Delay-dependent Robust Stabilization for Uncertain Singular Systems with State Delay

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Abstract This paper considers the problem of delay-dependent robust stabilization for uncertain singular delay systems. In terms of linear matrix inequality (LMI) approach, a delay-dependent stability criterion is given to ensure that the nominal system is regular, impulse free, and stable. Based on the criterion, the problem is solved *via* state feedback controller, which guarantees that the resultant closed-loop system is regular, impulse free, and stable for all admissible uncertainties. An explicit expression for the desired controller is also given. Some numerical examples are provided to illustrate the validity of the proposed methods.

Key words Singular time-delay system, delay-dependent, robust stabilization, linear matrix inequality

1 Introduction

Over the past decades, much attention has been focused on the problems of stability analysis and stabilization for singular delay systems. Especially, with the development of the robust control theory, many robust stabilization methods have been proposed for uncertain singular delay systems. The existing results can be classified into two types: delay-independent stabilization and delay-dependent stabilization. Generally, the delay-independent case is more conservative than the delay-dependent case, especially when the time delay is comparatively small. The delay-independent case has been extensively studied (see, *e.g.* [1, 2] and the references therein); however, there are only few papers on the delay-dependent case[3,4]. [3] discussed the problem of delay-dependent robust stability analysis, and a delay-dependent robust stability criterion was obtained. But the considered system was assumed to be necessarily regular and impulse free; moreover, a matrix describing the relationship between fast and slow subsystems was needed and an improper choice of the matrix would make the results unreliable. In [4], the problem of delay-dependent robust stabilization was solved *via* state feedback controller, and an expression for the desired controller was given by solving a set of nonlinear matrix inequalities with an equation constraint, which would result in some numerical problems and make the design procedure complex and unreliable. To the best of our knowledge, the problem of delay-dependent robust stabilization for uncertain singular delay systems has not been fully studied in the literature and still remains open.

In this paper, we investigate the problem of delay-dependent robust stabilization for uncertain singular systems with state delay. The considered systems are not assumed to be necessarily regular and impulse free. The considered problem is to design a state feedback controller such that the resultant closed-loop system is robustly stable. In terms of two linear matrix inequalities (LMIs), a sufficient condition for the solvability of the problem is derived. When this condition is satisfied, the desired state feedback controller is obtained.

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2 Problem formulation

Consider the uncertain singular system with state delay described by

$$\begin{cases} E\dot{\boldsymbol{x}}(t) = (A+\Delta A)\boldsymbol{x}(t) + (A_d+\Delta A_d)\boldsymbol{x}(t-d)+ \\ \qquad (B+\Delta B)\boldsymbol{u}(t), \\ \boldsymbol{x}(t)=\boldsymbol{\phi}(t), \quad t\in[-\bar{d},0] \end{cases} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbf{R}^n$ is the state, $\boldsymbol{u}(t)\in\mathbf{R}^m$ is the control input. d is an unknown but constant time delay and $\bar{d}$ is a constant satisfying $0\le d\le\bar{d}$. $\boldsymbol{\phi}(t)$ is a compatible vector valued initial function. The matrix $E\in\mathbf{R}^{n\times n}$ may be singular and $\operatorname{rank} E = r\le n$ is assumed. A, A_d, and B are known real constant matrices with appropriate dimensions. ΔA, ΔA_d, and ΔB are unknown matrices representing norm-bounded parametric uncertainties and are assumed to be of the form:

$$\begin{bmatrix} \Delta A & \Delta A_d & \Delta B \end{bmatrix} = MF(t)\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \tag{2}$$

where M, N_1, N_2, and N_3 are known real constant matrices with appropriate dimensions, and $F(t)\in\mathbf{R}^{q\times k}$ is an unknown real and possibly time-varying matrix satisfying

$$F^{\mathrm{T}}(t)F(t)\le I \tag{3}$$

The parametric uncertainties ΔA, ΔA_d, and ΔB are said to be admissible if both (2) and (3) hold.

