu - r)^2}{2\sigma^2 R}]} \tag{18}$$

In the following figures we suppose that the time unit in the model is one year and let $x_0 = 1$, $R = K = \sigma = 0.5$.

In Fig. 1, we let the interest rate $r = 0.125$. When $\mu \geq 0.125$, $J_{\max}(\pi(\cdot))$ is an increasing function of μ. In particular, if $\mu = 0.125$, the optimal portfolio strategy (17) implies that the investor should save all his wealth $x(\cdot)$ in bank at the interest rate 0.125. Then, the investor′s maximum expected utility is $J_{\max}(0) = \mathrm{e}^{0.0625}$.

We plot the relationships between $J_{\max}(\pi(\cdot))$ and r in Figs. 2∼4. Four points are worth noting:

1) From (18), when $r < \mu - \sigma^2 R$, $J_{\max}(\pi(\cdot))$ is a decreasing function of r; when $r > \mu - \sigma^2 R$, $J_{\max}(\pi(\cdot))$ is an increasing function of r; when $r = \mu - \sigma^2 R$, $J_{\max}(\pi(\cdot))$ attains its minimum value $K/(1-R)x_0^{1-R}\mathrm{e}^{\frac{1}{2}T(1-R)(2\mu - \sigma^2 R)}$. The above Figs. 2∼4 explicitly illustrate these theoretical results.

2) In Fig. 2, we let $\mu = 0.2$ and suppose $1/2\mu < \sigma^2 R < \mu$ and $\mu^2/(2\sigma^2 R) > \mu - 1/2\sigma^2 R$ hold. When the interest rate is 0.2, we know that by (15) the investor′s optimal portfolio strategy is to save all his wealth $x(\cdot)$ in bank at the interest rate 0.2. Then, his maximum expected utility is equal to $\mathrm{e}^{0.1}$.

3) In Fig. 3, we let $\mu = 0.4$ and assume that $\mu > 2\sigma^2 R$. If $r = 0$, the investor′s optimal portfolio strategy is $\pi(\cdot) = 3.2x(\cdot)$. It is to say that the investor should borrow $\pi(\cdot) - x(\cdot) = 2.2x(\cdot)$ from bank at the interest rate zero and invest

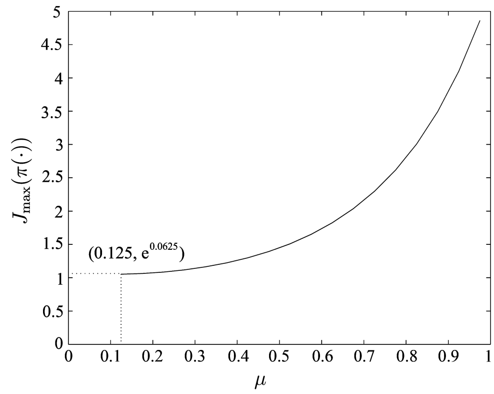

Fig. 1 The relationship between $J_{\max}(\pi(\cdot))$ and μ

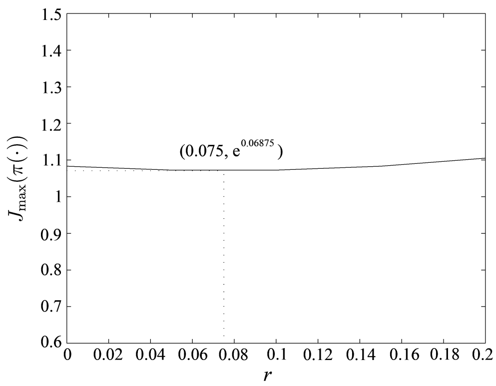

Fig. 2 The relationship between $J_{\max}(\pi(\cdot))$ and r

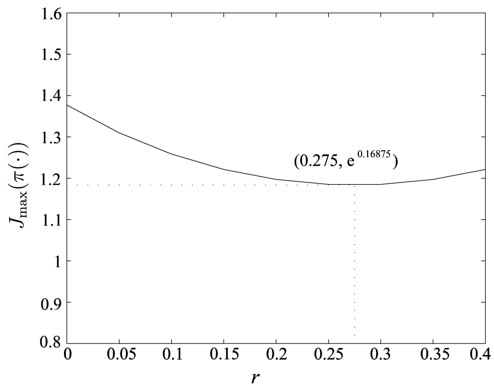

Fig. 3 The relationship between $J_{\max}(\pi(\cdot))$ and r

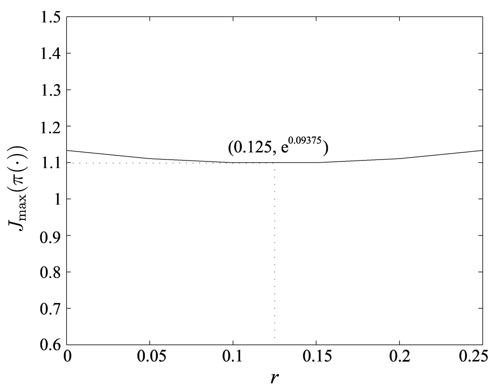

Fig. 4 The relationship between $J_{\max}(\pi(\cdot))$ and r

$\pi(\cdot) = 3.2x(\cdot)$ in the stock. Then, the maximum expected utility in this case is $e^{0.32}$. Fig. 3 also describes the fact that if $r \leq 0.15$ then the optimal portfolio strategy is to borrow money from bank and invest all his wealth in the stock.

4) In Fig. 4, we let $\mu = 0.25$. It shows a special case under the condition $\mu = 2\sigma^2 R$. When the interest rate r is equal to zero or 0.25, the investor′s maximum expected utility has the same value of $e^{0.125}$. It implies that the investor has two kinds of different portfolio choice chances to obtain the same maximum expected utility. One is to borrow $x(\cdot)$ from bank at the interest rate zero and invests all his wealth $2x(\cdot)$ in the stock. The other is to save all his wealth $x(\cdot)$ in bank at the interest rate $r = 0.25$.

5 Conclusion

In this paper, using the classical convex variational technique, we derived the maximum principle for a kind of risk-sensitive optimal control problem rising from a kind of optimal portfolio choice problem in some financial market. As an application of the risk-sensitive maximum principle obtained in Section 2, we studied a kind of optimal portfolio choice problem. Some computational results and figures provided in Section 4 further support our viewpoints.

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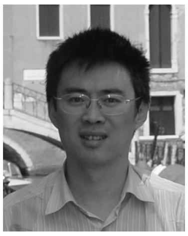

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