Stability Analysis of PI TCP/AQM Networks: A Parameter Space Approach

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Abstract This work focuses on deriving the necessary and sufficient stability condition of transmission control protocol (TCP) network with a proportional-integral (PI) active queue management (AQM) scheme *via* a parameter space approach. First, a fluid-flow TCP′s nonlinear model is converted into a linear time-delay system of neutral type. Second, the stability of the closed-loop system is characterized in terms of the network's and the controller′s parameters. By simulation studies we investigate the boundary′s relations of these two kinds of parameters in the parameter space and illustrate how PI controller′s parameters affect the stability. Finally, different stability conditions are compared to show the less conservatism of our necessary- and sufficient-condition-based method, and simulation experiments by both Matlab and NS-2 are conducted to prove our claim.

Key words Congestion control, TCP, PI control, stability

1 Introduction

With the development of the internet, the amount of data packets transferred has increased rapidly. To avoid the performance degradation of the internet, the network transmission control protocol (TCP) itself has a mechanism of adjusting the packet sending rates by probing the congestion signal at routers[1]. But, such an ability of TCPs is limited. The active queue management (AQM) routers play a key role in meeting the demand for performances in internet application and for preventing the internet from collapsing due to heavy traffic load. Some typical AQM schemes include drop tail[2], random early detection (RED)[3], random exponential marking (REM)[4], proportional (P), and proportional-integral (PI)[5]. From the viewpoint of control theory, the AQM scheme amounts to a feedback control law that enables us to apply the control principles to analyze and to design the AQM scheme in the Internet environment. This study is based on the use of a recently developed dynamic model of TCP[6]. The network simulator (NS) simulations showed that this model accurately captured the TCP dynamic behavior. In [5], this nonlinear model was linearized about an operating point to gain insight into feedback control, and P and PI AQM schemes were introduced by comparison to RED. In [7], rate-based PI AQM schemes were used for high-speed links with small buffers. In [8], the case of multiple TCP sources and multiple congested routers was considered for the stability analysis based on the multiple variable frequency domain theory. The robust property against the round-trip time delay was analyzed in [9] and a new TCP feedback structure was proposed to further improve the tolerance of network-induced delays. In the present paper, we directly employ the necessary and sufficient stability condition in the parameter space to investigate the relations of network parameters bounds and the PI controller parameters bounds and attempt to present the complete stability regions in the pa-

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rameter space.

2 TCP model

In [6], a dynamic model of TCP behavior was proposed using fluid-flow and stochastic differential equation analysis. The NS simulation results showed that this model accurately captured the dynamics of TCP. Under some approximate conditions[5], this model is described by the following coupled, nonlinear differential equations

$$\begin{cases}\dot{w}(t) = \dfrac{1}{R} - \dfrac{w^2(t)}{2R}p_R \\ \\ \dot{q}(t) = \dfrac{N}{R}w(t) - C\end{cases} \tag{1}$$

where $w(t)$ is the average TCP window size, N the number of TCP sessions (load factor), $p_R = p(t-R)$ denotes the delayed packet marking probability, and $p \in [0,1], R = \dfrac{q}{C} + T_p$ stands for the round-trip time delay composed of both the queuing and the transport delays with C being the link capacity, T_P the propagation delay, and q the queue length of congested routers.

The objective of AQM scheme in the network is to relate the queue length q in the bottleneck router to the packet marking probability p, to signal the TCP source adjusting their packet sending rates earlier, and to adapt the queue length restriction at bottleneck routers.

The small-signal linearization of model (1) about the operating point (w_0, q_0, p_0) is[5]

$$\begin{cases}\delta\dot{w}(t) = -\dfrac{2N}{R_0^2C}\delta w(t) - \dfrac{R_0C^2}{2N^2}\delta p(t-R_0) \\ \delta\dot{q}(t) = \dfrac{N}{R_0}\delta w(t) - \dfrac{1}{R_0}\delta q(t)\end{cases} \tag{2}$$

where $\delta w(t) = w(t) - w_0$, $\delta q(t) = q(t) - q_0$, $\delta p(t) = p(t) - p_0$.