The nominal unforced singular delay system of (1) can be written as

$$E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d) \tag{4}$$

Definition 1[5]**.**

1. The pair (E, A) is said to be regular if $\det(sE-A)$ is not identically zero.

2. The pair (E, A) is said to be impulse free if $\deg(\det(sE-A)) = \operatorname{rank} E$.

Definition 2. For a given scalar $\bar{d}>0$, the singular delay system (4) is said to be regular and impulse free for any constant time delay d satisfying $0\le d\le\bar{d}$, if the pairs (E,A) and $(E,A+A_d)$ are regular and impulse free.

Remark 1. The regularity and the absence of impulses of the pair (E,A) ensures the system (4) with time delay $d\ne 0$ to be regular and impulse free, while the fact that the pair $(E,A+A_d)$ is regular and impulse free ensures the system (4) with time delay $d=0$ to be regular and impulse free.

Definition 3. The uncertain singular delay system (1) is said to be robustly stable, if the system with $\boldsymbol{u}(t)=0$ is regular, impulse free, and stable for all admissible uncertainties ΔA and ΔA_d.

In this paper, we shall address the following problem.

Delay-dependent robust stabilization problem. For a given scalar $\bar{d}>0$, design a state feedback controller $\boldsymbol{u}(t)=K\boldsymbol{x}(t)$, $K\in\mathbf{R}^{m\times n}$ for system (1) such that the resultant closed-loop system is robustly stable for any constant time delay d satisfying $0\le d\le\bar{d}$. In this case, the system is said to be robustly stabilizable.

We conclude this section by presenting several preliminary results, which will be used in the proof of our main results.

Lemma 1[6]**.** The singular system $E\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)$ is regular, impulse free, and stable, if and only if there exists a matrix P such that

$$P^{\mathrm{T}}E=E^{\mathrm{T}}P\ge 0 \tag{5}$$

$$P^{\mathrm{T}}A+A^{\mathrm{T}}P<0 \tag{6}$$

Lemma 2[7]**.** Given matrices Ω, Γ, and Ξ with appropriate dimensions and with Ω symmetrical, $\Omega+\Gamma F\Xi+\Xi^{\mathrm{T}}F^{\mathrm{T}}\Gamma^{\mathrm{T}}<0$ for any F satisfying $F^{\mathrm{T}}F\le I$, if and only if there exists a scalar $\varepsilon>0$ such that $\Omega+\varepsilon\Gamma\Gamma^{\mathrm{T}}+\varepsilon^{-1}\Xi^{\mathrm{T}}\Xi<0$.

Lemma 3[8]**.** For symmetric positive-definite matrix Q and matrices P and R with appropriate dimensions, matrix inequality $P^{\mathrm{T}}R+R^{\mathrm{T}}P\le R^{\mathrm{T}}QR+P^{\mathrm{T}}Q^{-1}P$ holds.

3 Main results

Initially, we present the following theorem for the singular delay system (4), which will play a key role in the proof of our main results.

Theorem 1. For a prescribed scalar $\bar{d}>0$, the singular delay system (4) is regular, impulse free, and stable for any constant time delay d satisfying $0\le d\le\bar{d}$, if there exist symmetric positive-definite matrices P_1, Q, Z and matrices S, P_2, P_3, X_{11}, X_{12}, X_{22}, Y_1, and Y_2 such that

$$\Omega=\begin{bmatrix}\Omega_{11} & \Omega_{12} & P_2^{\mathrm{T}}A_d-Y_1E\\ * & -P_3-P_3^{\mathrm{T}}+\bar{d}X_{22}+\bar{d}Z & P_3^{\mathrm{T}}A_d-Y_2E\\ * & * & -Q\end{bmatrix}<0 \tag{7}$$

$$\begin{bmatrix}X_{11} & X_{12} & Y_1\\ * & X_{22} & Y_2\\ * & * & Z\end{bmatrix}>0 \tag{8}$$

where $R\in\mathbf{R}^{n\times l}$ is any matrix satisfying $E^{\mathrm{T}}R=0$ and
$\Omega_{11}=P_2^{\mathrm{T}}A+A^{T}P_2+\bar{d}X_{11}+Q+Y_1E+E^{\mathrm{T}}Y_1^{\mathrm{T}}$
$\Omega_{12}=E^{\mathrm{T}}P_1+SR^{\mathrm{T}}-P_2^{\mathrm{T}}+A^{\mathrm{T}}P_3+E^{\mathrm{T}}Y_2^{\mathrm{T}}+\bar{d}X_{12}$

