# Peak-to-Peak Gain Minimization for Uncertain Linear Discrete Systems: A Matrix Inequality Approach

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**Abstract** A matrix inequality approach to peak-to-peak gain minimization for a class of uncertain linear discrete systems is studied. We minimize the $*$-norm, which is the best upper bound on the induced $L_\infty$ norm obtained by bounding the reachable set with inescapable ellipsoids, instead of minimizing the induced $L_\infty$ norm directly. Based on this idea, the problems of robust peak-to-peak gain minimization and controller synthesis are reduced to solving the feasibility problems of a set of matrix inequalities. A numerical example is used to demonstrate the feasibility and effectiveness of the presented method.

**Key words** Uncertain linear discrete system, peak-to-peak gain, robust control

## 1 Introduction

The $L_1$ control theory is of great importance which is attributed to the fact that design specifications for practical control problems are often expressed in terms of time-domain bounds on the amplitude of signals (exogenous disturbances and regulated outputs). For example, there is typically strict specification on the peak tracking error in disk-drive servo systems. In [1], Vidyasagar first formulated the $L_1$ optimal control problem. In contrast with the $H_\infty$ control problem, the objective of $L_1$ optimal design is to minimize the maximum peak-to-peak gain of a closed-loop system that is driven by bounded amplitude disturbances. Numerical results on this topic were obtained in both discrete and continuous contexts, see, *e.g.*, [2∼4]. These results were also extended to uncertain systems in [5] and [6]. It should be pointed out that the current synthesis approaches to peak-to-peak gain minimization require solving a sequence of linear programming problems of increasing size and thus suffer due to complexity problems. The design methods tend to be complex and thus introduce computational problems that must be solved in order to implement algorithm in engineering.

In this paper, we consider the problem of peak-to-peak gain minimization for a class of uncertain linear discrete systems. Based on the idea in [7], we seek to avoid the complexity problems by minimizing the $*$-norm, which is the upper bound on the induced $L_\infty$ norm, rather than minimizing the induced $L_\infty$ norm directly. This upper bound comes from approximating the set of states reachable with norm-bounded input with inescapable ellipsoids. It will be shown that the $*$-norm is the tightest upper bound obtainable by approximating the reachable set with inescapable ellipsoids. Based on this method, the solutions to the problems of robust $L_\infty$ gain analysis and robust controller synthesis are given. The obtained results are proved to be necessary and sufficient and formulated in terms of matrix inequalities. When the matrix inequalities are feasible, an explicit expression of the desired state-feedback control law is given.

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**Notations.** Throughout this paper, $\mathbf{R}^n$ denotes the set of all $n$-dimensional vectors, and $\mathbf{R}^{m\times n}$ represents the set of real matrices with $m$ rows and $n$ columns. If not explicitly indicated, the norm is taken to be a 2-norm. Thus, for $\boldsymbol{x}\in\mathbf{R}^n$, $\|\boldsymbol{x}\|=\sqrt{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$ and for a matrix $Q$, $\|Q\|$ is the largest singular value of $Q$. For the signal $\boldsymbol{\omega}(t):[0,\infty)\to\mathbf{R}^m$, the $L_\infty$ norm is defined as $\|\boldsymbol{\omega}(t)\|_\infty := \mathrm{ess}\sup_{t\geq 0}\|\boldsymbol{\omega}(t)\|$. $L_\infty^m$ refers to the set of all $m$-dimensional signal that belongs to $L_\infty$, that is, $L_\infty^m=\{\boldsymbol{\omega}\in\mathbf{R}^m:\|\boldsymbol{\omega}\|_\infty\leq\rho<\infty\}$. We also use $BL_\infty^m$ to denote the closed unit ball in $L_\infty^m$, that is, $BL_\infty^m=\{\boldsymbol{\omega}\in L_\infty^m:\|\boldsymbol{\omega}\|_\infty\leq 1\}$.

