# Robust $H_\infty$ Filtering for a Class of Uncertain Lurie Time-delay Singular Systems

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**Abstract** This paper deals with the problem of the robust $H_\infty$ filtering for a class of Lurie singular systems with state time-delays, parameter uncertainties and unknown statistics characteristics but with limited power disturbance, aiming to design a robustly stable filter such that the uncertain Lurie time-delay singular systems are not only regular, impulse free and stable, but also have a prescribed level of $H_\infty$ performance for the filtering error dynamics for all admissible uncertainties. A sufficient condition for the existence of such a filter is proposed in terms of linear matrix inequalities (LMIs). When a solution to this set of LMIs exists, the parametric matrices of a desired filter can be easily obtained using LMI toolbox.

**Key words** Lurie singular system, robust $H_\infty$ filtering, linear matrix inequality (LMI)

## 1 Introduction

Lurie system is a typical nonlinear system. In this paper, we study the problem of robust $H_\infty$ filtering for Lurie singular systems with both time-delays and parameter uncertainties. The parametric uncertainties are assumed to be time-varying, but norm-bounded; moreover, time-delay is also assumed to be unknown time-varying, but bounded; the disturbance input has unknown statistical characteristics but limited power. The design methodology presented in this paper is a 2-step procedure; that is, firstly, a sufficient condition based on LMIs which guarantees both the robust stability and a prescribed level of $H_\infty$ performance for uncertain Lurie time-delay singular systems is obtained; secondly, the synthesis of a robust $H_\infty$ filter based on the above-derived sufficient condition is given such that the filtering error system is robustly stable and has a prescribed $H_\infty$ performance level for all admissible uncertainties is obtained.

The suitable $H_\infty$ filter can be constructed through a convex optimization problem, which can be effectively handled by Matlab/LMI Toolbox. Finally, an illustrative example is given to show the effectiveness of the proposed approach.

## 2 System description and definitions

Consider the following uncertain Lurie singular systems with time-delays:

$$\begin{aligned}\Sigma: E\dot{\boldsymbol{x}}(t) =&(A_0+\Delta A_0(\boldsymbol{x},t))\boldsymbol{x}(t)+\\ &(A_1+\Delta A_1(\boldsymbol{x},t))\boldsymbol{x}(t-h(t))+\\ &E_0\boldsymbol{f}(\boldsymbol{\sigma}(t))+(B_0+\Delta B_0(\boldsymbol{x},t))\boldsymbol{w}(t)\end{aligned}\tag{1}$$

$$\begin{aligned}\boldsymbol{y}(t) =&(C_0+\Delta C_0(\boldsymbol{x},t))\boldsymbol{x}(t)+\\ &(C_1+\Delta C_1(\boldsymbol{x},t))\boldsymbol{x}(t-h(t))+\\ &E_1\boldsymbol{f}(\boldsymbol{\sigma}(t))+(B_1+\Delta B_1(x,t))\boldsymbol{w}(t)\\ \boldsymbol{z}(t) =&L\boldsymbol{x}(t)\end{aligned}\tag{2}$$

$$\boldsymbol{\sigma}(t)=C\boldsymbol{x}(t), \boldsymbol{x}(t)=\boldsymbol{\phi}(t), \boldsymbol{w}(t)=\boldsymbol{0}, t\in[-h,0]\tag{3}$$

where $\boldsymbol{x}(t)\in R^n$ is the state vector, $\boldsymbol{w}(t)\in R^p$ is the disturbance input vector from $L_2[0,\infty)$, $\boldsymbol{y}(t)\in R^{n_1}$ is the measurement output vector, and $\boldsymbol{z}(t)\in R^{n_2}$ is the estimated state vector. The matrix $E\in R^{n\times n}$ may be singular. We also assume that rank $E=r\leq n$. $A_0$, $A_1$, $B_0$, $B_1$, $C_0$, $C_1$, $L$, $D$, $E_0$, $E_1$, and $C$ are known real constant matrices. $\Delta A_0(\cdot)$,$\Delta A_1(\cdot)$, $\Delta B_0(\cdot)$, $\Delta B_1(\cdot)$, $\Delta C_0(\cdot)$ and $\Delta C_1(\cdot)$ are time-variant matrices representing norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\begin{aligned}&\left[\begin{array}{ccc}\Delta A_0(\cdot) & \Delta A_1(\cdot) & \Delta B_0(\cdot)\\ \Delta C_0(\cdot) & \Delta C_1(\cdot) & \Delta B_1(\cdot)\end{array}\right]=\\ &\left[\begin{array}{c}G_1\\ G_2\end{array}\right]F(x,t)\left[\begin{array}{ccc}H_1 & H_2 & H_3\end{array}\right]\end{aligned}\tag{4}$$

where $G_1$, $G_2$, $H_1$, $H_2$ and $H_3$ are known real constant matrices. The uncertain matrix $F(\boldsymbol{x},t)$ with Lebesgue measurable elements satisfies

$$F^{\mathrm{T}}(\boldsymbol{x},t)F(\boldsymbol{x},t)\leq I, \forall t\tag{5}$$

$h(t)$ is time-varying bounded delay satisfying $0\leq h(t)\leq h, \dot{h}(t)\leq d\leq 1$. $\boldsymbol{\phi}(t)$ is smooth vector-valued continuous initial function defined in the Banach space $C_h$. It is assumed that $\boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\phi}(t)\leq\nu$, where $\nu>0$ can be seen as an upper bound on the initial states. In this paper, every nonlinear term is assumed to be of the form

$$\begin{aligned}&f_j(\cdot)\in K_j[0,k_j]=\{f_j(\sigma_j)|f_j(0)=0,\\ &\|\dot{f}_j(\sigma_j)\|\leq\alpha, 0<\sigma_jf_j(\sigma_j)\leq k_j\sigma_j^2(\sigma_j\neq 0)\}\end{aligned}\tag{6}$$

where $\alpha$ and $k_j$ $(j=1,2,\cdots,n)$ are positive scalars.