Consider the following PI controller

$$\delta p(t) = K_{PI}[\delta q(t) + \frac{1}{T_I}\int_0^t \delta q(s)\mathrm{d}s]$$

Replacing t by $t - R_0$ yields

$$\delta p(t-R_0) = K_{PI}[\delta q(t-R_0) + \frac{1}{T_I}\int_{-R_0}^{t-R_0}\delta q(s)\mathrm{d}s] \tag{3}$$

Substituting (3) into the first equation of (2) gives

$$\delta\dot{w}(t) = -\frac{2N}{R_0^2C}\delta w(t) - \frac{R_0C^2}{2N^2}K_{PI}[\delta q(t-R_0) + \frac{1}{T_I}\int_{-R_0}^{t-R_0}\delta q(s)\mathrm{d}s] \tag{4}$$

Taking the time derivative on both sides of (4), we obtain

$$\delta\ddot{w}(t) = -\frac{2N}{R_0^2C}\delta\dot{w}(t) - \frac{R_0C^2}{2N^2}K_{PI}[\delta\dot{q}(t-R_0) + \frac{1}{T_I}\delta q(t-R_0)] \tag{5}$$

Let $x_1(t) = \delta w(t)$, $x_2(t) = \delta\dot{w}(t)$, and $x_3(t) = \delta q(t)$, from (5) and the second equation of (2) it follows that

$$\dot{\boldsymbol{X}}(t) - K_{PI}C\dot{\boldsymbol{X}}(t-R_0) = A\boldsymbol{X}(t) + K_{PI}B\boldsymbol{X}(t-R_0) \tag{6}$$

where

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{2N}{R_0^2C} & 0 \\ \dfrac{N}{R_0} & 0 & -\dfrac{1}{R_0}\end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{R_0C^2}{2N^2T_I} \\ 0 & 0 & 0\end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{R_0C^2}{2N^2} \\ 0 & 0 & 0\end{bmatrix}$$

Note that the linearized model (6) is a time-delay system of neutral type with the coefficients of A, B, and C depending on the network parameters, especially on the round-trip time delay.

3 Stability analysis

Consider the following time-delay system of neutral type

$$\dot{\boldsymbol{X}}(t) - C\dot{\boldsymbol{X}}(t-d) = A\boldsymbol{X}(t) + B\boldsymbol{X}(t-d), \quad d > 0 \tag{7}$$

Throughout this paper, we assume the following:

Assumption 1. The solutions of (7) always exist with initial condition $\boldsymbol{X}(t) = \boldsymbol{\phi}(t), t \in [-d, 0]$.

Assumption 2. Define the operator $D : C([-d,0], \mathbf{R}^n)$ as $D\boldsymbol{X}_t = \boldsymbol{X}(t) - C\boldsymbol{X}(t-d)$. All the eigenvalues of matrix C are inside the open unit circle, *i.e.*, $|\ \lambda_i(C)\ | < 1 (i = 1, 2, \cdots, n)$.

The stability property of (7) is determined by the roots of the following characteristic equation

$$\det F(s) = \det(s(I - Ce^{-ds}) - A - Be^{-ds}) = 0 \tag{8}$$

The approach adopted in this paper is to use the necessary and sufficient condition for stability in the parameter space $(\alpha_1, \alpha_2) \in \mathbf{R} \times \mathbf{R}$, where α_1, α_2 are system parameters (possibly the d) on which A, B, and C depend.

It is well known that the solution of (7) is asymptotically stable if and only if there is no root of characteristic equation in the right half plane (RHP) or on the imaginary axis. We study in the parameter space when the roots of characteristic equation are absent in the RHP or on the imaginary axis, leading to asymptotically stable response.

