Wavelet Inpainting Based on p-Laplace Operator

ZHANG Hong-Ying[1] PENG Qi-Cong[2] WU Yang-Dong[3]

Abstract The problem of filling in missing or damaged wavelet coefficients is considered in this paper. Chan, Shen, and Zhou have proposed two total variation (TV) wavelet inpainting models to solve this problem. The main benefit of TV model is that it can keep the edges very well, but this method suffers from the staircase effect. To overcome this defect, we analyze the physical characteristics of TV model and p-Laplace operator in local coordinates, and explain that diffusion performance of p-Laplace is superior to that of TV model in essence. Afterwards, an inpainting model based on p-Laplace operator for damaged wavelet coefficients is presented. This new model can effectively reduce the staircase effect in TV model whereas it can still keep sharp edges as well as TV model. Experiment results show that better inpaingting quality can be achieved with much less computing time with our model.

Key words Image inpainting, wavelet transform, p-Laplace operator, total variation model

1 Introduction

Image inpainting refers to filling in missing or damaged regions in images. Mathematically speaking, inpainting is essentially an interpolation problem, and it is widely used in computer vision and image processing, including image replacement[1], disocclusion[2,3], and error concealment[4]. After the release of the new image compression standard JPEG2000, which is largely based on wavelet transforms, including the famous Daubechies 7/9 biorthogonal wavelet decomposition, many images are formatted and stored in terms of wavelet coefficients. In the wireless communication of these images, it could happen that certain wavelet packets are randomly lost or damaged during the transmission process. To recover the original images from their incomplete wavelet transforms is an inpainting problem according to the universal definition proposed in [5]. But this task remarkably differs from the classical inpainting problems in that the inpainting regions are in the wavelet domain.

Chan, Shen, and Zhou[6] proposed two variational models for wavelets based image inpainting depending on whether or not noise needs to be suppressed in the image. Their ideas are to use the given regularization in the pixel domain to control and restore wavelet coefficients in the wavelet domain. In their models, they use the total variation (TV) norm because it can retain sharp edges while reducing noise and other oscillations. But the corresponding Euler-Lagrange equation is not trivial to compute since it is highly nonlinear and ill-posed in strong sense. Furthermore, these models suffer from the staircase effect, *i.e.*, smooth regions (ramp) are transformed into piecewise constant regions (stairs). To overcome these deficiencies, we first analyze the physical characteristic of TV model and p-Laplace operator in local coordinates and explain that diffusion performance of p-Laplace is superior to that of TV model in essence. Then we present a new inpainting model based on p-Laplace operator for damaged wavelet coefficients. The results of experiment show that, with our method, better inpaingting quality can be achieved with much less computing time.

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1. School of Information Engineering, Southwest University of Science and Technology, Mianyang 621010, P. R. China 2. School of Communication and Information Engineering, University of Electronic Science and Technology of China, Chengdu 610054, P. R. China 3. School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 610054, P. R. China
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2 Analysis of p-Laplace operator and TV model

As shown in Fig. 1, Ω represents the whole image region, Drepresents the inpainting region, $u_0\,|_{\Omega\backslash D}$ is the available image information, u is the original image. The image inpainting model based on TV proposed by Tony Chan[7] is as follows

$$\min\ J[u] = \int_{\Omega} |\nabla u|\,\mathrm{d}x\mathrm{d}y + \frac{\lambda}{2}\int_{\Omega\backslash D}(u-u_0)^2\,\mathrm{d}x\mathrm{d}y \tag{1}$$

where λ is a constant. According to variational theory, the Euler-Lagrange equation corresponding to (1) is

$$-div\left(|\nabla u|^{-1}\nabla u\right) + \lambda_D(x,y)(u-u_0) = 0 \tag{2}$$

where div represents the divergence operator, $\lambda_D(x,y) = \lambda\cdot 1_{\Omega\backslash D}(x,y) = \begin{cases} \lambda\ (x,y)\in\Omega\backslash D \\ 0\ (x,y)\in D\end{cases}$. This model is rooted in the total variational denoising model proposed by Rudin-Osher-Fatemi[8]. In this model, diffusion performance is dependent on $div\left(|\nabla u|^{-1}\nabla u\right)$. We define

$$\Delta u = div\left(|\nabla u|^{-1}\nabla u\right) \tag{3}$$

as diffusion operator of TV model. In [9], one defines p-Laplace operator as

$$\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right),\ 1<p<\infty \tag{4}$$