## 2 Problem formulation

Consider the following uncertain linear discrete system

$$\begin{aligned}\boldsymbol{x}(k+1)&=(A+\Delta A)\boldsymbol{x}(k)+(B+\Delta B)\boldsymbol{u}(k)+B_\omega\boldsymbol{\omega}(k)\\ \boldsymbol{z}(k)&=C\boldsymbol{x}(k)+D\boldsymbol{u}(k)+D_\omega\boldsymbol{\omega}(k)\end{aligned}\tag{1}$$

where $\boldsymbol{x}(k)\in\mathbf{R}^n$ is the state vector, $\boldsymbol{u}(k)\in\mathbf{R}^m$ is the control input vector, $\boldsymbol{\omega}(k)\in\mathbf{R}^p$ is the disturbance input vector that belongs to $BL_\infty^p$, and $\boldsymbol{z}(k)\in\mathbf{R}^q$ is the controlled output vector. $A$, $B$, $B_\omega$, $C$, $D$, and $D_\omega$ are constant matrices with appropriate dimensions. $\Delta A$ and $\Delta B$ represent the admissible parameter uncertainties that has the following form

$$\begin{bmatrix}\Delta A & \Delta B\end{bmatrix}=MF(k)\begin{bmatrix}N_a & N_b\end{bmatrix}\tag{2}$$

where $M$, $N_a$ and $N_b$ are constant matrices, and $F(k)$ is unknown but satisfies

$$F^{\mathrm{T}}(k)F(k)\leq I\tag{3}$$

For simplicity, we first study system (1) with $\boldsymbol{u}(k)=0$ and $F(k)=0$, that is,

$$\begin{aligned}\boldsymbol{x}(k+1)&=A\boldsymbol{x}(k)+B_\omega\boldsymbol{\omega}(k)\\ \boldsymbol{z}(k)&=C\boldsymbol{x}(k)+D_\omega\boldsymbol{\omega}(k)\end{aligned}\tag{4}$$

The induced $L_\infty$ norm of system (4) can be defined as $\|G(z)\|_{i\infty}=\mathrm{ess}\sup_{\boldsymbol{\omega}(k)\in BL_\infty^p}\|G(z)\boldsymbol{\omega}(k)\|$, where $G(z)$ is the transform function of system (4), that is, $G(z)=C(zI-A)^{-1}B_\omega+D_\omega$. It is easy to see that $\|G(z)\|_{i\infty}=\mathrm{ess}\sup_{\boldsymbol{x}(k)\in\Phi}\|C\boldsymbol{x}(k)+D_\omega\boldsymbol{\omega}(k)\|$, where $\Phi$ is a reachable set defined as the set of all states reachable from the origin in finite time by some input $\boldsymbol{\omega}(k)\in BL_\infty^p$. This reachable set obviously is an example of an inescapable set. A set $\Omega$ is said to be inescapable if 1) $\Omega$ contains the origin and 2) $\boldsymbol{x}(0)\in\Omega$ and $\boldsymbol{\omega}(k)\in BL_\infty^p$ implies that $\boldsymbol{x}(k)\in\Omega$ for all future time $k>0$. It can be easily shown that any inescapable set must contain a reachable set and thus any inescapable set $\Omega$ gives rise to the upper bound $\mathrm{ess}\sup_{\boldsymbol{x}(k)\in\Omega}\|C\boldsymbol{x}(k)+D_\omega\boldsymbol{\omega}(k)\|$ on the induced $L_\infty$ norm of $G(z)$. Then, we can consider the upper bounds on the induced $L_\infty$ norm obtained from a certain inescapable set and this tightest upper bound is defined as $*$-norm, that is,

$$\|G(s)\|_*=\inf\max_{\boldsymbol{x}(k)\in\Omega}\|C\boldsymbol{x}(k)+D_\omega\boldsymbol{\omega}(k)\|\tag{5}$$

The following lemma is required in the proof of our main results.

**Lemma 1**[8]**.** Given a symmetric matrix $\Omega$ and matrices $\Gamma$ and $\Xi$ with appropriate dimensions

$$\Omega + \Gamma F \Xi + \Xi^{\mathrm{T}} F^{\mathrm{T}} \Gamma^{\mathrm{T}} < 0$$

for any matrix $F$ satisfying $F^{\mathrm{T}} F \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$\Omega + \varepsilon^{-1} \Gamma \Gamma^{\mathrm{T}} + \varepsilon \Xi^{\mathrm{T}} \Xi < 0.$$

## 3 Main results

In this section, the result for the peak-to-peak minimization problem is given. First, we present the following theorem that will play a key role in the derivation of our main results.