Proof. From (7), it is easy to show that

$$\tilde{E}^{\mathrm{T}}\tilde{P}=\tilde{P}^{\mathrm{T}}\tilde{E}\ge 0 \tag{9}$$

$$\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}^{\mathrm{T}}\tilde{A}+\bar{d}\tilde{X}+\tilde{Q}+\tilde{Y}\tilde{E}+\tilde{E}^{\mathrm{T}}\tilde{Y}^{\mathrm{T}}+(\tilde{P}^{\mathrm{T}}\tilde{A}_d-\tilde{Y}\tilde{E})\tilde{Q}^{-1}(\tilde{P}^{\mathrm{T}}\tilde{A}_d-\tilde{Y}\tilde{E})^{\mathrm{T}}<0 \tag{10}$$

where

$$\tilde{E}=\begin{bmatrix}E & 0\\ 0 & 0\end{bmatrix},\ \tilde{A}=\begin{bmatrix}0 & I\\ A & -I\end{bmatrix},\ \tilde{A}_d=\begin{bmatrix}0 & 0\\ A_d & 0\end{bmatrix}$$

$$\tilde{P}=\begin{bmatrix}P_1E+RS^{\mathrm{T}} & 0\\ P_2 & P_3\end{bmatrix},\ \tilde{X}=\begin{bmatrix}X_{11} & X_{12}\\ * & X_{22}\end{bmatrix}$$

$$\tilde{Y}=\begin{bmatrix}Y_1 & 0\\ Y_2 & 0\end{bmatrix},\ \tilde{Q}=\begin{bmatrix}Q & 0\\ 0 & \bar{d}Z\end{bmatrix}$$

By Lemma 3, it follows from (10) that

$$\begin{aligned}0>{}&\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}^{\mathrm{T}}\tilde{A}+\bar{d}\tilde{X}+\tilde{Q}+\tilde{Y}\tilde{E}+\tilde{E}^{\mathrm{T}}\tilde{Y}^{\mathrm{T}}+\\ &\tilde{P}^{\mathrm{T}}\tilde{A}_d-\tilde{Y}\tilde{E}+(\tilde{P}^{\mathrm{T}}\tilde{A}_d-\tilde{Y}\tilde{E})^{\mathrm{T}}-\tilde{Q}\ge\\ &(\tilde{A}+\tilde{A}_d)^{\mathrm{T}}\tilde{P}+\tilde{P}^{\mathrm{T}}(\tilde{A}+\tilde{A}_d)\end{aligned} \tag{11}$$

According to Lemma 1, we can deduce from (9) and (11) that the pair $(\tilde{E},\tilde{A}+\tilde{A}_d)$ is regular and impulse free.

Since $\operatorname{rank}\tilde{E}=\operatorname{rank}E=r\le n$, there exist nonsingular matrices M and N such that

$$\hat{E}=M\tilde{E}N=\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix} \tag{12}$$

Denote

$$\begin{aligned}\hat{A}&=M\tilde{A}N=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\\ \hat{P}&=M^{-T}\tilde{P}N=\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}\\ \hat{Y}&=N^{T}\tilde{Y}M^{-1}=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}\end{aligned} \tag{13}$$

From (9) and using the expressions of $\hat{E}$ and $\hat{P}$ in (12) and (13), it is easy to obtain $P_{12}=0$, $P_{11}\ge 0$; therefore,

$$\hat{P}=\begin{bmatrix}P_{11} & 0\\ P_{21} & P_{22}\end{bmatrix}$$

From (10), we get

$$\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}^{\mathrm{T}}\tilde{A}+\tilde{Y}\tilde{E}+\tilde{E}^{\mathrm{T}}\tilde{Y}^{\mathrm{T}}<0 \tag{14}$$

Now, pre-multiplying and post-multiplying (14) by N^{T} and N, respectively, we can obtain $A_{22}^{\mathrm{T}}P_{22}+P_{22}^{\mathrm{T}}A_{22}<0$. This implies that A_{22} is nonsingular, and thus the pair $(\tilde{E},\tilde{A})$ is regular and impulse free.

