Robust Control System Design Using Proportional Plus Partial Derivative State Feedback

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Abstract Based on a general parametric eigenstructure assignment result proposed for descriptor linear systems via proportional plus partial derivative state feedback and a result for generalized eigenvalue sensitivity problem of matrix pairs, parametric representation of the closed-loop eigenvalue sensitivities to the perturbed elements in the open-loop system matrices is obtained. An effective algorithm for eigenvalue assignment with minimum sensitivity in descriptor linear systems via proportional plus partial derivative state feedback is then proposed. The algorithm does not contain 'going back' procedures, and allows the closed-loop eigenvalues to be conveniently optimized within desired regions. An example demonstrates its effectiveness and simplicity.

Key words Descriptor linear systems, eigenvalue assignment, eigenvalue sensitivities, proportional plus partial derivative state feedback

1 Introduction

Eigenvalue assignment with minimum sensitivity in multivariable linear systems is an important problem in the field of robust control, and has been intensively studied in the last decades. However, most of the results are obtained for the case of conventional linear systems[1~10]. For the case of descriptor linear systems, this problem has only been investigated by a few researchers[11~15]. Kautsky *et al.*[11] extended their earlier well-known techniques in [3] for conventional linear systems to the case of descriptor systems, and laid a special emphasis on the closed-loop regularity. Syrmos and Lewis[12] proposed a robustness theory for the generalized spectrum of descriptor linear systems, and presented a compact theory for the robust eigenvalue assignment problem in descriptor linear systems using the concept of chordal metric. Duan and Patton [13] studied robust pole assignment in descriptor linear systems via proportional plus partial derivative state feedback. Due to the capacity of derivative feedback, their work concentrated on the case that the closed-loop system possesses n (=the system order) finite closed-loop eigenvalues. Recently, Duan *et al.*[14] investigated robust pole assignment in descriptor linear systems via state feedback, and considered robust pole assignment in descriptor linear systems via output feedback[15]. These works are based on the eigenstructure assignment results respectively proposed by Duan[16] and Duan[17], and, like Duan and Patton[13], they realized robust pole assignment by minimizing the condition numbers associated with the closed-loop eigenvalues.

Solutions to eigenvalue assignment with minimum sensitivity can be classified into two categories: one is to minimize the eigenvalue sensitivities to model parameter variations in all the elements of the open-loop system matrices[1~4,9~15]; the other is to minimize the eigenvalue sensitivities to model parameter variations in some, but not all, of the elements of the open-loop system matrices[1,5~8]. Different from [13], this paper considers the problem of minimizing the eigenvalue sensitivities to model parameter variations in part of the elements of the open-loop system matrices using proportional plus partial derivative state feedback. Based on a result for generalized eigenvalue sensitivity problem of matrix pairs proposed by Haley[18] and a general parametric eigenstructure assignment result for descriptor linear systems via proportional plus partial derivative state feedback proposed by Duan and Patton[19], parametric representation of the closed-loop eigenvalue sensitivities to the perturbed elements in the open-loop system matrices is established, and an effective algorithm for robust eigenvalue assignment in descriptor linear systems via proportional plus partial derivative state feedback is then proposed. Due to the advantages of the eigenstructure assignment approach used, the approach proposed for the robust pole assignment problem possesses several features: 1) The procedures for solution of the proposed robust pole assignment problem are in a sequential order, and no "going back" procedures are needed. 2) The eigenvalues may be easily included in the design parameters and are optimized within certain desired fields on the complex plane to improve the robustness. 3) The optimality of the solution to the whole robust pole assignment problem is solely dependent on the optimality of the solution to the minimization problem converted. A numerical example will demonstrate the above advantages.

The paper is divided into five sections. In the next section, the problem of eigenvalue assignment with minimum sensitivity in descriptor linear systems via proportional plus partial derivative state feedback is formulated. Section 3 states a result, proposed by Duan[19], on eigenstructure assignment in descriptor linear systems via proportional plus partial derivative state feedback. The algorithm for solving the proposed robust eigenvalue assignment problem is presented in Section 4. An example is examined in Section 5.