Given a characteristic equation

$$\det F(\alpha_1, \alpha_2, s) = 0, \quad s = \sigma + j\omega$$

which is analytic in s, we define

$$\begin{cases} G_1(\alpha_1, \alpha_2, \sigma, \omega) = \mathrm{Re}(\det F(\alpha_1, \alpha_2, \sigma + j\omega)) \\ G_2(\alpha_1, \alpha_2, \sigma, \omega) = \mathrm{Im}(\det F(\alpha_1, \alpha_2, \sigma + j\omega))\end{cases}$$

Suppose that there is a point $(\alpha_1^0, \alpha_2^0, 0, \sigma)$ on the imaginary axis such that $G_1(\alpha_1^0, \alpha_2^0, 0, \omega) = 0$, $G_2(\alpha_1^0, \alpha_2^0, 0, \omega) = 0$, *i.e.*, there is a root on the imaginary axis. According to the Existence Theorem of Implicit Function, if the Jacobi matrix

$$J = \begin{bmatrix} \dfrac{\partial G_1}{\partial \alpha_1} & \dfrac{\partial G_1}{\partial \alpha_2} \\ \dfrac{\partial G_2}{\partial \alpha_1} & \dfrac{\partial G_2}{\partial \alpha_2}\end{bmatrix}_{(\alpha_1^0, \alpha_2^0, 0, \omega)}$$

is nonsingular, the equations

$$\begin{cases} G_1(\alpha_1, \alpha_2, 0, \omega) = 0 \\ G_2(\alpha_1, \alpha_2, 0, \omega) = 0\end{cases}$$

have local continuous unique solution curve $(\alpha_1(\omega), \alpha_2(\omega))$. Moreover, the critical roots are in the RHP in the parameter space to the left of the curve $(\alpha_1(\omega), \alpha_2(\omega))$, when we follow this curve along the direction of increasing ω, whenever $\det J < 0$ and to the right when $\det J > 0$[10]. This rule allows us to detect the stable region in the parameter space in which there are no critical roots in the RHP.

4 Stability boundaries

In this section, we apply the necessary and sufficient stability condition developed in the parameter space in the previous section to analyze the linearized network dynamics. The characteristic equation of the closed-loop network system (6) is

$$\det F(s) = e^{R_0 s} s(s+\alpha_1)(s+\alpha_2) + \alpha_3 s + \alpha_4 = 0 \qquad (9)$$

where $\alpha_1 = \alpha_2^2 C/\beta$, $\alpha_2 = 1/R_0$, $\alpha_3 = K_{PI}\beta$, $\alpha_4 = K_{PI}\beta/T_I, and\ \beta = C^2/2N$. Note that all these four parameters are non-negative. Following the analysis in Section 3, we let $s = \sigma + j\omega$ in (9), decompose the equation (9) into real and imaginary parts, and investigate the stability problem in the parameter space (α_3, α_4) taking α_1 and α_2 as fixed parameters. Then, setting $\sigma \equiv 0$, *i.e.*, along the imaginary axis, we solve for α_3 and α_4 in terms of $(\alpha_1, \alpha_2, \omega)$,

$$\begin{cases} \alpha_3 = ((\omega^2 - \alpha_1\alpha_2)\omega\cos(R_0\omega) + (\alpha_1+\alpha_2)\omega^2\sin(R_0\omega))/\omega \\ \alpha_4 = (\alpha_1\alpha_2 - \omega^2)\omega\sin(R_0\omega) + (\alpha_1+\alpha_2)\omega^2\cos(R_0\omega) \end{cases} \qquad (10)$$

As $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and R_0 are real, if s is a root of (9), then the complex conjugate of s is also a root. So we only need to consider $\omega \in [0, \infty)$. In addition, the determinant of the Jacobi matrix J is

$$\det J = \begin{vmatrix} 0 & 1 \\ \omega & 0 \end{vmatrix} = -\omega$$

Obviously, $\det J < 0$ along the imaginary axis regardless of the values α_3, α_4 take.