Noting the fact that $\det(sE-A)=\det(s\tilde{E}-\tilde{A})$, $\deg(\det(sE-A)))=\deg(\det(s\tilde{E}-\tilde{A})))=\operatorname{rank}\tilde{E}=\operatorname{rank}E$, $\det(sE-(A+A_d))=\det(s\tilde{E}-(\tilde{A}+\tilde{A}_d))$, and $\deg(\det(sE-(A+A_d)))=\deg(\det(s\tilde{E}-(\tilde{A}+\tilde{A}_d)))=\operatorname{rank}\tilde{E}=\operatorname{rank}E$, we can easily see that the pairs (E,A) and $(E,A+A_d)$ are regular and impulse free, and thus system (4) is regular and impulse free.

Next, we will show that system (4) is stable. To the end, we propose the following function.

$$V(\boldsymbol{x}_t)=V_1(\boldsymbol{x}_t)+V_2(\boldsymbol{x}_t)+V_3(\boldsymbol{x}_t)$$

where
$V_1(\boldsymbol{x}_t)=\boldsymbol{x}(t)^{\mathrm{T}}E^{\mathrm{T}}P_1E\boldsymbol{x}(t)$,
$V_2(\boldsymbol{x}_t)=\int_{-d}^{0}\int_{t+\beta}^{t}\dot{\boldsymbol{x}}(\alpha)^{\mathrm{T}}E^{\mathrm{T}}ZE\dot{\boldsymbol{x}}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta$

$V_3(\boldsymbol{x}_t) = \int_{t-d}^{t} \boldsymbol{x}(\alpha)^{\mathrm{T}} Q \boldsymbol{x}(\alpha) \mathrm{d}\alpha$

Differentiating $V(\boldsymbol{x}_t)$ with respect to t, we have

$$\dot{V}_1(\boldsymbol{x}_t) = 2 \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix}^{\mathrm{T}} \tilde{P}^{\mathrm{T}} \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix} + 2 \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix}^{\mathrm{T}} \tilde{P}^{\mathrm{T}} \begin{bmatrix} 0 \\ A_d \end{bmatrix} \boldsymbol{x}(t-d)$$

$$\dot{V}_2(\boldsymbol{x}_t) \leq \bar{d}\dot{\boldsymbol{x}}(t)^{\mathrm{T}} E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(t) - \int_{t-d}^{t} \dot{\boldsymbol{x}}(\alpha)^{\mathrm{T}} E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(\alpha) \mathrm{d}\alpha$$

$$\dot{V}_3(\boldsymbol{x}_t) = \boldsymbol{x}(t)^{\mathrm{T}} Q \boldsymbol{x}(t) - \boldsymbol{x}(t-d)^{\mathrm{T}} Q \boldsymbol{x}(t-d)$$

It is clear that

$$\int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_{11} & X_{12} & Y_1 \\ * & X_{22} & Y_2 \\ * & * & Z \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} \mathrm{d}\alpha \geq 0$$

Thus,

$$\dot{V}_2(\boldsymbol{x}_t) \leq \bar{d}\dot{\boldsymbol{x}}(t)^{\mathrm{T}} E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(t) - \int_{t-d}^{t} \dot{\boldsymbol{x}}(\alpha)^{\mathrm{T}} E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(\alpha) \mathrm{d}\alpha + \int_{t-d}^{t} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_{11} & X_{12} & Y_1 \\ * & X_{22} & Y_2 \\ * & * & Z \end{bmatrix} \times \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \\ E\dot{\boldsymbol{x}}(\alpha) \end{bmatrix} \mathrm{d}\alpha \leq$$
$$\bar{d} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix} + 2 \begin{bmatrix} \boldsymbol{x}(t) \\ E\dot{\boldsymbol{x}}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 E \\ Y_2 E \end{bmatrix} [\boldsymbol{x}(t) - \boldsymbol{x}(t-d)] + \bar{d}\dot{\boldsymbol{x}}(t)^{\mathrm{T}} E^{\mathrm{T}} Z E \dot{\boldsymbol{x}}(t).$$