2 Problem formulation

Consider the following descriptor linear system

$$(E+\Delta E)\dot{\boldsymbol{x}} = (A+\Delta A)\boldsymbol{x} + (B+\Delta B)\boldsymbol{u} \qquad (1)$$

where $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{u} \in \mathbf{R}^r$ are, respectively, the state vector and the input vector; E, A and B are matrices of appropriate dimensions with $\mathrm{rank}(E) = n_0 \leq n$ and $\mathrm{rank}(B) = r$, and they satisfy the following R-controllability assumption:

Assumption 1. $\mathrm{rank}[sE - A \;\; B] = n$, for $\forall\, s \in \mathbf{C}$.

The matrices ΔE, ΔA and ΔB are the system parameter perturbations which possess the following forms

$$\Delta E = \sum_{i=1}^{l} E_i\,\varepsilon_i, \quad \Delta A = \sum_{i=1}^{l} A_i\,\varepsilon_i, \quad \Delta B = \sum_{i=1}^{l} B_i\,\varepsilon_i \quad (2)$$

Received November 29, 2005; in revised form March 20, 2006
Supported in part by the Chinese Outstanding Youth Science Foundation (69925308) and supported by Program for Changjiang Scholars and Innovative Research Team in University
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DOI: 10.1360/aas-007-0506

where $E_i,\ A_i$ and $B_i,\ i=1,2,\ldots,l$, are known real matrices of proper dimensions; $\varepsilon_i,\ i=1,2,\ldots,l$, are small perturbation variables.

When the following proportional plus partial derivative state feedback controller

$$\boldsymbol{u}=K\boldsymbol{x}-LC\dot{\boldsymbol{x}} \tag{3}$$

is applied to system (1), where $C\in\mathbf{R}^{m\times n}$ is the measurement matrix with full row rank, and $K\in\mathbf{R}^{r\times n},\ L\in\mathbf{R}^{r\times m}$ are respectively the proportional state feedback gain matrix and the partial derivative state feedback gain matrix, the closed-loop system is obtained in the following form

$$(E_c+\Delta E_c)\dot{\boldsymbol{x}}=(A_c+\Delta A_c)\boldsymbol{x} \tag{4}$$

with

$$E_c=E+BLC,\quad A_c=A+BK \tag{5}$$

$$\Delta E_c=\Delta E+\Delta BLC,\quad \Delta A_c=\Delta A+\Delta BK \tag{6}$$

It has been shown in [19] that under the condition

$$\mathrm{rank}[E\ \ B]=n\quad \text{or}\quad \mathrm{rank}[E^{\mathrm{T}}\ \ C^{\mathrm{T}}]=n \tag{7}$$

there exists a real matrix L such that E_c is non-singular. Due to this fact, we aim to assign n finite closed-loop relative eigenvalues, and leave no infinite eigenvalues to the closed-loop system. Further, in view of the fact that non-defective matrix pair $[A_c\ \ E_c]$ possesses relative eigenvalues with less sensitivities to the matrix parameter perturbations[11], the closed-loop finite eigenvalues are restricted to be a set of n distinct, but self-conjugate complex numbers. The problem of robust pole assignment to be solved in this paper can be stated as follows.

Problem RPA. Given system (1) and (2) satisfying Assumption 1, and a set of regions $\varOmega_i,\ i=1,2,\ldots,n$, on the complex plane, which are symmetric about the real axis, find a proportional plus partial derivative state feedback controller in the form of (3), such that the following requirements are met:

1) The matrix pair $[A_c\ \ E_c]$ is regular, that is, $\det(sE_c-A_c)$ is not identically zero.

2) The matrix pair $[A_c\ \ E_c]$ has n distinct finite relative eigenvalues $s_i,\ i=1,2,\ldots,n$, satisfying $s_i\in\varOmega_i,\ i=1,2,\ldots,n$.

3) The eigenvalues of the matrix pair $[A_c+\Delta A_c\ \ E_c+\Delta E_c]$ at $\varepsilon_i=0,\ i=1,2,\ldots,l$, are as insensitive as possible to small variations in $\varepsilon_i,\ i=1,2,\ldots,l$.