Throughout this paper, we will use the following concepts and introduce the following useful definitions.

**Definition 1**[1]**.**

1) The pair $(E,A)$ is said to be regular if $\det(sE-A)$ is not identically zero.

2) The pair $(E,A)$ is said to be impulse free if $\deg(\det(sE-A))$=rank $E$.

**Definition 2**[1]**.**

1) The uncertain Lur′e time-delay singular system (1) is said to be regular and impulse free if the pair $(E,A)$ is regular and impulse free for all admissible uncertainties described in (4) and (5).

2) The uncertain Lurie time-delay singular system (1) is said to be stable if for any $\varepsilon>0$ there exists a scalar $\delta(\varepsilon)>0$ such that for any compatible initial conditions $\boldsymbol{\phi}(t)$ satisfying $\sup_{-\tau\leq t\leq 0}\|\boldsymbol{\phi}(t)\|\leq\delta(\varepsilon)$, the solution $\boldsymbol{x}(t)$ of system (1) satisfies $\|\boldsymbol{x}(t)\|\leq\varepsilon$, for $t\geq 0$; furthermore, $\lim\limits_{t\to\infty}\|\boldsymbol{x}(t)\|=0$ for all admissible uncertainties described in (4) and (5).

**Definition 3.** The uncertain Lurie time-delay singular systems (1) is said to be robustly stable if system (1) with

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$\boldsymbol{w}(t) = \boldsymbol{0}$ is regular, impulse free and stable for all admissible uncertainties.

**Definition 4.** (The problem of robust $H_\infty$ filtering) The uncertain Lurie singular system $\Sigma$ is said to have a robust $H_\infty$ performance if there exists a singular and robustly stable filter of order $n$ with the following form:

$$\Sigma_f : \boldsymbol{0} = A_f\boldsymbol{x}_f(t) + B_f\boldsymbol{y}(t) \tag{7}$$

$$\boldsymbol{z}_f(t) = L_f\boldsymbol{x}_f(t), \boldsymbol{x}_f(0) = \boldsymbol{0} \tag{8}$$

such that not only the filtering error system is robustly stable in the sense of Definition 3 , but also the filtering error

$$\tilde{\boldsymbol{z}}(t) = \boldsymbol{z}(t) - \boldsymbol{z}_f(t) \tag{9}$$

satisfies the following condition:

$$\int_0^\infty \tilde{\boldsymbol{z}}^{\mathrm{T}}\tilde{\boldsymbol{z}}\mathrm{d}t \le \gamma^2(\int_0^\infty \boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}\mathrm{d}t + \tau) \tag{10}$$

for given scalars $\gamma > 0$, and $\sigma > 0$, for all non-zero $\boldsymbol{w}(t) \in L_2[0, \infty)$ and for all parameter uncertainties. Note that the scalar $\sigma > 0$ is induced by the initial condition of the system.

## 3 Main results

In this section, the problems of robust stability and robust $H_\infty$ filtering based on LMI approach for systems (1)$\sim$(4) are discussed. The following theorems are useful for derivation of the main result.

**Theorem 1.** Consider the uncertain Lurie time-delay singular system $\Sigma$. This system is robustly stable and

$$\int_0^\infty z^{\mathrm{T}}z dt \le \gamma^2(\int_0^\infty w^{\mathrm{T}}w\mathrm{d}t + \tau) \tag{11}$$

for given scalars $\gamma > 0$, and $\sigma > 0$, if there exist a matrix $P$, a positive definite symmetric matrix $Q > 0$, and scalars $\varepsilon > 0$ and $\theta > 0$ such that the following conditions hold, i.e.,

$$EP^{\mathrm{T}} = PE^{\mathrm{T}} \ge 0 \tag{12}$$

$$\lambda_1 + h\lambda_2 \le \tau\gamma^2/\nu \tag{12$'$}$$

$$M = \begin{bmatrix} \hat{M}_1 & \hat{M}_2 \\ \hat{M}_3 & \hat{M}_4 \end{bmatrix} < 0 \tag{12$''$}$$

where

$$\hat{M}_1 = \begin{bmatrix} M_{11} & B_0P^{\mathrm{T}} & A_1P^{\mathrm{T}} \\ PB_0^{\mathrm{T}} & -\gamma^2 I & 0 \\ PA^{\mathrm{T}} & 0 & -(1-d)Q \end{bmatrix}$$

$$M_{11} = A_0P^{\mathrm{T}} + PA_0^{\mathrm{T}} + Q$$

$$\hat{M}_2 = \begin{bmatrix} E_0 & PC^{\mathrm{T}}K^{\mathrm{T}} & PL^{\mathrm{T}} & PH_1^{\mathrm{T}} & G_1 \\ 0 & 0 & 0 & PH_3^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & PH_2^{\mathrm{T}} & 0 \end{bmatrix}$$

$$\hat{M}_3 = \begin{bmatrix} E_0^{\mathrm{T}} & 0 & 0 \\ KCP^{\mathrm{T}} & 0 & 0 \\ LP^{\mathrm{T}} & 0 & 0 \\ H_1P^{\mathrm{T}} & H_3P^{\mathrm{T}} & H_2P^{\mathrm{T}} \\ G_1^{\mathrm{T}} & 0 & 0 \end{bmatrix}$$