Hence,

$$\dot{V}(\boldsymbol{x}_t) \leq \boldsymbol{\xi}(t)^{\mathrm{T}} \Omega \boldsymbol{\xi}(t)$$

where $\boldsymbol{\xi}(t) = [\boldsymbol{x}(t)^{\mathrm{T}} \; (E\dot{\boldsymbol{x}}(t))^{\mathrm{T}} \; \boldsymbol{x}(t-d)^{\mathrm{T}}]^{\mathrm{T}}$. From (7), we get $\dot{V}(\boldsymbol{x}_t) < 0$, and thus system (4) is stable. □

Remark 2. In the proof of Theorem 1, it is noted that neither model transformation nor bounding technique for cross terms, which are usually used in the existing results, is required. Hence, the derivation procedure is simpler and the condition of Theorem 1 is less conservative than those of existing ones, which will be demonstrated by examples.

If the matrices, in (8), $Y_1 = 0$, $Y_2 = 0$, $X_{12} = 0$ and $X_{11} = X_{22} = Z = \varepsilon I/\bar{d}$ ($\varepsilon \to 0$), then Theorem 1 provides the result on delay-independent stability analysis, which is stated as follows.

Corollary 1. The singular delay system (4) is regular, impulse free, and stable, if there exist symmetric positive-definite matrices P_1, Q, Z and matrices S, P_2, and P_3 such that

$$\begin{bmatrix} P_2^{\mathrm{T}} A + A^{\mathrm{T}} P_2 + Q & \Pi_{12} & P_2^{\mathrm{T}} A_d \\ * & -P_3 - P_3^{\mathrm{T}} & P_3^{\mathrm{T}} A_d \\ * & * & -Q \end{bmatrix} < 0 \quad (15)$$

where R follows the same definition as that in Theorem 1 and $\Pi_{12} = E^{\mathrm{T}} P_1 + S R^{\mathrm{T}} - P_2^{\mathrm{T}} + A^{\mathrm{T}} P_3$.

Remark 3. As we have seen above, the result of Theorem 1 is powerful in the sense that it provides sufficient conditions for both the delay-dependent and delay-independent cases.

Since the solution of $\det(sE - A - e^{-sd} A_d) = 0$ is same as that of $\det(sE^{\mathrm{T}} - A^{\mathrm{T}} - e^{-sd} A_d^{\mathrm{T}}) = 0$, system (4) is regular, impulse free, and stable, if and only if the system

$$E^{\mathrm{T}} \dot{\boldsymbol{\zeta}}(t) = A^{\mathrm{T}} \boldsymbol{\zeta}(t) + A_d^{\mathrm{T}} \boldsymbol{\zeta}(t-d) \quad (16)$$

is regular, impulse free, and stable. Hence, using Theorem 1 for system (16) leads to the following theorem:

Theorem 2. For a prescribed scalar $\bar{d} > 0$, the singular delay system (4) is regular, impulse free, and stable for any constant time delay d satisfying $0 \leq d \leq \bar{d}$, if there exist symmetric positive-definite matrices P_1, Q, Z and matrices S, P_2, P_3, X_{11}, X_{12}, X_{22}, Y_1, and Y_2 such that both (8) and the following linear matrix inequality hold.

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & P_2^{\mathrm{T}} A_d^{\mathrm{T}} - Y_1 E^{\mathrm{T}} \\ * & -P_3 - P_3^{\mathrm{T}} + \bar{d} X_{22} + \bar{d} Z & P_3^{\mathrm{T}} A_d^{\mathrm{T}} - Y_2 E^{\mathrm{T}} \\ * & * & -Q \end{bmatrix} < 0 \quad (17)$$

where $R \in \mathbf{R}^{n \times l}$ is any matrix satisfying $ER = 0$ and

$\Omega_{11} = P_2^{\mathrm{T}} A^{\mathrm{T}} + A P_2 + \bar{d} X_{11} + Q + Y_1 E^{\mathrm{T}} + E Y_1^{\mathrm{T}}$

$\Omega_{12} = E P_1 + S R^{\mathrm{T}} - P_2^{\mathrm{T}} + A P_3 + E Y_2^{\mathrm{T}} + \bar{d} X_{12}$

Remark 4. Sufficient conditions of Theorem 1 and Theorem 2 may lead to different results. Hence, we can separately apply Theorem 1 and Theorem 2 and then choose the less conservative one.