3 Preliminaries

It is shown in [20] that, when Assumption 1 is satisfied, there exist a pair of right coprime polynomial matrices $N(s)\in\mathbf{R}^{n\times r}[s]$ and $D(s)\in\mathbf{R}^{r\times r}[s]$ satisfying

$$(A-sE)N(s)+BD(s)=0 \tag{8}$$

Lemma 1[19]**.** Given system (1) and (2) satisfying Assumption 1. Let $N(s)\in\mathbf{R}^{n\times r}[s]$ and $D(s)\in\mathbf{R}^{r\times r}[s]$ be polynomial matrices satisfying (8). Then

1) There exist a group of distinct, self-conjugate complex numbers $s_i,\ i=1,2,\cdots,n$, a non-singular matrix $V\in\mathbf{C}^{n\times n}$, two real matrices $K\in\mathbf{R}^{r\times n}$ and $L\in\mathbf{R}^{r\times m}$, such that $\det[(s(E+BLC)-(A+BK))]$ is not identically zero, and

$$(A+BK)V=(E+BLC)V\varLambda \tag{9}$$

holds for

$$\varLambda=\mathrm{diag}(s_1,\ s_2,\ \cdots,\ s_n) \tag{10}$$

if and only if

a) there exists a matrix $L\in\mathbf{R}^{r\times m}$ satisfying

Constraint 1. $\det(E+BLC)\neq 0$;

b) there exists a group of parameter vectors $\boldsymbol{f}_i,\ i=1,2,\ldots,n$, satisfying

Constraint 2. $\boldsymbol{f}_i=\bar{\boldsymbol{f}}_l$ if $s_i=\bar{s}_l$

and

Constraint 3. $\det[N(s_1)\boldsymbol{f}_1\ \ N(s_2)\boldsymbol{f}_2\ \ \cdots\ \ N(s_n)\boldsymbol{f}_n]\neq 0$.

2) When the above conditions a) and b) are satisfied, the partial derivative feedback gain L may be taken to be an arbitrary real matrix satisfying Constraint 1, the matrix V are given by

$$V=[N(s_1)\boldsymbol{f}_1\ \ N(s_2)\boldsymbol{f}_2\ \ \cdots\ \ N(s_n)\boldsymbol{f}_n] \tag{11}$$

and the corresponding matrix gain K is given by

$$K=(W+LCV\varLambda)V^{-1} \tag{12}$$

with

$$W=[D(s_1)\boldsymbol{f}_1\ \ D(s_2)\boldsymbol{f}_2\ \ \cdots\ \ D(s_n)\boldsymbol{f}_n] \tag{13}$$

where in (11) and (13), $\boldsymbol{f}_i,\ i=1,2,\ldots,n$, is a group of design parameter vectors satisfying Constraints 2 and 3.

Lemma 2[19]**.** Let the conditions a) and b) in Lemma 1 hold, matrices $K,\ L$ and V be given according to the second conclusion of Lemma 1, and U be defined by

$$U^{\mathrm{T}}=V^{-1}(E+BLC)^{-1} \tag{14}$$

Then

$$U^{\mathrm{T}}E_cV=I,\quad U^{\mathrm{T}}A_cV=\varLambda \tag{15}$$

4 Solution to problem RPA

In order to solve the robust pole assignment problem formulated in Section 2, proper sensitivity measures for the closed-loop eigenvalues need to be established. To achieve this purpose, we first state the following lemma.

Lemma 3[18]**.** Let $M,N\in\mathbf{R}^{n\times n}$ be matrix functions of some scalar parameter ε, λ be a simple finite relative eigenvalue of the non-defective matrix pair $[M\ \ N]$, and $\boldsymbol{x}$ and $\boldsymbol{y}$ be a pair of right and left eigenvectors of the matrix pair $[M\ \ N]$ associated with eigenvalue λ. Then

$$\frac{\partial\lambda}{\partial\varepsilon}=\frac{\boldsymbol{f}^{\mathrm{T}}\left(\dfrac{\partial M}{\partial\varepsilon}-\lambda\dfrac{\partial N}{\partial\varepsilon}\right)\boldsymbol{x}}{\boldsymbol{y}^{\mathrm{T}}N\boldsymbol{x}} \tag{16}$$

Let $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ be respectively the ith columns of matrices V and U. With Lemma 1 and Lemma 2, we can prove the following theorem.