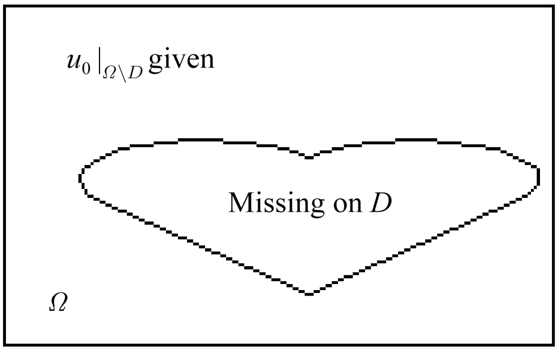

Fig. 1 Inpainting is to paint the missing $u_0\,|_D$ on an inpainting domain D based on what is available $\Omega\backslash D$

From (3) and (4), we can see that TV diffusion operator is the limit of p-Laplace operator when $p\to 1$. So, in order to analyze the physical characteristics of TV model and p-Laplace operator, we only deduce the expression of p-Laplace operator in local coordinates. Choose a local orthogonal coordinates system(ξ,η), as shown in Fig. 2, such that the η-axis is parallel to the gradient direction at a point and the ξ-axis is perpendicular, *i.e.*,

$$\xi = \frac{(-u_y,u_x)}{|\nabla u|} = \frac{\nabla^{\perp}u}{|\nabla u|},\ \eta = \frac{(u_x,u_y)}{|\nabla u|} = \frac{\nabla u}{|\nabla u|} \tag{5}$$

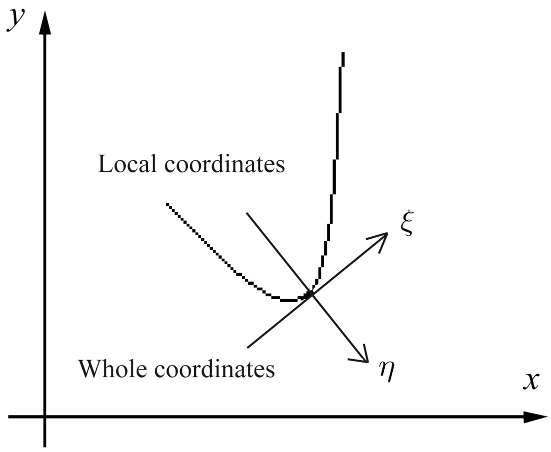

Fig. 2 The relation between the whole coordinates and the local coordinates

According to the second-order directional derivative method in [10, 11], we deduce the representation of (4) in the local orthogonal coordinates system (ξ, η). That is, in the local coordinates (ξ, η), (4) can be represented as

$$\Delta_p u = |\nabla u|^{p-2} u_{\xi\xi} + (p-1)|\nabla u|^{p-2} u_{\eta\eta} \tag{6}$$

Expression (6) illustrates that diffusion equation $\partial u/\partial t = \Delta_p u$ is essentially nonlinear anisotropic diffusion equation. The diffusion performance of diffusion equation is controlled by the diffusion coefficients $|\nabla u|^{p-2}$ and $(p-1)|\nabla u|^{p-2}$.

Now, we take into account two limit situations of (6) in the following.

1) When $p \equiv 1$, $\Delta_1 u = |\nabla u|^{-1} u_{\xi\xi}$, this is the expression of (5) in local coordinates. From this expression, we can see that TV diffusion operator only diffuses to the orthogonal direction of ∇u, with diffusion coefficient of $|\nabla u|^{-1}$, while no diffusion in the gradient direction. It is essentially anisotropic diffusion. The main benefit of TV model is that it does not penalize discontinuity in the image, thus keeping the edges very well. But it suffers from the staircase effect, *i.e.*, smooth regions (ramp) are transformed into piecewise constant regions (stairs).

2) When $p \equiv 2$, $\Delta_2 u = u_{\xi\xi} + u_{\eta\eta}$, this is isotropic diffusion because of the same diffusion coefficients. It is essentially the diffusion factor of harmonical model. This model can smooth image, while blur sharp edges.

According to the above analysis, we assure that the model with $1 < p < 2$ can reduce the staircase effect whereas it can still keep the sharp edges effectively. On the other hand, because TV model diffuses only in one direction while p-Laplace operator in two directions, it is apparent that the latter is faster than the former in diffusion speed. Consequently, better inpaingting quality can be achieved with much less computing time when applying p-Laplace operator to inpainting model. In the following section, we will present a new wavelet image inpaing model based on p-Laplace operator.