Next, based on Theorem 1, we give the following delay-dependent robust stability criterion.

Theorem 3. For a prescribed scalar $\bar{d} > 0$, the uncertain singular delay system (1) is robustly stable for any constant time delay d satisfying $0 \leq d \leq \bar{d}$, if there exist scalar $\varepsilon > 0$ and symmetric positive-definite matrices P_1, Q, Z and matrices S, P_2, P_3, X_{11}, X_{12}, X_{22}, Y_1, and Y_2 such that the linear matrix inequalities (8) and (18) hold.

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_2^{\mathrm{T}} A_d - Y_1 E + \varepsilon N_1^{\mathrm{T}} N_2 & P_2^{\mathrm{T}} M \\ * & \Xi_{22} & P_3^{\mathrm{T}} A_d - Y_2 E & P_3^{\mathrm{T}} M \\ * & * & -Q + \varepsilon N_2^{\mathrm{T}} N_2 & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (18)$$

where $R \in \mathbf{R}^{n \times l}$ is any matrix satisfying $E^{\mathrm{T}} R = 0$ and

$\Xi_{11} = P_2^{\mathrm{T}} A + A^{\mathrm{T}} P_2 + \bar{d} X_{11} + Q + Y_1 E + E^{\mathrm{T}} Y_1^{\mathrm{T}} + \varepsilon N_1^{\mathrm{T}} N_1$

$\Xi_{12} = E^{\mathrm{T}} P_1 + S R^{\mathrm{T}} - P_2^{\mathrm{T}} + A^{\mathrm{T}} P_3 + E^{\mathrm{T}} Y_2^{\mathrm{T}} + \bar{d} X_{12}$

$\Xi_{22} = -P_3 - P_3^{\mathrm{T}} + \bar{d} X_{22} + \bar{d} Z$

Proof. Using Schur complement, we obtain from (18) that

$$\Omega + \varepsilon^{-1} M M^{\mathrm{T}} + \varepsilon N^{\mathrm{T}} N < 0 \quad (19)$$

where Ω is same as that on the left side of (7) and $M = [M^{\mathrm{T}} P_2 \quad M^{\mathrm{T}} P_3 \quad 0]^{\mathrm{T}}$, $N = [N_1 \quad 0 \quad N_2]$.

By Lemma 2, it follows from (19) that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & P_2^{\mathrm{T}}(A_d + \Delta A_d) - Y_1 E \\ * & \Xi_{22} & P_3^{\mathrm{T}}(A_d + \Delta A_d) - Y_2 E \\ * & * & -Q \end{bmatrix} < 0 \quad (20)$$

where

$\Psi_{11} = P_2^{\mathrm{T}}(A + \Delta A) + (A + \Delta A)^{\mathrm{T}}P_2 + \bar{d}X_{11} + Q +$
$\quad Y_1E + E^{\mathrm{T}}Y_1^{\mathrm{T}}$

$\Psi_{12} = E^{\mathrm{T}}P_1 + SR^{\mathrm{T}} - P_2^{\mathrm{T}} + (A + \Delta A)^{\mathrm{T}}P_3 + E^{\mathrm{T}}Y_2^{\mathrm{T}} + \bar{d}X_{12}$

According to Theorem 1 and Definition 3, we have the desired result immediately. □

Now, we are in the position to present the result on the problem of delay-dependent robust stabilization.

Theorem 4. For a prescribed scalar $\bar{d} > 0$, the uncertain singular delay system (1) is robustly stabilizable for any constant time delay d satisfying $0 \leq d \leq \bar{d}$, if there exist scalars $\varepsilon > 0$ and ϵ, symmetric positive-definite matrices P_1, Q, Z and matrices S, P_2, X, X_{11}, X_{12}, X_{22}, Y_1 and Y_2 such that the linear matrices inequalities (8) and (21) hold,