Theorem 1. Let the conditions in Lemma 2 hold. Then the eigenvalue sensitivities of the closed-loop system (4)~(6) to variations ε_i at $\varepsilon_i=0,\ i=1,2,\ldots,l$, are given as follows

$$\frac{\partial s_i}{\partial\varepsilon_j}=\boldsymbol{u}_i^{\mathrm{T}}(A_jN(s_i)+B_jD(s_i)-s_iE_jN(s_i))\boldsymbol{f}_i \tag{17}$$

$$i=1,2,\ldots,n,\quad j=1,2,\ldots,l$$

where $\boldsymbol{u}_i$ is the ith column of the matrix U given in Lemma 2.

Proof. In view of (2) and (6), we have

$$\Delta E_c=\sum_{p=1}^{l}(E_p+B_pLC)\varepsilon_p,\quad \Delta A_c=\sum_{p=1}^{l}(A_p+B_pK)\varepsilon_p \tag{18}$$

It is clear that the eigenvalues and eigenvectors of perturbed matrix pairs $[A_c+\Delta A_c \quad E_c+\Delta E_c]$ are all functions of perturbation parameters $\varepsilon_i,\ i=1,2,\dots,l$. Denote $\omega=(\varepsilon_1,\ \varepsilon_2,\ \cdots,\ \varepsilon_l)$. By applying Lemma 3 and using (18), we have

$$\frac{\partial s_i(\omega)}{\partial\varepsilon_j}=\frac{\boldsymbol{u}_i^{\mathrm{T}}(\omega)\left(\dfrac{\partial\Delta A_c}{\partial\varepsilon_j}-s_i(\omega)\dfrac{\partial\Delta E_c}{\partial\varepsilon_j}\right)\boldsymbol{v}_i(\omega)}{\boldsymbol{u}_i^{\mathrm{T}}(\omega)(E_c+\Delta E_c)\boldsymbol{v}_i(\omega)}$$

$$=\frac{\boldsymbol{u}_i^{\mathrm{T}}(\omega)\left[A_j+B_jK-s_i(\omega)(E_j+B_jLC)\right]\boldsymbol{v}_i(\omega)}{\boldsymbol{u}_i^{\mathrm{T}}(\omega)(E_c+\Delta E_c)\boldsymbol{v}_i(\omega)} \tag{19}$$

$$i=1,2,\dots,n,\quad j=1,2,\dots,l$$

Letting $\omega=0$ and noticing that $\boldsymbol{u}_i(\mathbf{0})=\boldsymbol{u}_i,\ \boldsymbol{v}_i(\mathbf{0})=\boldsymbol{v}_i,\ s_i(0)=s_i$ and $\Delta E_c|_{\omega=0}=0$, we have

$$\frac{\partial s_i}{\partial\varepsilon_j}=\frac{\boldsymbol{u}_i^{\mathrm{T}}\left[A_j+B_jK-s_i(E_j+B_jLC)\right]\boldsymbol{v}_i}{\boldsymbol{u}_i^{\mathrm{T}}E_c\boldsymbol{v}_i} \tag{20}$$

$$i=1,2,\dots,n,\quad j=1,2,\dots,l$$

From (12), we can obtain

$$K\boldsymbol{v}_i=\boldsymbol{w}_i+s_iLC\boldsymbol{v}_i \tag{21}$$

By using (21), (20) can be turned into the following form

$$\frac{\partial s_i}{\partial\varepsilon_j}=\frac{\boldsymbol{u}_i^{\mathrm{T}}\left(A_j\boldsymbol{v}_i+B_j\boldsymbol{w}_i-s_iE_j\boldsymbol{v}_i\right)}{\boldsymbol{u}_i^{\mathrm{T}}E_c\boldsymbol{v}_i} \tag{22}$$

$$i=1,2,\dots,n,\quad j=1,2,\dots,l$$

Finally by applying Lemma 1 and Lemma 2, we can convert (22) into the form of (17). □