$$\hat{M}_4 = \begin{bmatrix} -\varepsilon^{-1}I & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -\theta I & 0 \\ 0 & 0 & 0 & 0 & -\theta^{-1}I \end{bmatrix}$$

and $\beta > 0$ is the unique root of the equation

$$\beta(\lambda_1 + \lambda_2 he^{\beta h}) = \lambda_{\tilde{M}} \tag{13}$$

with

$$\lambda_1 = \lambda_{\max}(P^{-1}E)$$
$$\lambda_2 = \lambda_{\max}(P^{-1}QP^{-\mathrm{T}})$$
$$\lambda_{\tilde{M}} = \lambda_{\max}(P^{-1}\tilde{M}_1P^{-\mathrm{T}})$$
$$\tilde{M}_1 = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{13} & \tilde{M}_{14} \end{bmatrix}$$

where

$$\tilde{M}_{11} = M_{11} + \varepsilon E_0E_0^{\mathrm{T}} + \theta G_1G_1^{\mathrm{T}} + \theta^{-1}PH_1^{\mathrm{T}}H_1P^{\mathrm{T}}$$
$$\tilde{M}_{12} = A_1P^{\mathrm{T}} + \theta^{-1}PH_1^{\mathrm{T}}H_2P^{\mathrm{T}}$$
$$\tilde{M}_{13} = PA_1^{\mathrm{T}} + \theta^{-1}PH_2^{\mathrm{T}}H_1P^{\mathrm{T}}$$
$$\tilde{M}_{14} = -(1-d)Q + \theta^{-1}PH_2^{\mathrm{T}}H_2P^{\mathrm{T}}$$

**Proof.** First, we prove the robust stability of system $\Sigma$. To this end, we consider the system $\Sigma$ with $\boldsymbol{w}(t) = \boldsymbol{0}$; that is

$$E\dot{\boldsymbol{x}}(t) = A_{0\Delta}\boldsymbol{x}(t) + A_{1\Delta}\boldsymbol{x}(t - h_i(t)) + E_0\boldsymbol{f}(\boldsymbol{\sigma}(t)) \tag{14}$$

where $A_{0\Delta} = A_0 + \Delta A_0()$, and $A_{1\Delta} = A_1 + \Delta A_1()$.

By Schur complement argument, the LMI (12$''$) is equivalent to $A_0P^{\mathrm{T}} + PA_0^{\mathrm{T}} + Q < 0$, and $Q > 0$. Then it is easy to see that $A_0P^{\mathrm{T}} + PA_0^{\mathrm{T}} < 0$. From Dai[2], it follows that the pair $(E, A_0)$ is regular and impulse free. Therefore, by Definition 2, the singular delay system $\Sigma$ is regular and impulse free.

Dai[2] showed that if the pair $(E, A_0)$ is regular and impulse free, then there exist two orthogonal matrices $L_1$ and $L_2 \in R^{n\times n}$ such that

$$\begin{aligned} \bar{E} &:= L_1EL_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{A} &:= L_1A_0L_2 = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix} \end{aligned} \tag{15}$$

where $I_r \in R^{r\times r}$, and $I_{n-r} \in R^{(n-r)\times(n-r)}$ are identity matrices, $A_r \in R^{r\times r}$. According to (15), let

$$\begin{aligned} \bar{A}_1 &:= L_1A_1L_2 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \\ \bar{B}_0 &:= L_1B_0L_2, \bar{E}_0 := L_1E_0 = \begin{bmatrix} \bar{E}_{01} \\ \bar{E}_{02} \end{bmatrix} \\ \bar{C} &:= L_2^{-1}CL_2, \bar{K} := KL_2 \\ \bar{Q}_1 &:= L_1QL_1^{\mathrm{T}} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \end{aligned} \tag{16}$$

Pre-multiplying and post-multiplying the left and right-hand sides of (12), (12$'$), and (12$''$) by diag$\{L_1, L_1, L_1, I, I, I, I, I\}$ and diag$\{L_1^{\mathrm{T}}, L_1^{\mathrm{T}}, L_1^{\mathrm{T}}, I, I, I, I, I\}$, respectively, we have

$$\bar{E}\bar{P}^{\mathrm{T}} = \bar{P}\bar{E}^{\mathrm{T}} \ge 0 \tag{17}$$

and

$$\bar{M} = \begin{bmatrix} \bar{\hat{M}}_1 & \bar{\hat{M}}_2 \\ \bar{\hat{M}}_3 & \bar{\hat{M}}_4 \end{bmatrix} < 0 \tag{18}$$

where

$$\bar{M}_1 = \begin{bmatrix} \bar{M}_{11} & \bar{B}_0\bar{P}^{\mathrm{T}} & \bar{A}_1\bar{P}^{\mathrm{T}} \\ \bar{P}\bar{B}_0^{\mathrm{T}} & -\gamma^2 I & 0 \\ \bar{P}\bar{A}^{\mathrm{T}} & 0 & -(1-d)\bar{Q} \end{bmatrix}$$

$$\bar{M}_{11} = \bar{A}_0\bar{P}^{\mathrm{T}} + \bar{P}\bar{A}_0{}^{\mathrm{T}} + \bar{Q}$$

$$\bar{M}_2 = \begin{bmatrix} \bar{E}_0 & \bar{P}\bar{C}^{\mathrm{T}}\bar{K}^{\mathrm{T}} & \bar{P}\bar{L}^{\mathrm{T}} & \bar{P}\bar{H}_1^{\mathrm{T}} & \bar{G}_1 \\ 0 & 0 & 0 & \bar{P}\bar{H}_3^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & \bar{P}\bar{H}_2^{\mathrm{T}} & 0 \end{bmatrix}$$