**Theorem 1.** Consider the discrete system (4). Let $P \in \mathbf{R}^{n\times n}$ be symmetric positive-definite. The closed ellipsoid $\Omega = \{\boldsymbol{x}(k) \in \mathbf{R}^n : \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \leq 1\}$ is inescapable if and only if there exists a scalar $\alpha \geq 0$ such that

$$\begin{bmatrix} -P+\alpha P & 0 & A^{\mathrm{T}}P \\ 0 & -\alpha I & B_\omega^{\mathrm{T}}P \\ PA & PB_\omega & -P \end{bmatrix} \leq 0 \tag{6}$$

**Proof.**

**Sufficiency.** Suppose that there exists a scalar $\alpha \geq 0$ such that (6) holds. First, we define the functional $V(\boldsymbol{x}(k)) = \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k)$ and prove that the forward difference of $V(\boldsymbol{x}(k))$ satisfies $\Delta V(\boldsymbol{x}(k)) \leq 0$ whenever $\boldsymbol{\omega}(k) \in BL_\infty^p$ and $\boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \geq 1$. By using Schur complement, it follows from (6) that

$$\begin{aligned} \psi(\boldsymbol{x}(k),\boldsymbol{\omega}(k)) = &\, \boldsymbol{x}^{\mathrm{T}}(k)(A^{\mathrm{T}}PA-P)\boldsymbol{x}(k) + \\ & 2\boldsymbol{x}^{\mathrm{T}}(k)A^{\mathrm{T}}PB_\omega\boldsymbol{\omega}(k) + \boldsymbol{\omega}^{\mathrm{T}}(k)B_\omega^{\mathrm{T}}PB_\omega\boldsymbol{\omega}(k) \\ & \leq \alpha\big(\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) - \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k)\big) \end{aligned} \tag{7}$$

and thus the forward difference of $V(\boldsymbol{x}(k))$ along the trajectory of system (4) satisfies

$$\begin{aligned} \Delta V(\boldsymbol{x}(k)) &= \boldsymbol{x}^{\mathrm{T}}(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \\ &\leq \alpha\big(\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) - \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k)\big) \\ &\leq 0 \end{aligned} \tag{8}$$

Now, we prove that any $\boldsymbol{x}(T) \in \Omega$ will remain in $\Omega$. By contradiction, we assume that set $\Omega$ is escapable, *i.e.*, there exists an $\boldsymbol{x}(T) \in \Omega$ such that $\boldsymbol{x}^{\mathrm{T}}(T+1)P\boldsymbol{x}(T+1) > 1$. Then, noting (6) and using Schur complement again, we have

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}(T+1)P\boldsymbol{x}(T+1) = \begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\omega}(T) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}PA & A^{\mathrm{T}}PB_\omega \\ B_\omega^{\mathrm{T}}PA & B_\omega^{\mathrm{T}}PB_\omega \end{bmatrix} \cdot \\ \begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\omega}(T) \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\omega}(T) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P-\alpha P & 0 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(T) \\ \boldsymbol{\omega}(T) \end{bmatrix} \end{aligned} \tag{9}$$

Then

$$(1-\alpha)\boldsymbol{x}^{\mathrm{T}}(T)P\boldsymbol{x}(T) + \alpha\boldsymbol{\omega}^{\mathrm{T}}(T)\boldsymbol{\omega}(T) > 1 \tag{10}$$

and therefore

$$\boldsymbol{x}^{\mathrm{T}}(T)P\boldsymbol{x}(T) > \frac{1-\alpha\boldsymbol{\omega}^{\mathrm{T}}(T)\boldsymbol{\omega}(T)}{1-\alpha} \geq 1 \tag{11}$$

which is a contradiction.

**Necessity.** It is easy to verify that if $\Omega = \{\boldsymbol{x}(k) \in \mathbf{R}^n : \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \leq 1\}$ is inescapable, then $\psi(\boldsymbol{x}(k),\boldsymbol{\omega}(k)) \leq 0$ whenever $\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) \leq \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k)$. Now, assume that this condition is not true. Then, there exist $\boldsymbol{x}_0$ and $\boldsymbol{\omega}_0$ such that $\boldsymbol{\omega}_0^{\mathrm{T}}\boldsymbol{\omega}_0 \leq \boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0$ but $\psi(\boldsymbol{x}_0,\boldsymbol{\omega}_0) > 0$. This implies that $\boldsymbol{x}_0 \neq 0$. Otherwise, we have $\boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0 = 0$ and $\boldsymbol{\omega}_0 = 0$, thus $\psi(\boldsymbol{x}_0,\boldsymbol{\omega}_0) = 0$. Then, we have $\boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0 > 0$ and hence we can define the following vectors

$$\boldsymbol{x}_1 = \frac{\boldsymbol{x}_0}{\sqrt{\boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0}},\ \boldsymbol{\omega}_1 = \frac{\boldsymbol{\omega}_0}{\sqrt{\boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0}}$$