It follows from Lemma 1 that the matrices K and L given according to the second conclusion of Lemma 1 meet the first two requirements in Problem RPA. In order to solve Problem RPA, we need to seek proper choices of design parameters, $L,\ s_i,\ \boldsymbol{f}_i,\ i=1,2,\dots,n$, to further meet the third requirement in Problem RPA stated in Section 2. This can be realized by minimizing the closed-loop eigenvalue sensitivities given in (17). We may define an objective as

$$J(L,\ s_i,\ \boldsymbol{f}_i,\ i=1,2,\dots,n)=\sum_{i=1}^{n}\sum_{j=1}^{l}\alpha_{ij}\left|\frac{\partial s_i}{\partial\varepsilon_j}\right|^2 \tag{23}$$

where $\frac{\partial s_i}{\partial\varepsilon_j},\ i=1,2,\dots,n,\ j=1,2,\dots,l$, are given by Theorem 1, and $\alpha_{ij},\ i=1,2,\dots,n,\ j=1,2,\dots,l$, are a group of positive scalars representing the weighting factors. Therefore, the parameters $L,\ s_i,\ \boldsymbol{f}_i,\ i=1,2,\dots,n$, can be sought by the following optimization problem:

$$\begin{aligned}&\text{minimize } J(L,\ s_i,\ \boldsymbol{f}_i,\ i=1,2,\dots,n)\\&\quad\text{s.t. } s_i\in\Omega_i,\ i=1,2,\dots,n\\&\quad\text{Constraints 1}\sim3\end{aligned} \tag{24}$$

Based on the above deduction and analysis, an algorithm for solution to Problem RPA can be given as follows.

Algorithm RPA.

1) Solve the right coprime matrix polynomials $N(s)$ and $D(s)$ satisfying (8).

2) Solve the parametric expressions for matrices V, W and U according to (11), (13) and (14).

3) Find the optimal design parameters $L,\ \boldsymbol{f}_i,\ s_i,\ i=1,2,\dots,n$, by solving the minimization problem (24).

4) Calculate matrices V and W according to (11) and (13) based on the parameters $\boldsymbol{f}_i,\ s_i,\ i=1,2,\dots,n$, obtained in Step 3.

5) Calculate the state feedback gain matrix K by formula (12) based on matrices V and W obtained in Step 4 and the parameter matrix L obtained in Step 3.

5 Example

Consider a descriptor linear system with the following coefficient matrices[13,19]:

$$E=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\1&0&0&0&0&0\end{bmatrix}$$

$$A=\begin{bmatrix}0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&1&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\1&0&0&0&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}1&0&0&0&1&0\\0&0&0&0&1&0\end{bmatrix}^{\mathrm{T}}$$

The measurement matrix of the state derivatives is given by

$$C=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix}$$

It is easy to verify that Assumption 1 and condition (7) are satisfied. By a method given by Duan and Patton[19], the following solution to the right coprime factorization (8) is obtained:

$$N(s)=\begin{bmatrix}s^2&-s^2\\s&-s\\1&0\\0&s\\0&1\\s^2(s-1)&-s^2(s-1)\end{bmatrix}$$

$$D(s)=\begin{bmatrix}s^3-1&-s^3\\-s^3+1&s^3-1\end{bmatrix}$$

Restrict the closed-loop eigenvalues $s_i,\ i=1\sim6$, to be distinct and real, and let

$$\boldsymbol{f}_i=\begin{bmatrix}e_i\\d_i\end{bmatrix},\quad i=1\sim6$$

Then the general parametric solutions for the closed-loop eigenvectors are given by

$$\boldsymbol{v}_i=\begin{bmatrix}(e_i-d_i)s_i^2\\(e_i-d_i)s_i\\e_i\\s_id_i\\d_i\\(e_i-d_i)(s_i-1)s_i^2\end{bmatrix},\quad i=1\sim6$$

and the general forms for the corresponding vectors $\boldsymbol{w}_i$'s are given by

$$\boldsymbol{w}_i=\begin{bmatrix}-e_i+(e_i-d_i)s_i^3\\(e_i-d_i)(1-s_i^3)\end{bmatrix},\quad i=1\sim6$$

Since all the closed-loop eigenvalues are real, we can also restrict the parameters e_i's and d_i's to be real. Therefore, Constraint 2 holds automatically.