$$\bar{M}_3 = \begin{bmatrix} \bar{E}_0^{\mathrm{T}} & 0 & 0 \\ \bar{K}\bar{C}\bar{P}^{\mathrm{T}} & 0 & 0 \\ \bar{L}\bar{P}^{\mathrm{T}} & 0 & 0 \\ \bar{H}_1\bar{P}^{\mathrm{T}} & \bar{H}_3\bar{P}^{\mathrm{T}} & \bar{H}_2\bar{P}^{\mathrm{T}} \\ \bar{G}_1^{\mathrm{T}} & 0 & 0 \end{bmatrix}$$

$$\bar{M}_4 = \begin{bmatrix} -\varepsilon^{-1}I & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -\theta I & 0 \\ 0 & 0 & 0 & 0 & -\theta^{-1}I \end{bmatrix}$$

Now, let

$$\boldsymbol{\xi}(t) = L_2^{-1}\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{\xi}_1(t) \\ \boldsymbol{\xi}_2(t) \end{bmatrix} \tag{19}$$

where $\boldsymbol{\xi}_1(t) \in R^r$ and $\boldsymbol{\xi}_2(t) \in R^{n-r}$. Substituting (15) and (16) into the singular delay system (14), we get

$$\bar{E}\dot{\boldsymbol{\xi}}(t) = \bar{A}_{0\Delta}\boldsymbol{x}(t) + \bar{A}_{1\Delta}\boldsymbol{x}(t-h(t)) + \bar{E}_0\boldsymbol{f}(\boldsymbol{\eta}(t)) \tag{20}$$

where $\boldsymbol{\eta}(t) = \bar{C}\boldsymbol{\xi}(t)$. It is easy to see that the stability of the uncertain Lurie time-delay singular system (20) is equivalent to that of system (17). The singular delay system (20) can also be decomposed into

$$\begin{aligned} \dot{\boldsymbol{\xi}}_1(t) =& A_{r\Delta}\boldsymbol{\xi}_1(t) + \bar{A}_{11\Delta}\boldsymbol{\xi}_1(t-h(t)) \\ &+ \bar{A}_{12\Delta}\boldsymbol{\xi}_2(t-h(t)) + \bar{E}_{01}\boldsymbol{f}(\boldsymbol{\eta}(t)) \end{aligned} \tag{21}$$

$$\begin{aligned} \boldsymbol{0} =& \xi_2(t) + \bar{A}_{21\Delta}\boldsymbol{\xi}_1(t-h(t)) \\ &+ \bar{A}_{22\Delta}\boldsymbol{\xi}_2(t-h(t)) + \bar{E}_{02}\boldsymbol{f}(\boldsymbol{\eta}(t)) \end{aligned} \tag{22}$$

Define

$$\begin{aligned} V(\boldsymbol{\xi}(t)) =& \boldsymbol{\xi}_1^{\mathrm{T}}(t)\bar{P}_{11}^{-1}\boldsymbol{\xi}_1(t) + \int_{t-h(t)}^{t} \boldsymbol{\xi}^{\mathrm{T}}(s)\bar{P}^{-1}\bar{Q}\bar{P}^{-\mathrm{T}}\boldsymbol{\xi}(s)\mathrm{d}s \\ =& \boldsymbol{\xi}^{\mathrm{T}}(t)\bar{P}^{-1}\bar{E}\boldsymbol{\xi}(t) + \int_{t-h(t)}^{t} \boldsymbol{\xi}^{\mathrm{T}}(s)\bar{P}^{-1}\bar{Q}\bar{P}^{-\mathrm{T}}\boldsymbol{\xi}(s)\mathrm{d}s \end{aligned} \tag{23}$$

According to, for example [1,3∼5], we can immediately deduce that the following inequality holds

$$\dot{V}(\boldsymbol{\xi}(t)) \le \tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\bar{P}^{-1}\bar{M}_{1\Delta}\bar{P}^{-\mathrm{T}}\tilde{\boldsymbol{\xi}}(t) \tag{24}$$

where $\tilde{\boldsymbol{\xi}}(t) = \begin{bmatrix} \boldsymbol{\xi}^{\mathrm{T}}(t) & \boldsymbol{\xi}^{\mathrm{T}}(t-h(t)) \end{bmatrix}$

$$\bar{M}_{1\Delta} = \begin{bmatrix} \bar{M}_{11\Delta} & \bar{A}_{1\Delta}\bar{P}^{\mathrm{T}} \\ \bar{P}\bar{A}_{1\Delta}^{\mathrm{T}} & (1-d)\bar{Q} \end{bmatrix}$$

$$\begin{aligned} \bar{M}_{11\Delta} =& \bar{A}_{0\Delta}\bar{P}^{\mathrm{T}} + \bar{P}\bar{A}_{0\Delta}^{\mathrm{T}} + \bar{Q} + \\ & \varepsilon\bar{E}_0\bar{E}_0^{\mathrm{T}} + \varepsilon^{-1}\bar{P}\bar{C}^{\mathrm{T}}\bar{K}^{\mathrm{T}}\bar{K}\bar{C}\bar{P}^{\mathrm{T}} \end{aligned}$$