Taking the network parameters (R_0, N, C) as $(0.25, 60, 3750)$, we plot the curve (α_3, α_4) for $\omega \in [0, \infty)$ according to (10). This is shown in Fig. 1 (Fig. 1(b) is a zoomed-in version of Fig. 1(a) around the origin). From the sign of the determinant of the Jacobi matrix, we confirm that the stability parameter region is to the right of the curve when we follow this curve along the direction of increasing ω. We observe that when $\omega = 0$, the starting point is $(-\alpha_1\alpha_2\ ,\ 0) = (-2N/R_0^3 C\ , 0) = (-2.048\ , 0)$, and as ω increases, the whole curve looks like a "spiral" one (see Fig. 1(a)).

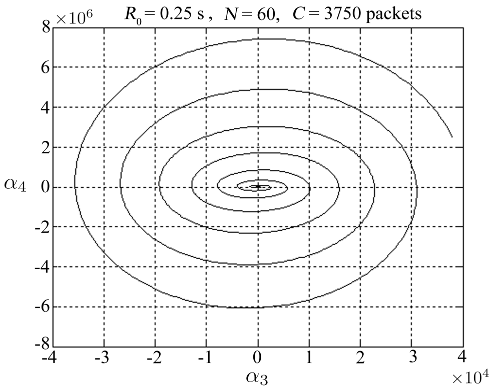
From the above discussion, we finally conclude that the stable parameter region for TCP-PI system is the one restricted by the lines $\alpha_3 = 0$ and $\alpha_4 = 0$ together with the curve located in the first quadrant (see Fig. 1(b)).

In the following, we turn our attention to the boundary′s relations analysis of the controller′s parameters and the network′s parameters. According to the expression of α_3 and α_4 in (9), we have

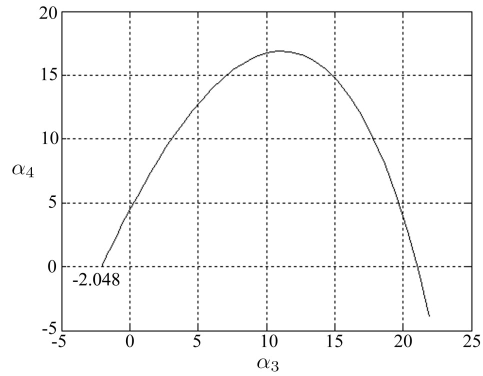
$$T_I\alpha_3 = \alpha_4 \qquad (11)$$

$$K_{PI} = \alpha_3/\beta = 2N\alpha_3/C^2 \qquad (12)$$

from which, it is possible for us to utilize the search method to find the curves of (K_{PI}, R_0), (K_{PI}, N), and (K_{PI}, C), respectively, for a fixed integral constant T_I. For instance, if we want to plot the curve (K_{PI}, R_0), the following steps can be followed. Given the network parameters (N, C), fixing a T_I and taking a series of values of R_{0i}, we solve equation (11) to search for the corresponding roots ω_i, and then, substitute these ω_i into (12), respectively, and we immediately get the corresponding K_{PIi}. From the series of couple (K_{PIi}, R_{0i}), the plotting of the curve (K_{PI}, R_0) follows easily as shown in Fig. 2. The solution ω_i of equation (11) increases as R_{0i} increases. According to the stability criterion given in the previous section, we confirm that the region below the curve in Fig. 2 corresponds to the stable parameter region. In a manner similar to plotting the curve (K_{PI}, R_0), we can depict the other two curves (K_{PI}, N) and (K_{PI}, C) taking T_I to be a parametric variable. Figs. 3 and 4 exhibit such curves.