It can be easily verified that $\boldsymbol{x}_1$ and $\boldsymbol{\omega}_1$ satisfy $\boldsymbol{x}_1 \in \Omega$, $\psi(\boldsymbol{x}_1,\boldsymbol{\omega}_1) > 0$ and $\boldsymbol{\omega}_1^{\mathrm{T}}\boldsymbol{\omega}_1 \leq \boldsymbol{x}_1^{\mathrm{T}}P\boldsymbol{x}_1$. Consider the following system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B_\omega\boldsymbol{\omega}_1,\ \boldsymbol{x}(0) = \boldsymbol{x}_1$$

And it can be shown that $V(\boldsymbol{x}(0)) = 1$, $\Delta V(\boldsymbol{x}(0)) > 0$ and therefore $V(\boldsymbol{x}(1)) > 1$, which implies $\boldsymbol{x}(1) \notin \Omega$. Thus, $\Omega$ is escapable.

Obviously, this theorem clearly holds if $P = 0$. If $P \neq 0$, there must exist $\boldsymbol{x}_2$ and $\boldsymbol{\omega}_2$ such that $\boldsymbol{\omega}_2^{\mathrm{T}}\boldsymbol{\omega}_2 < \boldsymbol{x}_2^{\mathrm{T}}P\boldsymbol{x}_2$. This, together with the claim mentioned above, by using S-procedure in [9], implies that there must exist an $\alpha \geq 0$ such that $\psi(\boldsymbol{x}(k),\boldsymbol{\omega}(k)) \leq \alpha(\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) - \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k))$. This is equivalent to (6) in the sense of Schur complement, which concludes the proof. □

**Theorem 2.** Consider the discrete system (4). For a given scalar $\gamma > 0$, this system is stable with $L_\infty$ gain less than $\gamma$, if and only if there exist a symmetric positive-definite matrix $P$ and scalars $\alpha \geq 0$ and $\sigma > 0$ such that (6) and (12) hold, *i.e.*,

$$\begin{bmatrix} -\sigma P & 0 & C^T \\ 0 & -(\gamma^2-\sigma)I & D_{\boldsymbol{\omega}}^{\mathrm{T}} \\ C & D_{\boldsymbol{\omega}} & -I \end{bmatrix} < 0 \tag{12}$$

**Proof.** If there exist a symmetric positive-definite matrix $P$ and a scalar $\alpha \geq 0$ such that (6) holds, it follows that $\Omega = \{\boldsymbol{x}^{\mathrm{T}}(k) \in \mathbf{R} : \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \leq 1\}$ is an inescapable set. Also, by using Schur complement, it follows from (6) that $A^{\mathrm{T}}PA - P < 0$. Then system (4) with $\boldsymbol{\omega}(k) = 0$ is stable.

For notational simplicity, we define

$$u = \max_{\substack{\boldsymbol{x}\in\Omega \\ \boldsymbol{\omega}(k)\in BL_\infty^p}} \|C\boldsymbol{x}(k) + D_\omega\boldsymbol{\omega}(k)\|$$

and

$$U = \{\gamma : \exists \sigma \in \mathbf{R} \text{ such that (12) holds}\}$$

Our objective is to prove that $u = \inf U$.

Note $\boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) \leq 1$ and $\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) \leq 1$. Then, by using Schur complement again, it follows from (12) that

$$\begin{aligned} \|\boldsymbol{z}(k)\|^2 &= \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{\omega}(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C^{\mathrm{T}}C & C^{\mathrm{T}}D_{\boldsymbol{\omega}} \\ D_{\boldsymbol{\omega}}^{\mathrm{T}}C & D_{\boldsymbol{\omega}}^{\mathrm{T}}D_{\boldsymbol{\omega}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{\omega}(k) \end{bmatrix} \\ &< \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{\omega}(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma P & 0 \\ 0 & (\gamma^2-\sigma)I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{\omega}(k) \end{bmatrix} \\ &= \sigma\boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) + (\gamma^2-\sigma)\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k) \\ &\leq \sigma + (\gamma^2-\sigma) \\ &= \gamma^2 \end{aligned} \tag{13}$$

and hence $\|\boldsymbol{z}(k)\| < \gamma$. Thus, $u < \gamma$.