Consider uncertain expressions (4) and (5), the matrix $\bar{M}_{1\Delta}$ can be rewritten as

$$\bar{M}_{1\Delta} = \bar{M}_1 + \Omega_1 F(\boldsymbol{x},t)\Omega_2 + (\Omega_1 F(\boldsymbol{x},t)\Omega_2)^{\mathrm{T}} \tag{25}$$

where $\bar{M}_1$ is $\bar{M}_{1\Delta}$ without uncertainties $\Delta\bar{A}_0()$ and $\Delta\bar{A}_1()$, and

$$\Omega_1^{\mathrm{T}} = \begin{bmatrix} \bar{G}_1^{\mathrm{T}} & 0 \end{bmatrix}, \Omega_2 = \begin{bmatrix} \bar{H}_1\bar{P}^{\mathrm{T}} & \bar{H}_2\bar{P}^{\mathrm{T}} \end{bmatrix}$$

On the other hand, by Schur complement and (18) we have

$$\bar{M}_1 + \theta\Omega_1\Omega_1^{\mathrm{T}} + \theta^{-1}\Omega_2^{\mathrm{T}}\Omega_2 < 0$$

According to [6] and (25), it can be deduced that

$$\bar{M}_{1\Delta} \le \bar{M}_1 + \theta\Omega_1\Omega_1^{\mathrm{T}} + \theta^{-1}\Omega_2^{\mathrm{T}}\Omega_2 < 0 \tag{26}$$

impling that for all $\boldsymbol{\xi}(t) \ne \boldsymbol{0}$

$$\dot{V}(\boldsymbol{\xi}(t)) \le -\lambda_{\hat{M}}\|\tilde{\boldsymbol{\xi}}(t)\|^2 \le -\lambda_{\hat{M}}\|\boldsymbol{\xi}(t)\|^2 \tag{27}$$

where $\lambda_{\hat{M}} = -\lambda_{\max}[\bar{P}^{-1}(\bar{M}_1 + \theta\Omega_1\Omega_1^{\mathrm{T}} + \theta^{-1}\Omega_2^{\mathrm{T}}\Omega_2)\bar{P}^{-\mathrm{T}}]$.
Then we obtain, for a scalar $\beta > 0$,

$$\begin{aligned} \frac{de^{\beta t}V(\boldsymbol{\xi}(t))}{dt} =& e^{\beta t}[\beta V(\boldsymbol{\xi}(t)) + \dot{V}(\boldsymbol{\xi}(t))] \le \\ & e^{\beta t}[(\bar{\lambda}_1\beta - \lambda_{\hat{M}})\|\boldsymbol{\xi}(t)\|^2 + \\ & \bar{\lambda}_2\beta\int_{t-h(t)}^{t}\|\boldsymbol{\xi}(s)\|^2 ds] \end{aligned}$$

where $\bar{\lambda}_1 = \lambda_{\max}(\bar{P}^{-1}\bar{E})$, $\bar{\lambda}_2 = \lambda_{\max}(\bar{P}^{-1}\bar{Q}\bar{P}^{-\mathrm{T}})$.
Since $L_1$ and $L_2$ are orthogonal matrices, it follows from (13) that $\beta > 0$ is the unique root of the equation

$$\beta(\bar{\lambda}_1 + \bar{\lambda}_2 h e^{\beta h}) = \lambda_{\hat{M}}$$

Integrating both sides of this inequality from 0 to $t$ and then considering the following inequality[7]

$$\int_0^t e^{\beta t}(\int_{t-h(t)}^{t}\|\boldsymbol{\xi}(s)\|^2 ds)\mathrm{d}t \le he^{\beta h}\int_0^t e^{\beta s}\|\boldsymbol{\xi}(s)\|^2\mathrm{d}s$$

we have

$$V(\boldsymbol{\xi}(t)) \le e^{-\beta t}V(\bar{\boldsymbol{\phi}}(t))$$

where $\bar{\boldsymbol{\phi}}(t) = L_2^{-1}\boldsymbol{\phi}(t)$. Therefore,

$$\|\boldsymbol{\xi}_1(t)\|^2 \le (e^{-\beta t}/\lambda_{\min}(\bar{P}_{11}^{-1}))V(\bar{\boldsymbol{\phi}}(t))$$

which implies that

$$\lim_{t\to\infty}\|\boldsymbol{\xi}_1(t)\| = 0 \tag{28}$$

Furthermore, by (23) and (24),

$$\begin{aligned} \lambda_{\min}(\bar{P}_{11}^{-1})\|\boldsymbol{\xi}_1(t)\|^2 - V(\bar{\phi}(t)) \le& \int_0^t \dot{V}(\boldsymbol{\xi}(s))\mathrm{d}s \le \\ & -\lambda_{\hat{M}}\int_0^t\|\boldsymbol{\xi}_2(s)\|^2\mathrm{d}s \end{aligned}$$

So, we can deduce that

$$\int_0^t\|\boldsymbol{\xi}_2(s)\|^2\mathrm{d}s \le (1/\lambda_{\hat{M}})V(\bar{\boldsymbol{\phi}}(t)) \tag{29}$$

According to (27), we have

$$\int_0^t\|\bar{E}_{02}\boldsymbol{f}(\boldsymbol{\eta})\|\mathrm{d}t \le \int_0^t\sqrt{\lambda_3}(\|\boldsymbol{\xi}_1(t)\| + \|\boldsymbol{\xi}_2(t)\|)\mathrm{d}t$$

where $\lambda_3 = \lambda_{\max}[\bar{C}^{\mathrm{T}}\bar{K}^{\mathrm{T}}\bar{E}_{02}^{\mathrm{T}}\bar{E}_{02}\bar{K}\bar{C}] \ge 0$.