3 Wavelet inpainting model based on $\boldsymbol{p}$-Laplace operator

3.1 Wavelet inpainting model

In [9], one defines p-Laplace operator as $\Delta_p v = \nabla \cdot \left(|\nabla v|^{p-2} \nabla v\right)$, and the corresponding p-Laplace function is defined as

$$\begin{cases} \nabla_p u - f(x,u) = 0 & 1 < p < \infty, \quad x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{7}$$

here Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. To image restoration problem, $f(x,u) = \lambda\left(u - u^0\right)$, $f : \mathbf{R} \times \Omega \to \mathbf{R}$, where λ is a constant. $u^0 = u + n$ is the same as in the previous section. Then the energy functional corresponding to (7) is defined as

$$J[u] = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{\lambda}{2}\int_\Omega (u - u_0)^2 \mathrm{d}x \tag{8}$$

If $1 < p < 2$, we will get an image inpainting model based on p-Laplace operator, and we define it as

$$J[u] = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{\lambda}{2}\int_{\Omega\backslash D} (u - u_0)^2 \mathrm{d}x \quad 1 < p < 2 \tag{9}$$

where p can be adaptively selected based on the local gradient features of images. That is, away from edges, p will be approached to 2 to overcome the staircase effect; on the contrary, p will be approached to 1 to preserve edges. So this new model can effectively reduce the staircase effect in TV model whereas it can still retain the sharp edges as TV model. Motivated by [6], we present a novel inpainting model based on the p-Laplace operator for damaged wavelet coefficients. Our model is

$$\begin{cases} \min\limits_{\beta_{j,k}} F(u, u_0) = \frac{1}{p}\int_\Omega |\nabla_x u(\boldsymbol{\beta}, x)|^p \mathrm{d}x + \\ \sum\limits_{j,k} \frac{\lambda_{j,k}}{2}(\beta_{j,k} - \alpha_{j,k})^2 & 1 < p < 2 \\ u(\boldsymbol{\beta}, x) = \sum\limits_{j,k} \beta_{j,k}\psi_{j,k}(x), \ \boldsymbol{\beta} = (\beta_{j,k}) & j,k \in Z \\ u_0(\boldsymbol{\alpha}, x) = \sum\limits_{j,k} \alpha_{j,k}\psi_{j,k}(x) & j,k \in Z \\ \lambda_{j,k} = \begin{cases} 0 & (j,k) \in D \\ \lambda & (j,k) \in \Omega\backslash D \end{cases} \end{cases} \tag{10}$$

where Ω is the image domain, and D is the inpainting index region.

According to variational theory, the Euler-Lagrange equation corresponding to (10) is

$$-\nabla \cdot \left(|\nabla_x u(\boldsymbol{\beta}, x)|^{p-2} \nabla_x u(\boldsymbol{\beta}, x)\right) + \lambda_{j,k}(\beta_{j,k} - \alpha_{j,k}) = 0 \tag{11}$$

The gradient decent flow of this model is

$$(\beta_{j,k})_t = \nabla \cdot \left(|\nabla_x u(\boldsymbol{\beta}, x)|^{p-2} \nabla_x u(\boldsymbol{\beta}, x)\right) - \lambda_{j,k}(\beta_{j,k} - \alpha_{j,k}) \tag{12}$$

3.2 Wavelet inpainting algorithm

To find the minimizer of (10), we just need to solve for the solution of the above Euler-Lagrange equation (11). We can also solve it using the method of gradient flow, which is achieved by introducing an artificial time variable and solving the above equation (11) to the steady state for model (10). The steady state refers to $(\beta_{i,k})_t = 0$. In this case, gradient flow (12) is reduced to the Euler-Lagrange equation (11). Many numerical schemes can solve the above equation. In this paper, we also use the same numerical algorithm as in [6], the explicit finite difference algorithm, to find the minimizer. To simplify the formulation, we introduce the standard finite difference notations, such as

The forward differences:

$$D_1^+ u_{k,l} = u_{k+1,l} - u_{k,l}, \quad D_2^+ u_{k,l} = u_{k,l+1} - u_{k,l}$$

The backward differences:

$$D_1^- u_{k,l} = u_{k,l} - u_{k-1,l}, \quad D_2^- u_{k,l} = u_{k,l} - u_{k,l-1}$$

The time step size is denoted by Δ_t and space grid size is Δ_x.