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & P_2^{\mathrm{T}}A_d^{\mathrm{T}} - Y_1E^{\mathrm{T}} & (N_1P_2 + N_3X)^{\mathrm{T}} & (N_2P_2)^{\mathrm{T}} \\ * & \Pi_{22} & \epsilon P_2^{\mathrm{T}}A_d^{\mathrm{T}} - Y_2E^{\mathrm{T}} & \epsilon(N_1P_2 + N_3X)^{\mathrm{T}} & \epsilon(N_2P_2)^{\mathrm{T}} \\ * & * & -Q + \varepsilon MM^{\mathrm{T}} & 0 & 0 \\ * & * & * & -\varepsilon I & 0 \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{21}$$

where $R \in \mathbf{R}^{n\times l}$ is any matrix satisfying $ER = 0$ and

$\Pi_{11} = P_2^{\mathrm{T}}A^{\mathrm{T}} + X^{\mathrm{T}}B^{\mathrm{T}} + AP_2 + BX + \bar{d}X_{11} + Q +$
$\quad Y_1E^{\mathrm{T}} + EY_1^{\mathrm{T}} + \varepsilon MM^{\mathrm{T}}$

$\Pi_{12} = EP_1 + SR^{\mathrm{T}} - P_2^{\mathrm{T}} + \epsilon(AP_2 + BX) + EY_2^{\mathrm{T}} + \bar{d}X_{12}$

$\Pi_{22} = -\epsilon P_2 - \epsilon P_2^{\mathrm{T}} + \bar{d}X_{22} + \bar{d}Z$

In this case, a desired state feedback controller is given by

$$\boldsymbol{u}(t) = XP_2^{-1}\boldsymbol{x}(t) \tag{22}$$

Proof. Setting $\epsilon P_2 = P_3$ and $X = KP_2$ and using Schur complement, we obtain from (21) that

$$\Xi + \varepsilon MM^{\mathrm{T}} + \varepsilon^{-1}N^{\mathrm{T}}N < 0 \tag{23}$$

where

$$\Xi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & P_2^{\mathrm{T}}A_d - Y_1E^{\mathrm{T}} \\ * & \Psi_{22} & P_3^{\mathrm{T}}A_d - Y_2E^{\mathrm{T}} \\ * & * & -Q \end{bmatrix}$$

$$M = \begin{bmatrix} M^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & M^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$N = \begin{bmatrix} N_1P_2 + N_3KP_2 & N_1P_3 + N_3KP_3 & 0 \\ N_2P_2 & N_2P_3 & 0 \end{bmatrix}$$

and

$\Psi_{11} = P_2^{\mathrm{T}}(A + BK)^{\mathrm{T}} + (A + BK)P_2 + \bar{d}X_{11} + Q +$
$\quad Y_1E^{\mathrm{T}} + EY_1^{\mathrm{T}}$

$\Psi_{12} = EP_1 + SR^{\mathrm{T}} - P_2^{\mathrm{T}} + (A + BK)P_3 + EY_2^{\mathrm{T}} + \bar{d}X_{12}$

$\Psi_{22} = -P_3 - P_3^{\mathrm{T}} + \bar{d}X_{22} + \bar{d}Z$

By Lemma 2, it follows from (23) that

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & P_2^{\mathrm{T}}(A_d + \triangle A_d)^{\mathrm{T}} - Y_1E^{\mathrm{T}} \\ * & \Psi_{22} & P_3^{\mathrm{T}}(A_d + \triangle A_d)^{\mathrm{T}} - Y_2E^{\mathrm{T}} \\ * & * & -Q \end{bmatrix} < 0 \tag{24}$$

where

$\Theta_{11} = P_2^{\mathrm{T}}A_K^{\mathrm{T}} + A_KP_2 + \bar{d}X_{11} + Q + Y_1E^{\mathrm{T}} + EY_1^{\mathrm{T}}$

$\Theta_{12} = EP + SR^{\mathrm{T}} - P_2^{\mathrm{T}} + A_KP_3 + EY_2^{\mathrm{T}} + \bar{d}X_{12}$

$A_K = A + BK + \triangle A + \triangle BK$

According to Theorem 2 and Definition 3, the desired result follows immediately. □

4 Numerical examples

In this section, some examples are provided to illustrate the effectiveness and the less conservatism of the obtained results.