Taking into account the famous Barbalat′s Lemma[8], it follows from (6), (28), and (29) that

$$\lim_{t\to\infty}\bar{E}_{02}\boldsymbol{f}(\boldsymbol{\eta}) = 0 \tag{30}$$

Thus, combining (18) and (22) with (28) and (30), we can get

$$\lim_{t\to\infty}\|\boldsymbol{\xi}_2(t)\| = 0 \quad (31)$$

which implies that system $\Sigma$ is robustly stable.

Next, we will show the $H_\infty$ performance of system $\Sigma$. To this end, if the dimension of $w(t)$ $p < n$, substitute $\hat{\boldsymbol{w}}^{\rm T}(t) = \begin{bmatrix} \boldsymbol{w}^{\rm T}(t) & 0 \end{bmatrix}$ for $\boldsymbol{w}^{\rm T}(t)$ such that $\boldsymbol{w}(t) \in R^n$. At the same time, substitute the matrices $\begin{bmatrix} B_0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \Delta B_{10}() & 0 \end{bmatrix}$ for matrices $B_0$ and $\Delta B_{10}()$, respectively, such that matrices $B_0$ and $\Delta B_{10}() \in R^{n\times n}$. Consider the following system

$$\begin{aligned}&\bar{E}\dot{\boldsymbol{\xi}}(t) = \bar{A}_{0\Delta}\boldsymbol{\xi}(t) + \bar{A}_{1\Delta}\boldsymbol{\xi}(t-h(t)) + \bar{E}_0\boldsymbol{f}(\boldsymbol{\eta}(t)) + \bar{B}_{0\Delta}\bar{\boldsymbol{w}}(t)\\ &\boldsymbol{z}(t) = \bar{L}\boldsymbol{\xi}(t)\end{aligned}$$

where $\bar{\boldsymbol{w}}(t) = L_2^{-1}\boldsymbol{w}(t)$. Similar to the derivation of (27), according to [6,9,10], we can obtain

$$\begin{aligned}&\dot{V}(\boldsymbol{\xi}(t),\boldsymbol{w}(t)) + \boldsymbol{z}^{\rm T}(t)\boldsymbol{z}(t) - \gamma^2\bar{\boldsymbol{w}}^{\rm T}(t)\bar{\boldsymbol{w}}(t)\\ &= \dot{V}(\boldsymbol{\xi}(t),\boldsymbol{w}(t)) + \boldsymbol{z}^{\rm T}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{w}^{\rm T}(t)\boldsymbol{w}(t)\\ &\le \boldsymbol{\zeta}^{\rm T}(t)\bar{P}^{-1}\bar{M}_2\bar{P}^{-\rm T}\boldsymbol{\zeta}(t)\end{aligned} \quad (32)$$

where $\boldsymbol{\zeta}(t) = \begin{bmatrix} \boldsymbol{\xi}^{\rm T}(t) & \boldsymbol{\xi}^{\rm T}(t-h(t)) & \bar{\boldsymbol{w}}^{\rm T}(t) \end{bmatrix}^{\rm T}$, and

$$\begin{aligned}&\bar{M}_2 = \begin{bmatrix} \bar{M}_1 & \bar{B}_0\bar{P}^{\rm T}\\ \bar{P}\bar{B}_0^{\rm T} & -\gamma^2 I\end{bmatrix} + \boldsymbol{\theta}\tilde{\Omega}_1\tilde{\Omega}_1^{\rm T} + \boldsymbol{\theta}^{-1}\tilde{\Omega}_2^{\rm T}\tilde{\Omega}_2\\ &\tilde{\Omega}_1^{\rm T} = \begin{bmatrix} \Omega_1^{\rm T} & 0\end{bmatrix}, \tilde{\Omega}_2 = \begin{bmatrix} \Omega_2 & \bar{H}_3\bar{P}^{\rm T}\end{bmatrix}\end{aligned}$$

Integrating both sides of (32) from 0 to $\infty$ and taking into account $\lim\limits_{t\to\infty}\boldsymbol{\xi}(t) = 0$, (12c) and (18), we have

$$\int_0^\infty \boldsymbol{z}^{\rm T}\boldsymbol{z}{\rm d}t - \gamma^2\int_0^\infty \boldsymbol{w}^{\rm T}\boldsymbol{w}{\rm d}t \le V(\bar{\boldsymbol{\phi}}(t)) \le (\lambda_1 + h\lambda_2)\nu \le \tau\gamma^2,$$

which implies that (11) is satisfied. This completes the proof. □

Now, we are in the position to present the solution to the robust $H_\infty$ filtering problem for uncertain Lurie time-delay singular systems.

Defining $\boldsymbol{\varphi} = \begin{bmatrix} \boldsymbol{x}^{\rm T}(t) & \boldsymbol{x}_f^{\rm T}(t)\end{bmatrix}^{\rm T}$, noting (4) and (5), and after extending the dimension of $w(t)$, we derive the following augmented model from system $\Sigma$ and the filter $\Sigma_f$. where

$$\begin{aligned}\Sigma_e: \tilde{E}\dot{\boldsymbol{\varphi}}(t) =& (\tilde{A}_0 + \Delta\tilde{A}_0(\cdot))\boldsymbol{\varphi}(t)\\ &+ (\tilde{A}_1 + \Delta\tilde{A}_1(\cdot))\boldsymbol{\varphi}(t-h(t))\\ &+ \tilde{E}_0 f(\boldsymbol{\sigma}(t)) + (\tilde{B}_0 + \Delta\tilde{B}_0(\cdot))\tilde{\boldsymbol{w}}(t)\\ \tilde{\boldsymbol{z}}(t) =& \tilde{L}\boldsymbol{\varphi}(t), \boldsymbol{\sigma}(t) = \tilde{C}\boldsymbol{\varphi}(t)\\ \boldsymbol{\varphi}(t) =& \begin{bmatrix} \boldsymbol{\phi}^{\rm T}(t) & 0\end{bmatrix}^{\rm T}, t \in [-h, 0]\end{aligned} \quad (33)$$