We note that it is important to evaluate the nonlinear term, which we denote as

$$Wcurv = \nabla \cdot \left(|\nabla_x \hat{u}(\boldsymbol{\beta}, x)|^{p-2} \nabla_x \hat{u}(\boldsymbol{\beta}, x) \right) \qquad (13)$$

in (12). This term is the p-Laplace operator projected on the wavelet basis

$$Wcurv = \int \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) \psi_{j,k}(x) \mathrm{d}x \qquad (14)$$

However, the p-Laplace operator is defined in the pixel domain. In this paper, we calculate it straightforwardly by transforming the wavelet domain to the pixel domain to compute the p-Laplace operator, and then transform back to the wavelet domain. That is, we calculate

$$u = IWT(\boldsymbol{\beta}) \qquad (15)$$

where IWT is the inverse wavelet transform. For all (i, j), compute

$$\begin{aligned} \mathrm{curv}_{i,j} = D_1^- \frac{D_1^+ u_{i,j}}{\left(\left| D_1^+ u_{i,j} \right|^2 + \left| D_2^+ u_{i,j} \right|^2 + \varepsilon \right)^{\frac{2-p}{2}}} + \\ D_2^- \frac{D_2^+ u_{i,j}}{\left(\left| D_2^+ u_{i,j} \right|^2 + \left| D_2^+ u_{i,j} \right|^2 + \varepsilon \right)^{\frac{2-p}{2}}} \end{aligned} \qquad (16)$$

where ε is a small positive number which is used to prevent the numerical blow up when $\left| D_1^+ u_{i,j} \right|^2 + \left| D_2^+ u_{i,j} \right|^2 = 0$. Then we compute the curvature projection on the wavelet basis by

$$Wcurv = FWT(curv) \qquad (17)$$

where FWT is the forward wavelet transform.

The complete algorithm can be summarized by the following pseudo-code.

Algorithm.

1) Start with initial guess $\beta_{j,k}^{new} = \alpha_{j,k}\chi_{j,k}$. Set $\beta_{j,k}^{old} = 0$, and the initial error $E = \left\| \boldsymbol{\beta}^{new} - \boldsymbol{\beta}^{old} \right\|_2$.

2) While $i \leq N$ or $E \leq \delta$, do

a) Set $\boldsymbol{\beta}^{old} = \boldsymbol{\beta}^{new}$

b) Calculate $Wcurv$ by (15)~(17)

c) For all (i, j), update

$$\beta_{j,k}^{new} = \beta_{j,k}^{old} + \frac{\Delta_t}{\Delta_x} \left(\beta_{j,k}^{pTV} - \lambda_{j,k}(\beta_{j,k} - \alpha_{j,k}) \right)$$

d) Compute error $E = \left\| \boldsymbol{\beta}^{new} - \boldsymbol{\beta}^{old} \right\|_2$, and set $i = i+1$

e) End the while loop.

4 Simulation and result analysis

In this section, we simulate our algorithm on Matlab7.0. All experiments are run on a 1.8GHz AMD XP2200 with 512MB of RAM. To test the models, we use standard Peak Signal to Noise Ratio (PSNR) to quantify the performance of inpainting. As usual, the larger the PSNR value, the better the performance. In all examples shown here, we use db 7/9 biorthogonal wavelets with periodic extensions at the boundaries and set $p = 1.2$, $\lambda = 0$ for noiseless images, $\lambda = 0.08$ for noisy images.

In the first example, we apply the TV wavelet inpainting model and our wavelet inpainting model based on the p-Laplace operator to a real image shown in Fig. 3(a). The picture in Fig. 3(b) is the damaged image with 50% wavelet coefficients randomly damaged. Fig. 3(c) is the restored image with TV wavelet inpainting model after iterating 300 times (75.35s), which has PSNR=26.7320dB. Fig. 3(d) is the restored image with our model after iterating 200 times (59.32s), which has PSNR=29.2443dB. For comparison purpose, Fig. 4 shows the performance improvement measured by PSNR vs the severity of the damage. The x-axis represents the percentage of wavelet coefficients being damaged. The y-axis is the performance measured by PSNR. For the horizontal axis, for example, 0.8 means that 80% of wavelet coefficients are damaged. From this picture, we can see that our wavelet inpainting model based on the p-Laplace operator can dramatically improve image qualities better than TV wavelet inpainting model, especially in the large number of damaged coefficients.