Duan and Patton[13] have obtained a solution without consideration of robustness by a pole assignment approach, which is referred as Solution 1 in the following, while Duan and Patton[13] have also obtained three robust solutions by minimizing the closed-loop eigenvalue sensitivity measures $c_i = \|\boldsymbol{u}_i\|_2\|\boldsymbol{v}_i\|_2/(1+|s_i|^2)^{1/2},\ i = 1 \sim 6$, which are referred as Solutions 2∼4 in the following. Solutions 1∼3 were all obtained on the condition that the closed-loop eigenvalues are previously assigned to $s_i = -i,\ i = 1 \sim 6$, while Solution 4 was obtained by optimizing closed-loop eigenvalues within the following regions:

$$\Omega_1 = [-1 \quad -0.5], \quad \Omega_2 = [-2.5 \quad -1], \quad \Omega_3 = [-2.5 \quad -1]$$

$$\Omega_4 = [-4 \quad -2], \quad \Omega_5 = [-5.5 \quad -3], \quad \Omega_6 = [-6.5 \quad -4]$$

Corresponding to perturbations in the form of (2) with

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$E_2 = E_3 = E_4 = A_1 = A_4 = 0, \quad B_2 = B_3 = 0$$

we have worked out the following two solutions to the problem by using our algorithm. For the optimization problem involved in obtaining the solutions, the Matlab command *fmincon* is used.

Solution 5. As in Solutions 1∼3, the closed-loop eigenvalues are chosen as $s_i = -i,\ i = 1 \sim 6$. The parameters e_i, d_i and L are found through solving minimization problem (24). Taking $\alpha_i = 1,\ i = 1 \sim 6$, we obtain the parameters e_i and d_i as (see ∗ in the following) and the derivative feedback gain as

$$L = \begin{bmatrix} 0.8091586 & -0.0272887 \\ 3.6727977 & -1.0475362 \end{bmatrix}$$

The corresponding proportional feedback gain is then given by (see † in the following)

Solution 6. In this solution, all the three parts of parameters are optimized in the minimization problem (24), where the closed-loop eigenvalue regions are chosen as in Solution 4. Taking the same weighting factors as in Solution 5, we obtain the parameters s_i, e_i and d_i as (see ‡ in the following) and

$$L = \begin{bmatrix} 0.0684943 & 0.0693610 \\ -0.5730766 & 2.5431677 \end{bmatrix}$$

The corresponding proportional feedback gain is then given by (see § in the following)

Table 1 gives for each solution the robustness index value J and the spectral norm of the condition number vector $\boldsymbol{c} = [c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6]$, where $c_i = \|\boldsymbol{u}_i\|_2\|\boldsymbol{v}_i\|_2/(1 + |s_i|^2)^{1/2},\ i = 1 \sim 6$. The magnitude of each of the derivative feedback gain L and the proportional feedback gain K are also shown in Table 1. The practical closed-loop eigenvalues $s'_i,\ i = 1 \sim 6$, corresponding to these solutions are listed in Table 2 (see next page). The number σ in Table 2 is defined by

$$\sigma = \left(\sum_{i=1}^{6} \left(s_i - s'_i \right)^2 \right)^{1/2}$$

which represents the amount of drift of the closed-loop eigenvalues from the nominal ones. Table 3 shows the shifted closed-loop eigenvalues of the system under the perturbations with $\varepsilon_1 = \varepsilon_2 = 0.003$, $\varepsilon_3 = \varepsilon_4 = 0.001$, corresponding to Solutions 1∼6.