with $\tilde{\boldsymbol{w}}^{\rm T}(t) = \begin{bmatrix} \boldsymbol{w}^{\rm T}(t) & 0\end{bmatrix}^{\rm T}$, and

$$\begin{aligned}&\tilde{E} = \begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \tilde{A}_0 = \begin{bmatrix} A_0 & 0\\ B_fC_0 & A_f\end{bmatrix}\\ &\tilde{A}_1 = \begin{bmatrix} A_1 & 0\\ B_fC_1 & 0\end{bmatrix} \tilde{E}_0 = \begin{bmatrix} E_0\\ B_fE_1\end{bmatrix}\\ &\tilde{B}_0 = \begin{bmatrix} B_0 & 0\\ B_fB_1 & 0\end{bmatrix}, \tilde{L} = \begin{bmatrix} L & -L_f\end{bmatrix}\\ &\tilde{C} = \begin{bmatrix} C & 0\end{bmatrix}, \Delta\tilde{A}_0(\cdot) = \begin{bmatrix} \Delta A_0 & 0\\ B_f\Delta C_0 & 0\end{bmatrix}\\ &\Delta\tilde{A}_1(\cdot) = \begin{bmatrix} \Delta A_1 & 0\\ B_f\Delta C_1 & 0\end{bmatrix}, \Delta\tilde{B}_0(\cdot) = \begin{bmatrix} \Delta B_0 & 0\\ B_f\Delta B_1 & 0\end{bmatrix}\end{aligned}$$

The parameter uncertainties in system $\Sigma_e$ can be rewritten as

$$\begin{aligned}&\begin{bmatrix} \Delta\tilde{A}_0(\cdot) & \Delta\tilde{A}_1(\cdot) & \Delta\tilde{B}_0(\cdot)\end{bmatrix} =\\ &\tilde{G}_1F(x(t),t)\begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 & \tilde{H}_3\end{bmatrix}\end{aligned} \quad (34)$$

where

$$\begin{aligned}&\tilde{G}_1 = \begin{bmatrix} G_1\\ B_fG_2\end{bmatrix}, \tilde{H}_1 = \begin{bmatrix} H_1 & 0\end{bmatrix}\\ &\tilde{H}_2 = \begin{bmatrix} H_2 & 0\end{bmatrix}, \tilde{H}_3 = \begin{bmatrix} H_3 & 0\end{bmatrix}\end{aligned}$$

As in the work of Gahinet and Apkarian[11], it is assumed that the matrix $P$ has the following form.

$$P^{-1} = \mathcal{P} = \begin{bmatrix} I & 0\\ Y^{-1} & I\end{bmatrix}^{-1}\begin{bmatrix} Y & W\\ I & 0\end{bmatrix} = X_1^{-1}X_2 \quad (35)$$

where $Y > 0$ is a symmetric matrix and $W$ is a nonsingular matrix. Then, (12a) is equivalent to

$$X_1\mathcal{P}\tilde{E}X_1^{\rm T} = X_1\tilde{E}^{\rm T}\mathcal{P}^{\rm T}X_1^{\rm T}$$

which implies that

$$YE = E^{\rm T}Y \quad (36)$$

With the techniques in Theorem 1, we can deduce

$$\begin{aligned}\tilde{M}' =& {\rm diag}\{X_1, I, I, I, I, I, I, I\}{\rm diag}\{\mathcal{P}, \mathcal{P}, \mathcal{P}, I, I, I, I, I\}M\times\\ &{\rm diag}\{X_1^{\rm T}, I, I, I, I, I, I, I\}{\rm diag}\{\mathcal{P}^{\rm T}, \mathcal{P}^{\rm T}, \mathcal{P}^{\rm T}, I, I, I, I, I\} <\\ &0\end{aligned} \quad (37)$$

Set

$$\Upsilon = \mathcal{P}\mathcal{P}^{\rm T}\mathbb{S}, \Phi = \mathcal{P}\tilde{Q}\mathcal{P}^{\rm T} = \begin{bmatrix} \Phi_{11} & 0\\ 0 & \Phi_{22}\end{bmatrix}$$

$$(\Phi_{11} = \Phi_{11}^{\rm T} > 0 \text{ and } \Phi_{22} = \Phi_{22}^{\rm T} > 0)$$

Then, the following inequality holds.

$$\begin{aligned}\tilde{M} =& {\rm diag}\{{\rm diag}(I, Y), I, I, I, I, I, I, I\}M'\times\\ &{\rm diag}\{{\rm diag}(I, Y), I, I, I, I, I, I, I\} < 0\end{aligned} \quad (38)$$

where $\tilde{M} < 0$ is denoted by inequality (39), with $\tilde{\Phi}_{22} = Y\Phi_{22}Y$, and

$$\tilde{M} = \begin{bmatrix} \hat{\tilde{M}}_1 & \hat{\tilde{M}}_2\\ \hat{\tilde{M}}_3 & \hat{\tilde{M}}_4\end{bmatrix} < 0 \quad (39)$$

where

$$\hat{\tilde{M}}_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13}\\ \Theta_{12}^{\rm T} & -\gamma^2\Upsilon & 0\\ \Theta_{13}^{\rm T} & 0 & -(1-d)\Phi\end{bmatrix}$$