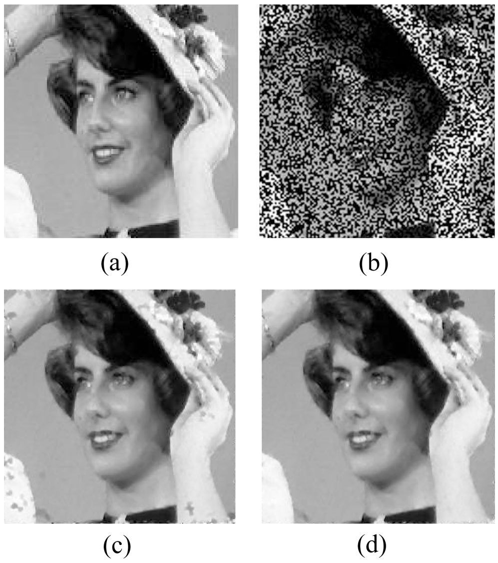

Fig. 3 Results of the real image by two different methods ((a) Original image; (b) Damaged image with 50% wavelet coefficients randomly damaged (PSNR=8.8233dB); (c) Restored image by TV wavelet inpainting model (75.35s, PSNR=26.7320dB); (d) Restored image by our model (59.32s, PSNR=29.2443dB))

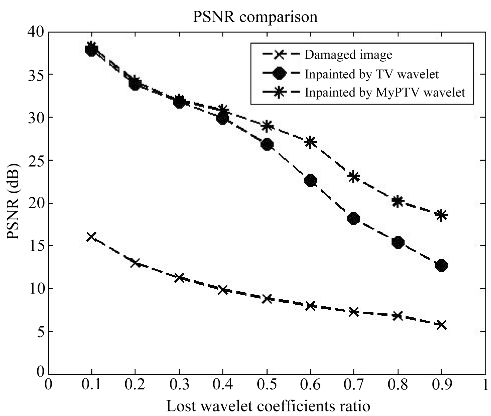

Fig. 4 Performance comparisons of two different methods

In the next experiment, we apply the two models to a synthetic image. To further compare the models, in addition to losing 50% coefficients randomly, we introduce Gaussian noise to the image as well. Fig. 5(b) shows the noisy image with losing 50% coefficients randomly and the restored images using the two models. From the pictures, we can see that our proposed model gives the better restored image than TV wavelet inpainting model.

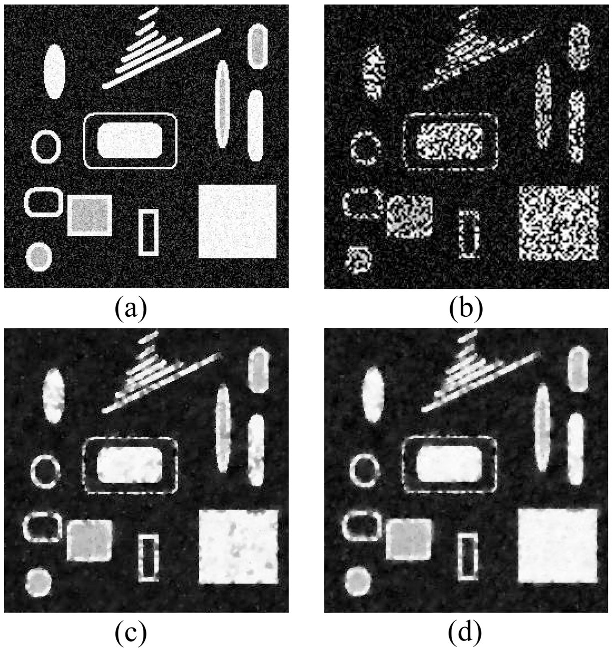

Fig. 5 Results of the synthetic image by two different methods ((a) Original noisy image; (b) 50% of the wavelet coefficients are randomly damaged (PSNR=10.9833dB); (c) Restored image by TV wavelet inpainting model (73.13s, PSNR= 18.5072dB); (d) Restored image by our model (62.65s, PSNR=18.9185dB))

5 Conclusion

In this paper, we have presented a wavelet inpainting model based on p-Laplace operator for restoring arbitrary number of coefficients in arbitrary locations of wavelets coefficients for images with or without noise. Comparing our model to Chan, Shen and Zhou′s TV wavelet inpainting model, we achieve the better inpaingting quality with much less computing time, especially with large number of damaged wavelet coefficients.

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ZHANG Hong-Ying Received her B. S. degree in Department of Mathematics from Northeast University in 2000 and her master degree from Southwest University of Science and Technology in 2003. She is currently working towards the Ph. D. degree in School of Communication and Information Engineering at University of Electronic Science and Technology of China. Her research interest covers image processing and image analysis. Corresponding author of this paper. E-mail: zhy0838@163.com

PENG Qi-Cong Professor in School of Communication and Information Engineering at University of Electronic Science and Technology of China. His interest covers communication signal processing, real time signal processing, and DSP.
E-mail: qpeng@uestc.edu.cn

WU Ya-Dong Ph. D. candidate in School of Computer Science and Engineering at University of Electronic Science and Technology of China. His research interest covers graphics and image processing, neural network, and parallel computation.
E-mail: wyd028@163.com