Following the same line as that in the proof of Theorem 2.4 in [7], we can show that $u$ is the least lower bound of $U$. This completes the proof. □

Based on the result of Theorem 2, we can steadily obtain the following result on robust $L_\infty$ gain analysis for the uncertain discrete system (1) with $\boldsymbol{u}(k)=0$.

**Theorem 3.** Consider the uncertain discrete system (1) with $\boldsymbol{u}(k)=0$. For a given scalar $\gamma>0$, this system is robustly stable with $L_\infty$ gain less than $\gamma$ for all admissible parameter uncertainties, if and only if there exist a symmetric positive-definite matrix $P$ and scalars $\alpha\geq 0$ and $\sigma>0$, $\varepsilon>0$ such that (12) and (14) hold, *i.e.*,

$$\begin{bmatrix} -P+\alpha P & 0 & A^{\mathrm{T}}P & 0 & \varepsilon N_a^{\mathrm{T}} \\ 0 & -\alpha I & B_\omega^{\mathrm{T}}P & 0 & 0 \\ PA & PB_\omega & -P & PM & 0 \\ 0 & 0 & M^{\mathrm{T}}P & -\varepsilon I & 0 \\ \varepsilon N_a & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} \leq 0 \quad (14)$$

**Proof.** The desired result follows immediately from the result of Theorem 3 and Lemma 1. □

The result for the robust $L_1$ controller synthesis problem is summarized in the following theorem.

**Theorem 4.** Consider the uncertain discrete system (1). For a given scalar $\gamma>0$, if there exist a symmetric positive-definite matrix $Q$, matrix $Y$, and scalars $\alpha\geq 0$, $\sigma>0$ and $\mu>0$ such that

$$\begin{bmatrix} -Q+\alpha Q & 0 & QA^{\mathrm{T}}+Y^{\mathrm{T}}B^{\mathrm{T}} & 0 & QN_a^{\mathrm{T}}+Y^{\mathrm{T}}N_b^{\mathrm{T}} \\ 0 & -\alpha I & B_\omega^{\mathrm{T}} & 0 & 0 \\ AQ+BY & B_\omega & -Q & \mu M & 0 \\ 0 & 0 & \mu M^{\mathrm{T}} & -\mu I & 0 \\ N_aQ+N_bY & 0 & 0 & 0 & -\mu I \end{bmatrix} \leq 0 \quad (15a)$$

$$\begin{bmatrix} -\sigma Q & 0 & QC^{\mathrm{T}}+Y^{\mathrm{T}}D^{\mathrm{T}} \\ 0 & -(\gamma^2-\sigma)I & D_\omega^{\mathrm{T}} \\ CQ+DY & D_\omega & -I \end{bmatrix} < 0 \quad (15b)$$

then, we can construct a state-feedback control law

$$\boldsymbol{u}(k)=YQ^{-1}\boldsymbol{x}(k)$$

such that the resultant closed-loop system is robustly stable with $L_\infty$ gain less than $\gamma$ for all admissible uncertainties.

**Proof.**

**Sufficiency.** Assume that (15) holds. Substituting the state-feedback control law $\boldsymbol{u}(k)=K\boldsymbol{x}(k)$ to system (1) results in the following closed-loop system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= (A+BK+MF(k)(N_a+N_bK))\boldsymbol{x}(k)+B_\omega\boldsymbol{\omega}(k) \\ \boldsymbol{z}(k) &= (C+DK)\boldsymbol{x}(k)+D_\omega\boldsymbol{\omega}(k) \end{aligned} \quad (16)$$

We multiply both sides of (15a) by $\mathrm{diag}\{Q^{-1},I_n,Q^{-1},\mu^{-1}I_n,\mu^{-1}I_n\}$, and both sides of (15b) by $\mathrm{diag}\{Q^{-1},I_n,I_n\}$. Defining $P=Q^{-1}$, $\varepsilon=\mu^{-1}$, $K=YQ^{-1}$, and noting (6) and (14), we can show that system (16) is robustly stable with $L_\infty$ gain less than $\gamma$.