$$\hat{\tilde{M}}_2 = \begin{bmatrix} \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18}\\ 0 & 0 & 0 & \Theta_{27} & 0\\ 0 & 0 & 0 & \Theta_{37} & 0\end{bmatrix}$$

$$\hat{\tilde{M}}_3 = \begin{bmatrix} \Theta_{14}^{\rm T} & 0 & 0\\ \Theta_{15}^{\rm T} & 0 & 0\\ \Theta_{16}^{\rm T} & 0 & 0\\ \Theta_{17}^{\rm T} & \Theta_{27}^{\rm T} & \Theta_{37}^{\rm T}\\ \Theta_{18}^{\rm T} & 0 & 0\end{bmatrix}$$

$$\hat{\tilde{M}}_4 = \begin{bmatrix} -\varepsilon^{-1}I & 0 & 0 & 0 & 0\\ 0 & -\varepsilon I & 0 & 0 & 0\\ 0 & 0 & -I & 0 & 0\\ 0 & 0 & 0 & -\theta I & 0\\ 0 & 0 & 0 & 0 & -\theta^{-1}I\end{bmatrix}$$

$$\Theta_{11} = \begin{bmatrix} \mathrm{sym}(YA_0 + \Gamma C_0) + \Phi_{11} & \mathrm{sym}(YA_0) + \Gamma C_0 + \Pi \\ \mathrm{sym}(YA_0) + C_0^{\mathrm{T}}\Gamma^{\mathrm{T}} + \Pi^{\mathrm{T}} & \mathrm{sym}(YA_0) + \Phi_{11} + \tilde{\Phi}_{22} \end{bmatrix}$$

$$\Theta_{13} = \begin{bmatrix} YA_1 + \Gamma C_1 & 0 \\ YA_1 & 0 \end{bmatrix}, \Theta_{37} = \begin{bmatrix} H_2^{\mathrm{T}} \\ 0 \end{bmatrix}$$

$$\Theta_{12} = \begin{bmatrix} YB_0 + \Gamma B_1 & 0 \\ YB_0 & 0 \end{bmatrix}, \Theta_{14} = \begin{bmatrix} YE_0 + \Gamma E_1 \\ YE_0 \end{bmatrix}$$

$$\Theta_{15} = \begin{bmatrix} C^{\mathrm{T}}K^{\mathrm{T}} \\ C^{\mathrm{T}}K^{\mathrm{T}} \end{bmatrix}, \Theta_{16} = \begin{bmatrix} L^{\mathrm{T}} \\ L^{\mathrm{T}} + \mathcal{L}^{\mathrm{T}} \end{bmatrix}$$

$$\Theta_{17} = \begin{bmatrix} H_1^{\mathrm{T}} \\ H_1^{\mathrm{T}} \end{bmatrix}, \Theta_{18} = \begin{bmatrix} YG_1 + \Gamma G_2 \\ YG_1 \end{bmatrix}, \Theta_{27} = \begin{bmatrix} H_3^{\mathrm{T}} \\ 0 \end{bmatrix}$$

Synthetizing the above analysis, we derive the main results in this section.

**Theorem 2.** Consider the uncertain Lurie time-delay singular system $\Sigma$ together with (7) and (8). For given scalars $\gamma > 0$, and $\sigma > 0$, So the robust $H_\infty$ filtering problem is solvable if there exist a matrix $\Gamma$, $\Pi$, a positive definite symmetric $Y$, $\Phi_{11}$, $\Phi_{22}$. $\tilde{\Phi}_{22} = Y\Phi_{22}Y$, and scalars $\varepsilon > 0$, and $\theta > 0$, such that equality (36), LMI (39) and (12$''$) hold, and $\beta > 0$ is the unique root of (40)

$$\beta(\lambda_1 + \lambda_2 h e^{\beta h}) = \lambda_{\tilde{M}'_{11}} \tag{40}$$

with

$$\lambda_1 = \lambda_{\max}(YE)$$
$$\lambda_2 = \max(\lambda_{\max}(\Phi_{11}), \lambda_{\max}(\Phi_{22}))$$
$$\lambda_{\tilde{M}'_{11}} = \lambda_{\max}(\tilde{M}'_{11})$$

where

$$\tilde{M}'_{11} = \begin{bmatrix} \tilde{M}'_{111} & \tilde{M}'_{112} \\ \tilde{M}'_{113} & \tilde{M}'_{114} \end{bmatrix}$$
$$\tilde{M}'_{111} = [\Theta_{11} + \varepsilon\Theta_{14}\Theta_{14}^{\mathrm{T}} + \theta\Theta_{18}\Theta_{18}^{\mathrm{T}} + \theta^{-1}\Theta_{17}\Theta_{17}^{\mathrm{T}}]$$
$$\tilde{M}'_{112} = \Theta_{13} + \theta^{-1}\Theta_{17}\Theta_{37}^{\mathrm{T}}$$
$$\tilde{M}'_{113} = \Theta_{13}^{\mathrm{T}} + \theta^{-1}\Theta_{37}\Theta_{17}^{\mathrm{T}}$$
$$\tilde{M}'_{114} = [-(1-d)\begin{bmatrix} \Phi_{11} & 0 \\ 0 & \Phi_{22} \end{bmatrix} + \theta^{-1}\Theta_{37}\Theta_{37}^{\mathrm{T}}]$$

In this case, a suitable $H_\infty$ filter in the form of (7) and (8) is given by

$$A_f = W^{-1}\Pi Y^{-1}, B_f = W^{-1}\Gamma, L_f = \mathcal{L}Y^{-1},$$

where $W$ is arbitrary nonsingular matrix.

**Proof.** From the above analysis and Theorem 1, the desired result follows immediately. □

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