(a) The curve $(\alpha_4(\omega)\ ,\ \alpha_3(\omega))$



(b) The initial parts of the curve

Fig. 1 The relation curve between $\alpha_4(\omega)$ and $\alpha_3(\omega)$

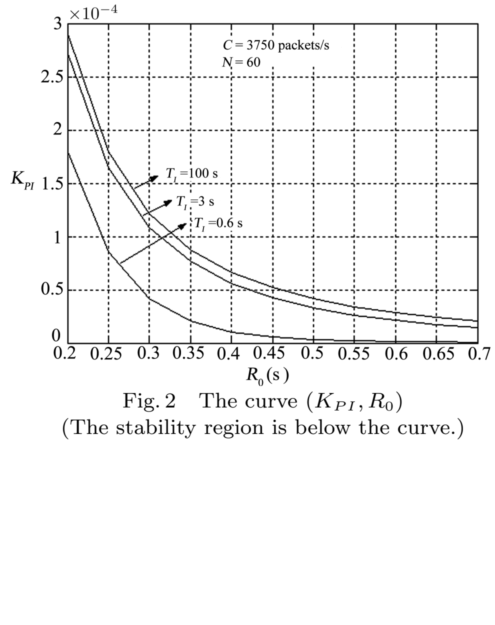


Fig. 2 The curve (K_{PI}, R_0)
(The stability region is below the curve.)

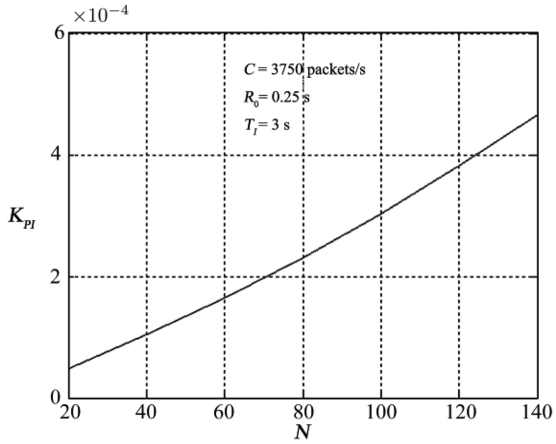


Fig. 3 The curve (K_{PI}, N)
(The stability region is below the curve.)

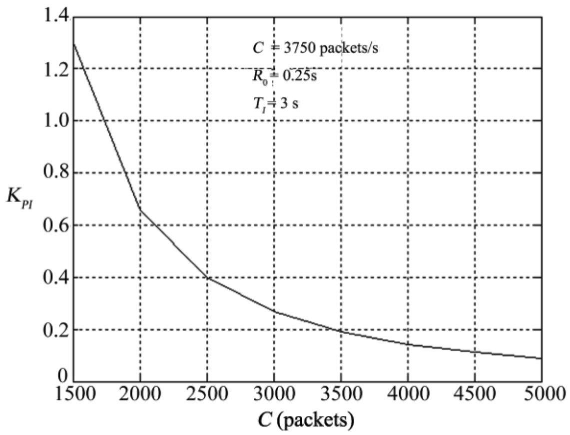


Fig. 4 The curve (K_{PI}, C)
(The stability region is below the curve.)

5 Simulations and comparisons

In this section, taking the network parameters as $N = 60$, $C = 3750$ packets/s, and (T_I=3 s, we first compare our result with the previous one in the literature to show the less conservative property of our work, and then, consider two kinds of simulations using Matlab and NS-2, respectively. First, we do the comparison between our result and the one in [11], a recently developed delay-dependent stability criterion using the Lyapunov sufficient condition. The maximum proportional gain bounds as a function of round-trip time delay R_0 are estimated in Table 1. It is clear that our necessary- and sufficient-condition-based result is less conservative than the Lyapunov sufficient condition based one.

Second, we examine the initial value responses of $\delta w(t)$ and $\delta q(t)$ taking R_0=0.25 s. The initial values of $\delta w(t)$ and $\delta q(t)$ are chosen as 5 packets and 20 packets, respectively. Setting the proportional gain $K_{PI} = 1.0 \times 10^{-4}$ results in stable responses as shown in Fig. 5 with solid line. Then, we select $K_{PI} = 1.6412 \times 10^{-4}$, which gives unstable responses with nondecaying oscillations as shown by dashed lines in Fig. 5.