**Necessity.** We can show that the result of Theorem 3 can be applied to analysis of the closed-loop system (16) by replacing $A$ by $A+BK$ and $N_a$ by $N_a+N_bK$. We multiply both sides of (6) by $\mathrm{diag}\{P^{-1},I_n,I_n\}$ and both sides of (14) by $\mathrm{diag}\{P^{-1},I_n,P^{-1},\varepsilon^{-1}I_n,\varepsilon^{-1}I_n\}$. By replacing $A$ and $N_a$ by $A+BK$ and $N_a+N_bK$, respectively, in the resultant inequalities and by introducing $Y=KQ$ and $\mu=\varepsilon^{-1}$, we can obtain matrix inequalities (15). This completes the proof. □

## 4 Numerical example

We consider the uncertain linear discrete system (1) with the following parameters:

$$A=\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, B_{\boldsymbol{\omega}}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, M=\begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \end{bmatrix}$$

$$N_a=\begin{bmatrix} 0.2 & 0.2 & 0.2 \end{bmatrix}, N_b=\begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, C=\begin{bmatrix} 0.5 & 0.5 & 1 \end{bmatrix}$$

$$D=\begin{bmatrix} 0.5 & 1 \end{bmatrix}, D_{\boldsymbol{\omega}}=0.5$$

For a given $L_\infty$ gain level $\gamma=0.6$, it can be easily verified that the matrix inequalities (15) possess a feasible solution when $\alpha=0.5$, $\sigma=0.05$, and a suitable state-feedback control law is given by

$$\boldsymbol{u}(k)=\begin{bmatrix} -2.9780 & 2.9095 & 1.9763 \\ 0.9860 & -1.9436 & -1.9857 \end{bmatrix}\boldsymbol{x}(k)$$

For the closed-loop system, the trajectories of the disturbance input vector $\boldsymbol{\omega}(k)$ and output vector $\boldsymbol{z}(t)$ are shown in Fig. 1 and Fig. 2, respectively.

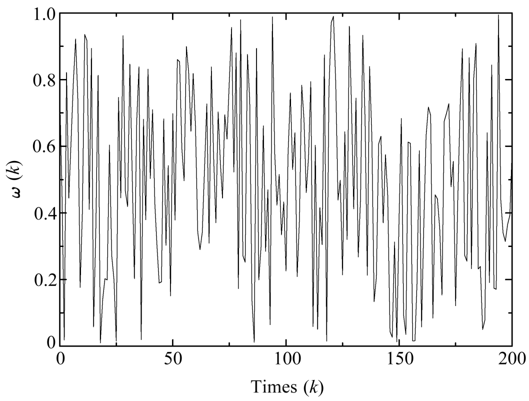


Fig. 1 The trajectory of input signal $\boldsymbol{\omega}(k)$

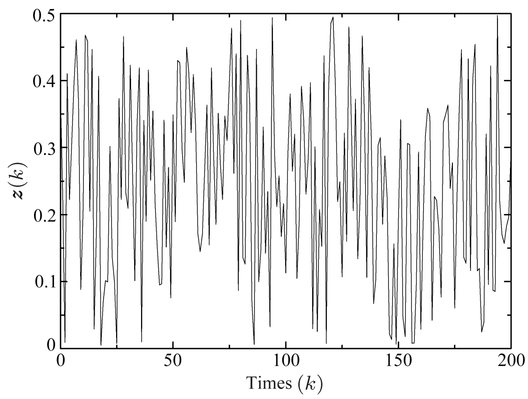


Fig. 2 The trajectory of output signal $\boldsymbol{z}(k)$

## 5 Conclusion

In this paper, we have taken an alternative approach to the problem of peak-to-peak gain minimization for a class of uncertain linear discrete systems. Instead of attempting to minimize the induced $L_\infty$ norm directly, we minimized an upper bound of the $*$-norm obtained by bounding the reachable set by inescapable ellipsoids. The robust $L_\infty$

gain analysis and controller synthesis problems have been solved and the obtained results have been proved to be necessary and sufficient. An illustrative example shows that the presented method is effective and applicable.

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