Example 1. Consider the following singular delay system[10]

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix} \cdot \boldsymbol{x}(t - d)$$

In this example, we choose $\boldsymbol{R} = [0 \quad 1]^{\mathrm{T}}$. The upper bounds on the time delay from Theorem 1 and Theorem 2 are shown in Table 1. For comparison, the table also lists the upper bounds obtained from the criteria in [3, 4, 9∼16]. It can be seen that our methods are less conservative.

Table 1 Comparison of delay-dependent stability conditions of Example 1

Methods	[3,9,11]	[4,12]	[13]	[14]	[15,16]	[10]	Theorem 1 & 2
Maximum $\bar{d}$ allowed	–	0.5567	0.8708	0.9091	0.9680	1.0423	1.0660

Example 2. Consider the following uncertain singular delay system.

$$E\dot{\boldsymbol{x}}(t) = (A + \triangle A)\boldsymbol{x}(t) + (A_d + \triangle A_d)\boldsymbol{x}(t - d)$$

where

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},\ A_d = \begin{bmatrix} -2.4 & 2 \\ 0 & 1 \end{bmatrix}$$

and the uncertain matrices $\triangle A$ and $\triangle A_d$ satisfy $\|\triangle A\| \leq \lambda$, $\|\triangle A_d\| \leq \lambda (\lambda > 0)$. This system is of the form in system (1) with $\boldsymbol{u}(t) = 0$. Then, we can write $M = \lambda I$, $N_1 = N_2 = 0.5I$.

In this example, we choose $\boldsymbol{R} = [0 \quad 1]^{\mathrm{T}}$. Table 2 gives the comparison of the maximum allowed delay $\bar{d}$ for various parameter λ. It is clear that the conditions in this paper gives better results than those in [12, 13].

Table 2 Comparison of delay-dependent stability conditions of Example 2

λ	0.25	0.30	0.35	0.40	0.45	0.50
[12]	0.4209	0.3939	0.3637	0.3279	0.2817	0.2106
[13]	0.8087	0.7942	0.7689	0.7262	0.6521	0.5054
Theorem 3	0.8514	0.8249	0.7924	0.7438	0.6641	0.5110

Example 3. Consider the uncertain singular delay system (1) with parameters as follows.

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -1.5 & 0.5 & -0.8 \\ 1 & 1 & 0.5 \\ 0.7 & 0.5 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} 0.4 & 0.3 & 0.1 \end{bmatrix}^{\mathrm{T}}, \boldsymbol{N}_1 = \begin{bmatrix} 0.2 & 0.4 & 0.5 \end{bmatrix}$$

$$\boldsymbol{N}_2 = \begin{bmatrix} 0.3 & 0.7 & 0.5 \end{bmatrix}, \boldsymbol{N}_3 = \begin{bmatrix} 0.4 & 0.5 \end{bmatrix}$$

In this example, we choose $\boldsymbol{R} = [-1 \quad 1 \quad 2]^{\mathrm{T}}$. For $\epsilon = 0.5$, Theorem 4 yields $\bar{d} = 3.1$, and the corresponding state feedback gain is

$$K = \begin{bmatrix} -2.3593 & 0.7100 & 4.9681 \\ -2.3048 & -1.4295 & -5.5923 \end{bmatrix}$$

Also, for $\epsilon = 1$, $\bar{d} = 1.32$ and

$$K = \begin{bmatrix} -0.1183 & 0.6422 & 1.2963 \\ -3.7556 & -2.2408 & -4.5535 \end{bmatrix}$$

5 Conclusion

In this paper, the problem of delay-dependent robust stabilization for singular delay systems with norm bounded parametric uncertainties has been studied. A delay-dependent robust stability condition is presented and a design procedure of the desired state feedback controller is given. All the obtained results are formulated in terms of strict LMIs involving no decomposition of the system matrices, which makes the design procedure relatively simple and reliable. Neither model transformation nor bounding technique is needed in the development of the results. Numerical examples show that the results of the proposed methods are less conservative than those of the existing methods.

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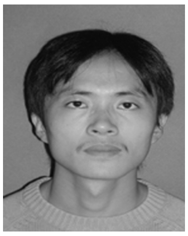

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