Finally, we conduct the experiments by NS-2 using the network topology shown in Fig. 6 with a sampling frequency of 160 Hz. The bottleneck router's buffer size is 800 packets and the target queue size is set at 190 packets, giving approximately 50 ms queuing delay. Together with the two-way propagation delay of 200 ms, the round-trip time delay is around 250 ms. Besides 60 long-lived FTP flows with packet size fixed at 500 bytes, the TCP uncontrolled CBR flows based on UDP is also added as disturbances to generate the realistic traffic scenarios in the network. The packet size of CBR flow is 1000 bytes and the transmission rate is 0.01 Mb. The number of CBR flows is kept at 60. The random flag of CBR is set to be false.

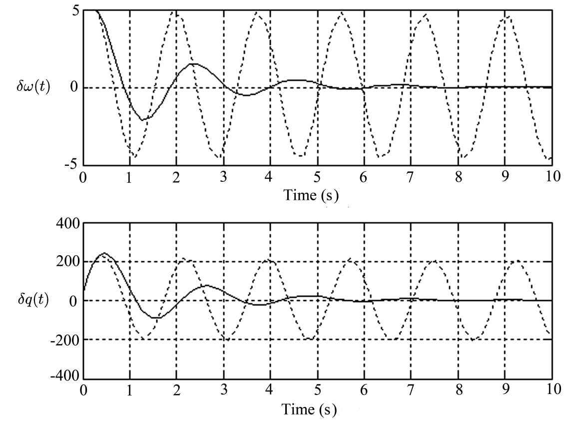


Fig. 5 The initial value responses of $\delta w(t)$ and $\delta q(t)$

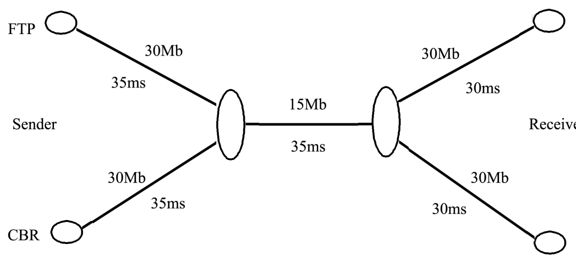


Fig. 6 Network topology

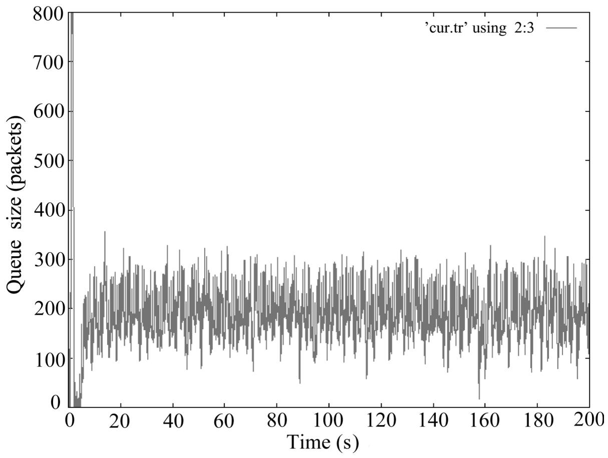
The simulation results are illustrated in Fig. 7. Fig. 7(a) corresponds to $K_{PI} = 0.5 \times 10^{-4}$ (which is located inside the stability region, see Fig. 2), and the PI controller regulates the queue size to the target 190 packets. Fig. 7(b) shows significant oscillations in the case of $K_{PI} = 2.2 \times 10^{-4}$ (which is located outside the stability region, see Fig. 2). According to Table 1, the result in [11] gives $K_{PI} = 1.5920 \times 10^{-4}$, and from Fig. 7(c), we observe that the PI controller still has regulation ability although the steady-state error is increased. For $K_{PI} = 1.6412 \times 10^{-4}$, the marginal proportional gain of our result listed in Table 1, the instantaneous queue size evolution is plotted in Fig. 7(d) in which case the oscillation increases compared to the case in Fig. 7(c).