Table 1 Robustness measures and magnitudes of solutions

Solutions	J	$\|\boldsymbol{c}\|_2$	$\|L\|_2$	$\|K\|_2$
1	5669694.9165	140.1221	1	173.3643
2	79836.2167	63.8404	0.3541	7.7286
3	51627.2658	60.9959	0.3472	7.0797
4	3651.1695	15.4776	0.2562	3.7894
5	3839.4225	94.5313	3.9111	76.4945
6	72.8407	46.3356	2.6350	93.0874

From Table 1 and Table 2, we can see that the solutions with smaller J values have reasonable $\|\boldsymbol{c}\|_2$ values and reasonable magnitudes, and the closed-loop eigenvalue drifts caused by the truncation errors in L and K for these solutions are also small. It can be seen from Table 3 that the robust solutions obtained by our algorithm, Solutions 5 and 6, have much smaller eigenvalue sensitivities than the non-robust one, Solution 1, and the robust solutions obtained by Duan and Patton[13], Solutions 2∼4. It can also be seen from Table 3 that inclusion of the closed-loop eigenvalues into the design parameters further improves the robustness of the closed-loop system.

$$\begin{array}{lllllll} e_i: & 0.2978396 & -0.1423776 & 0.0003916 & -0.0003332 & 0.0330662 & -0.1693062 \\ d_i: & 0.0279394 & -0.0795842 & 0.0000937 & -0.0000001 & 0.0290889 & -0.1509819 \end{array} \quad (*)$$

$$K = \begin{bmatrix} -7.2206467 & -18.2085216 & -12.0241388 & -5.8477275 & 0.7399704 & 0.2376506 \\ 35.4650184 & 30.9446253 & 15.1511768 & -24.3985087 & -50.7632703 & 9.1128192 \end{bmatrix} \quad (\dagger)$$

$$\begin{array}{lllllll} s_i: & -0.5 & -1.6914318 & -1 & -2 & -3 & -6.5 \\ e_i: & -227.6965931 & 4.1203370 & -57.4780905 & 471.1710735 & 90.1500257 & -170.6558778 \\ d_i: & -27.1273861 & -429.2019823 & -44.0334960 & 223.3387595 & 88.8614392 & -168.3274224 \end{array} \quad (\ddagger)$$

$$K = \begin{bmatrix} -5.5302626 & -5.8147489 & -2.7952875 & -0.3027807 & 1.4561108 & -0.4965289 \\ -74.3391045 & -46.5318848 & -12.1152356 & 2.2872871 & 13.5679474 & -23.9326232 \end{bmatrix} \quad (\S)$$

Table 2 Closed-loop eigenvalues

Solutions	s'_1	s'_2	s'_3	s'_4	s'_5	s'_6	σ
1	-1.0000000	-2.0000000	-3.0000000	-3.9999999	-5.0000006	-5.9999996	$7.21e-7$
2	-0.9999999	-2.0000005	-3.0000000	-3.9999989	-4.9999998	-6.0000008	$1.499e-6$
3	-0.9999999	-2.0000009	-3.0000000	-3.9999966	-4.9999991	-6.0000040	$5.457e-6$
4	-0.5000000	-1.0371611	-1.0632987	-3.2672928	-5.4999998	-6.5000005	$5.910e-7$
5	-1.0000000	-1.9999999	-2.9999976	-4.0000053	-5.0000108	-5.9999829	$2.105e-5$
6	-0.5000000	-1.6914333	-1.0000001	-1.9999989	-2.9999985	-6.4999981	$3.045e-6$

Table 3 Shifted closed-loop eigenvalues under system perturbations

Solutions	s_1	s_2	s_3	s_4	s_5	s_6
1	-1.0209925	-1.7313214	$-3.4718225 \pm 0.1020908j$		$-5.3416751 \pm 2.1466420j$	
2	-1.0153142	-1.9413655	-2.7143581	$-4.6580784 \pm 0.7373094j$		-6.2394378
3	-1.0128808	-1.8106250	-3.2311768	$-4.3095201 \pm 0.7873383j$		-6.4972430
4	-0.5061036	$-1.0308570 \pm 0.0359107j$		-3.3310303	-5.4024279	-6.8031855
5	-1.0020148	-1.9518119	-3.0038096	-4.1614917	-4.8849955	-6.1431884
6	-0.4984105	-1.5927183	-1.0007744	-2.1769518	-2.9169237	-6.5239623

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