6 Conclusion

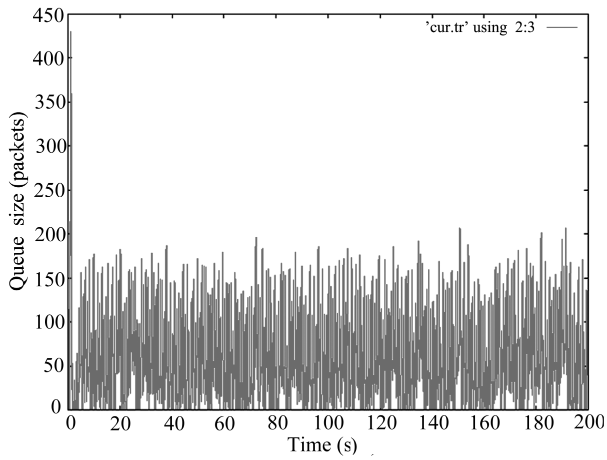
We have discussed the stability problem of TCP network using the PI AQM scheme based on the necessary and sufficient stability condition developed in the parameter space. The network′s and controller′s parameters bounds that give the marginal stability information have been determined, and the comparison with the other method has shown that our result is less conservative. The shortcoming of the method lies in the complexity of the calculation procedure. Further study in this field involves generalization of the result obtained with respect to the networks using the topology of multiple TCP sources and multiple congested routers.

Table 1 The comparison between the results in [11] and ours

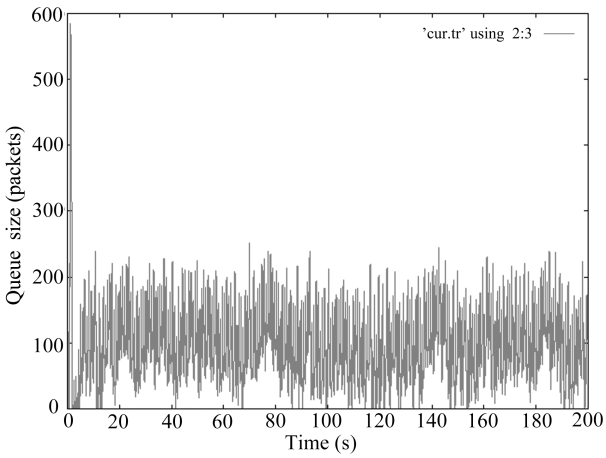
R_0 (s)	0.20	0.25	0.30	0.35
K_{PI}(in[11])	2.6314×10^{-4}	1.5920×10^{-4}	1.0546×10^{-4}	0.7425×10^{-4}
K_{PI}(ours)	2.7176×10^{-4}	1.6412×10^{-4}	1.0857×10^{-4}	0.7635×10^{-4}
Percent of improvement	3.28%	3.09%	2.95%	2.10%



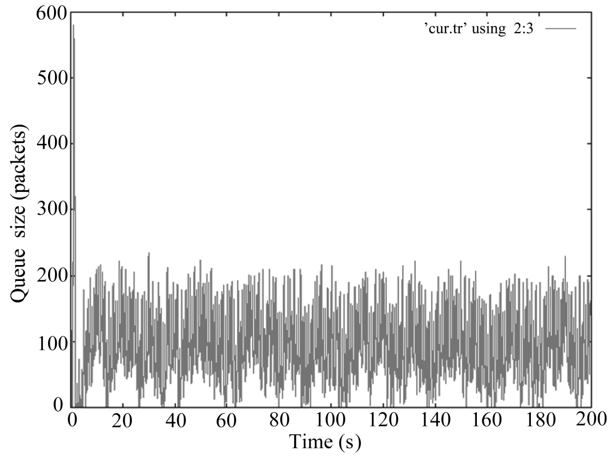
(a) $K_{PI} = 0.5 \times 10^{-4}$



(b) $K_{PI} = 2.2 \times 10^{-4}$



(c) $K_{PI} = 1.5920 \times 10^{-4}$



(d) $K_{PI} = 1.6412 \times 10^{-4}$

Fig. 7 Instantaneous queue size with different K_{PI}

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